§12.4–Comparison Tests

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Outline

1. The p-series test
2. Comparison tests

Definition

Let $p > 0$. A p-series is any series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p}.$$

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Theorem (The p-Series Test (PST))

A p-series converges if $p > 1$ and diverges if $0 < p \leq 1$.

Comparison tests

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- *If* $\sum a_n$ *diverges, then* $\sum b_n$ *diverges.*

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- If $\sum b_n$ converges, then t_n is bounded above and, hence, so is s_n; thus, s_n converges.
- If $\sum a_n$ diverges, then s_n is not bounded above and, hence, t_n is not bounded above either; thus, t_n diverges. $\square$

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- $\displaystyle\sum_{n=1}^{\infty} \frac{2-\sin(n)}{n}$

Theorem (The Limit Comparison Test (LCT))

If $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers with $a_n/b_n \to c$ and if $0 < c < \infty$, then the series $\sum a_n$ and $\sum b_n$ converge or diverge together.

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- Now the result follows from the SCT. $\square$

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- $\displaystyle\sum_{n=1}^{\infty} \frac{2n+4}{\sqrt{n^4+2}}$

Solution

- *We compare with* $1/2^n$*:*

$$\frac{1/(2^n-1)}{1/2^n} = \frac{2^n}{2^n-1} = \frac{1}{1-2^{-n}}$$

and $\lim_{n\to\infty} \frac{1}{1-2^{-n}} = 1$. *The comparison is valid. Since* $\sum 1/2^n$ *converges (being geometric),* $\sum 1/(2^n-1)$ *converges by LCT.*

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- *We compare with* $1/n$*:*

$$\frac{(2n+4)/\sqrt{n^4+2}}{1/n}=\frac{2n^2+4n}{\sqrt{n^4+2}}=\frac{2+\frac{4}{n}}{\sqrt{1+\frac{2}{n^4}}}$$

and $\lim_{n\to\infty}\frac{2+\frac{4}{n}}{\sqrt{1+\frac{2}{n^4}}}=2$. *The comparison is valid. Since* $\sum 1/n$ *diverges (PST,* $p=1$*),* $\sum(2n+4)/\sqrt{n^4+2}$ *diverges by LCT.*