§12.6 –Ratio Test, Root Test, Absolute Convergence

Mark Woodard

Furman U

Fall 2010

Outline

Theorem

If $\sum |a_n|$ *converges, then* $\sum a_n$ *converges.*

Theorem

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Proof.

Theorem

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Proof.

- We have $-|a_n| \leq a_n \leq |a_n|$; thus, $0 \leq a_n + |a_n| \leq 2|a_n|$.

Theorem

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Proof.

- We have $-|a_n| \leq a_n \leq |a_n|$; thus, $0 \leq a_n + |a_n| \leq 2|a_n|$.
- Thus the series $\sum(a_n + |a_n|)$ converges by SCT.

Theorem

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Proof.

- We have $-|a_n| \leq a_n \leq |a_n|$; thus, $0 \leq a_n + |a_n| \leq 2|a_n|$.
- Thus the series $\sum(a_n + |a_n|)$ converges by SCT.
- By hypothesis, the series $\sum -|a_n|$ converges.

Theorem

If $\sum |a_n|$ *converges, then* $\sum a_n$ *converges.*

Proof.

- We have $-|a_n| \leq a_n \leq |a_n|$; thus, $0 \leq a_n + |a_n| \leq 2|a_n|$.
- Thus the series $\sum(a_n + |a_n|)$ converges by SCT.
- By hypothesis, the series $\sum -|a_n|$ converges.
- Consequently,
$$\sum a_n = \sum \left((a_n + |a_n|) - |a_n|\right)$$
converges as well. $\square$

Problem

Examine the convergence of the following series:

Problem

Examine the convergence of the following series:

- $$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$$

Problem

Examine the convergence of the following series:

- $$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$$
- $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+2}{\sqrt{n^5+8}}$$

Definition

Definition

- If the series $\sum |a_n|$ converges, then we say that the series $\sum a_n$ converges *absolutely*.

Definition

- If the series $\sum |a_n|$ converges, then we say that the series $\sum a_n$ converges *absolutely*.
- if the series $\sum a_n$ converges but $\sum |a_n|$ diverges, then we say that the series $\sum a_n$ converges *conditionally*.

Problem

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converge absolutely, converge conditionally, or diverge?*

Problem

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converge absolutely, converge conditionally, or diverge?*

Solution

Problem

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converge absolutely, converge conditionally, or diverge?*

Solution

- *The series of absolute values diverges by PST,* $p = 1$.

Problem

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converge absolutely, converge conditionally, or diverge?*

Solution

- *The series of absolute values diverges by PST,* $p = 1$*.*
- *The series converges by the AST.*

Problem

Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converge absolutely, converge conditionally, or diverge?*

Solution

- *The series of absolute values diverges by PST,* $p = 1$.
- *The series converges by the AST.*
- *Thus the series converges conditionally.*

Theorem (The Ratio Test (RT))

Let $\{a_n\}$ *be a sequence of nonzero real numbers and suppose that*

$$\frac{|a_{n+1}|}{|a_n|} \to L$$

Theorem (The Ratio Test (RT))

Let $\{a_n\}$ be a sequence of nonzero real numbers and suppose that

$$\frac{|a_{n+1}|}{|a_n|} \to L$$

- *If $L < 1$, then $\sum a_n$ converges absolutely.*

Theorem (The Ratio Test (RT))

Let $\{a_n\}$ be a sequence of nonzero real numbers and suppose that

$$\frac{|a_{n+1}|}{|a_n|} \to L$$

- *If $L < 1$, then $\sum a_n$ converges absolutely.*
- *If $L > 1$, then $\sum a_n$ diverges.*

Theorem (The Ratio Test (RT))

Let $\{a_n\}$ be a sequence of nonzero real numbers and suppose that

$$\frac{|a_{n+1}|}{|a_n|} \to L$$

- *If $L < 1$, then $\sum a_n$ converges absolutely.*
- *If $L > 1$, then $\sum a_n$ diverges.*
- *If $L = 1$, then the test is inconclusive; the series may or may not converge.*

Problem

Determine the convergence or divergence of the following series:

Problem

Determine the convergence or divergence of the following series:

- $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Problem

Determine the convergence or divergence of the following series:

- $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Problem

Determine the convergence or divergence of the following series:

- $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Problem

Determine the convergence or divergence of the following series:

- $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Theorem (The Root Test)

Suppose $|a_n|^{1/n} \to L$.

Theorem (The Root Test)

Suppose $|a_n|^{1/n} \to L$.

- *If* $L < 1$, *then* $\sum a_n$ *converges absolutely.*

Theorem (The Root Test)

Suppose $|a_n|^{1/n} \to L$.

- *If* $L < 1$, *then* $\sum a_n$ *converges absolutely.*
- *If* $L > 1$, *then* $\sum a_n$ *diverges.*

Theorem (The Root Test)

Suppose $|a_n|^{1/n} \to L$.

- *If* $L < 1$, *then* $\sum a_n$ *converges absolutely.*
- *If* $L > 1$, *then* $\sum a_n$ *diverges.*
- *If* $L = 1$, *then the test is inconclusive; the series may or may not converge.*

Problem

Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{3n}{5n+6} \right)^n$ *converges.*

Problem

Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n}{5n+6}\right)^n$ *converges.*

Solution

Since

$$|a_n|^{1/n} = \frac{3n}{5n+6} \to \frac{3}{5} < 1,$$

the series converges by the Root Test.