§12.1–Sequences

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Outline

Definition

A *sequence* is a list of real numbers in a definite order: $a_1, a_2, a_3 \ldots$. The sequence $a_1, a_2, a_3, \ldots$ will be denoted by $\{a_n\}$.

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Solution

- $1, 1/2, 1/3, 1/4$.
- $2/3, 2/7, 2/13, 2/21$.

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Let T_n denote the nth triangular number. Find a formula for T_n.

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- *By a famous formula for the sum of an arithmetic sequence,*
$$T_n = \frac{n(n+1)}{2},$$
which gives us the desired formula.

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- This is an example of a *recursively defined sequence*.
- A recursively defined sequence does not give us an explicit formula for T_n in terms of n; nonetheless, we can compute quickly with the recursive formula.

Problem

Let $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. The elements of the sequence $\{f_n\}$ are called the Fibonacci numbers. Compute f_6.

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Solution

We proceed inductively:

$$\begin{aligned} f_3 &= 1 + 1 = 2 \\ f_4 &= 1 + 2 = 3 \\ f_5 &= 2 + 3 = 5 \\ f_6 &= 3 + 5 = 8 \\ f_7 &= 5 + 8 = 13. \end{aligned}$$

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Problem

What is the tendency of the sequence $\{a_n\}$ *as* $n \to \infty$.

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A sequence $\{a_n\}$ has limit L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty.$$

if for every $\varepsilon > 0$ there exists a corresponding integer N such that

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Theorem

If $\lim_{x\to\infty} f(x) = L$ *and if* $a_n = f(n)$ *for integers* $n \geq 1$*, then* $a_n \longrightarrow L$ *as* $n \longrightarrow \infty$*.*

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Let $a_n = \dfrac{n^2+1}{n^3+4n+9}$. *Determine the limit of the sequence* $\{a_n\}$, *if it exists.*

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Solution

- *Let* $f(x) = (x^2+1)/(x^3+4x+9)$. *By dividing numerator and denominator by* x^3*, we obtain*

$$f(x) = \frac{\frac{1}{x} + \frac{1}{x^3}}{1 + \frac{4}{x^2} + \frac{9}{x^3}}.$$

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- *As* $x \to \infty$*, we see that* $f(x) \to 0$. *Thus* $a_n \to 0$ *as* $n \to \infty$.

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If $a_n \to a$ *and* $b_n \to b$ *as* $n \to \infty$, *and if* c *is a constant, then*

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- $a_n/b_n \to a/b$ as $n \to \infty$, provided $b \neq 0$;

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- $a_n b_n \to ab$ *as* $n \to \infty$;
- $a_n/b_n \to a/b$ *as* $n \to \infty$, *provided* $b \neq 0$;
- $(a_n)^p \to a^p$ *as* $n \to \infty$, *provided* $a \geq 0$ *and* $p > 0$.

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- $a_n b_n \to ab$ as $n \to \infty$;
- $a_n / b_n \to a/b$ as $n \to \infty$, provided $b \neq 0$;
- $(a_n)^p \to a^p$ as $n \to \infty$, provided $a \geq 0$ and $p > 0$.

Theorem (Squeeze Theorem)

If there exists an integer m such that $a_n \leq b_n \leq c_n$ for $n \geq m$ and if $a_n \to L$ and $c_n \to L$ as $n \to \infty$, then $b_n \to L$ as $n \to \infty$.

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Proof.

- For $r > 0$, write $r^n = e^{n \ln(r)}$. If $0 < r < 1$, then $\ln(r) < 0$ and $r^n \to 0$. If $r = 1$, then $r^n = 1$ and $r^n \to 1$. If $r > 1$, then $\ln(r) > 0$ and $r^n \to \infty$.

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- If $-1 < r < 0$, then $|r^n| = |r|^n$. Since $|r|^n \to 0$, $r^n \to 0$.

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- If $-1 < r < 0$, then $|r^n| = |r|^n$. Since $|r|^n \to 0$, $r^n \to 0$.
- If $r \leq -1$, then $\{r^n\}$ oscillates and does not approach a fixed limit. $\square$

Problem

Let $a_n = \dfrac{2^n}{3^n + 8} + \dfrac{n}{n+4} + \dfrac{\sin(n)}{n}$. *Evaluate* $\lim_{n\to\infty} a_n$.

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Solution

- *Let* $b_n = \dfrac{2^n}{3^n + 8}$, $c_n = \dfrac{n}{n+4}$, *and* $d_n = \dfrac{\sin(n)}{n}$.

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- *Let* $b_n = \dfrac{2^n}{3^n+8}$, $c_n = \dfrac{n}{n+4}$, *and* $d_n = \dfrac{\sin(n)}{n}$.
- $b_n = (2/3)^n/(1+8/3^n) \to 0$ *as* $n \to \infty$.

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- $b_n = (2/3)^n/(1 + 8/3^n) \to 0$ *as* $n \to \infty$.
- $c_n = 1/(1 + 4/n) \to 1$ *as* $n \to \infty$.
- *Since* $-1 \leq \sin(n) \leq 1$, *it follows that* $-1/n \leq \sin(n)/n \leq 1/n$. *Since* $1/n \to 0$ *and* $-1/n \to 0$ *as* $n \to \infty$, $d_n = \sin(n)/n \to 0$ *as* $n \to \infty$ *by the Squeeze Theorem.*

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- *Since* $-1 \le \sin(n) \le 1$, *it follows that* $-1/n \le \sin(n)/n \le 1/n$. *Since* $1/n \to 0$ *and* $-1/n \to 0$ *as* $n \to \infty$, $d_n = \sin(n)/n \to 0$ *as* $n \to \infty$ *by the Squeeze Theorem.*
- *Finally* $a_n \to 0+1+0=1$ *as* $n \to \infty$.

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Show that the sequence $a_n = \dfrac{n}{n+1}$ *is bounded.*

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Problem

Let $a_n = \dfrac{n}{n+1}$. *Show that the sequence* $\{a_n\}$ *is monotone.*

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Problem

Let $a_1 = 1$ *and let* $a_n = 20 + \frac{1}{3}a_{n-1}$ *for* $n \geq 2$. *Show that* $\{a_n\}$ *is monotone increasing and bounded above. Find the limit of the sequence.*