§12.2–Infinite series

Mark Woodard

Furman U

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Outline

Problem

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Commentary

The preceding examples underscore the need for a proper definition of the sum of an infinite series.

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- Given a sequence $\{a_n\}$ of real numbers, we form a new sequence as follows:

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots
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$$

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  The sequence $\{s_n\}$ is called the *sequence of partial sums*.

- If $s_n \to L$, then we say that the infinite series $\sum_{k=1}^{\infty} a_k$ converges and we write

$$\sum_{k=1}^{\infty} a_k = L.$$

  If $\{s_n\}$ diverges, then we say that the infinite series $\sum_{k=1}^{\infty} a_k$ diverges.

Definition

A **series** $\sum a_n$ is a collection of two sequences: the sequence of terms $\{a_n\}$ and the sequence of partial sums $\{s_n\}$ where $s_n = a_1 + a_2 + \cdots + a_n$.

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- *Let* s_n *denote the nth partial sum of the series. It is easy to see that* $s_n = 1$ *if* n *is odd and* $s_n = 0$ *if* n *is even.*
- *Since the sequence* $0, 1, 0, 1, \ldots$ *does not converge, the infinite series* $\sum_{k=1}^{\infty}(-1)^{k+1}$ *diverges.*

Definition

Let $a, r \neq 0$. Any series of the form

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar^1 + ar^2 + ar^3 + \cdots$$

is called a *geometric series.* The number r is called the *common ratio* of the series.

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Theorem

$$\sum_{k=1}^{\infty} ar^{k-1} \begin{cases} = \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

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Solution

The series

$$5(1/3)^3 + 5(1/3)^4 + 5(1/3)^5 + \cdots$$

is geometric with $a = 5/27$ *and* $r = 1/3$. *Thus the series converges to*

$$\frac{(5/27)}{1-(1/3)} = \frac{5}{18}.$$

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Solution

The total distance is the infinite series

$$20 + \underbrace{2(.6)(20) + 2(.6)^2(20) + 2(.6)^3(20) + \cdots}_{\textit{geometric part}}$$

$$= 20 + \frac{24}{1 - .6}$$

$$= 20 + 60$$

$$= 80$$

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- *By a partial fraction decomposition,* $\frac{1}{k^2+k} = \frac{1}{k} - \frac{1}{k+1}$.
- *Thus it can be shown that* $s_n = 1 - \frac{1}{n+1}$.
- *Since* $s_n \to 1$ *as* $n \to \infty$, *we conclude that the series converges to 1.*

Problem

Show that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges.

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- *Consider s_8 and observe that*

$$\begin{aligned} s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &\geq 1 + \frac{1}{2} + 2\frac{1}{4} + 4\frac{1}{8} \\ &= 1 + \frac{3}{2} \end{aligned}$$

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- *In general, $s_{2^n} \geq 1 + \frac{n}{2}$, which show that $s_n \to \infty$ as $n \to \infty$.*

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If $a_n \not\to 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof.

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- Let us assume that $\sum_{k=1}^{\infty} a_k$ converges and let $\{s_n\}$ denote the corresponding sequence of partial sums.

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- Thus $s_n \to L$ for some limit L.
- It is also the case that $s_{n-1} \to L$.
- Thus $a_n = (s_n - s_{n-1}) \to L - L = 0$, as was to be shown. $\square$

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Solution

Since $(1+k^{-1})^k \to e \neq 0$, *the series diverges by the nth Term Test.*

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Find the sum of the series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k^2+k}+5(.5)^{k+3}\right)$.

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- *We have already observed that* $\sum_{k=1}^{\infty}\frac{1}{k^2+k}=1$.

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Solution

- *We have already observed that* $\sum_{k=1}^{\infty}\frac{1}{k^2+k}=1$.
- *The series* $\sum_{k=1}^{\infty}5(.5)^{k+3}=5/8$, *being geometric.*

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- *The series* $\sum_{k=1}^{\infty}5(.5)^{k+3}=5/8$, *being geometric.*
- *Thus the original series converges to* $1+5/8=13/8$.