§12.5 –Alternating Series

Mark Woodard

Furman U

Fall 2010

Outline

Definition

Let $\{a_n\}$ be a sequence of positive numbers. Infinite series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

$$\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$$

are called *alternating series*.

Definition

Let $\{a_n\}$ be a sequence of positive numbers. Infinite series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

$$\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$$

are called *alternating series*.

Disclaimer

We will concentrate on alternating series of the form $a_1 - a_2 + a_3 - \cdots$. All of our results apply to the series $-a_1 + a_2 - a_3 + \cdots$ as well.

Theorem

Let $\{a_n\}$ be a decreasing sequence of positive terms. If $a_n \to 0$, then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

Proof.

Proof.

- A simple argument shows that the partial sums satisfy:

$$\cdots s_2 \leq s_4 \leq s_6 \leq \cdots s_7 \leq s_5 \leq s_3 \leq s_1$$

Proof.

- A simple argument shows that the partial sums satisfy:

$$\cdots s_2 \leq s_4 \leq s_6 \leq \cdots s_7 \leq s_5 \leq s_3 \leq s_1$$

- In other words, the even numbered terms converge to a limit s_e and the odd numbered terms converge to a limit s_o. We need to show that these are the same.

Proof.

- A simple argument shows that the partial sums satisfy:

$$\cdots s_2 \leq s_4 \leq s_6 \leq \cdots s_7 \leq s_5 \leq s_3 \leq s_1$$

- In other words, the even numbered terms converge to a limit s_e and the odd numbered terms converge to a limit s_o. We need to show that these are the same.
- But

$$|s_o - s_e| = \lim_{n\to\infty} |s_{2n+1} - s_{2n}| = \lim_{n\to\infty} a_{2n+1} = 0,$$

which proves the claim. $\square$

Problem

Problem

- *Show that* $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}$ *converges.*

Problem

- *Show that* $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}$ *converges.*
- *Leibniz showed that the answer is* $\pi/4$.

Theorem (Alternating Series Estimation Theorem)

If an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$ *satisfies* $0 \leq a_{n+1} \leq a_n$ *and* $a_n \to 0$ *as* $n \to \infty$, *then if* s *is the sum of the series and* s_n *is the nth partial sum, then*

$$|R_n| = |s - s_n| \leq a_{n+1}.$$