§8.4–Partial Fractions

Mark Woodard

Furman U

Fall 2010

Outline

Example

Example

- Notice that

$$f(x) = \frac{x+1}{2x^2+7x+6} = \frac{1}{x+2} - \frac{1}{2x+3}.$$

The terms on the right form the *partial fraction decomposition* of the rational function f.

Example

- Notice that
$$f(x) = \frac{x+1}{2x^2+7x+6} = \frac{1}{x+2} - \frac{1}{2x+3}.$$
The terms on the right form the *partial fraction decomposition* of the rational function f.
- Thus
$$\begin{aligned}\int \frac{x+1}{2x^2+7x+6}\,dx &= \int \frac{1}{x+2}\,dx - \int \frac{1}{2x+3}\,dx \\ &= \ln|x+2| - \frac{1}{2}\ln|2x+3| + C\end{aligned}$$

Example

- Notice that

$$f(x) = \frac{x+1}{2x^2+7x+6} = \frac{1}{x+2} - \frac{1}{2x+3}.$$

  The terms on the right form the *partial fraction decomposition* of the rational function f.

- Thus

$$\begin{aligned}\int \frac{x+1}{2x^2+7x+6}dx &= \int \frac{1}{x+2}dx - \int \frac{1}{2x+3}dx \\ &= \ln|x+2| - \frac{1}{2}\ln|2x+3| + C\end{aligned}$$

- Does this suggest a general method for solving integrals of the form $\int P(x)/Q(x)dx$, where P and Q are polynomials?

Example

- Notice that
$$f(x) = \frac{x+1}{2x^2+7x+6} = \frac{1}{x+2} - \frac{1}{2x+3}.$$
The terms on the right form the *partial fraction decomposition* of the rational function f.
- Thus
$$\begin{aligned}\int \frac{x+1}{2x^2+7x+6}dx &= \int \frac{1}{x+2}dx - \int \frac{1}{2x+3}dx \\ &= \ln|x+2| - \frac{1}{2}\ln|2x+3| + C\end{aligned}$$
- Does this suggest a general method for solving integrals of the form $\int P(x)/Q(x)dx$, where P and Q are polynomials?
- If yes, how do we decompose $P(x)/Q(x)$ into partial fractions?

Definition

A *rational function* is any function of the form $f(x) = P(x)/Q(x)$, where P and Q are polynomials. The rational function f is said to be *proper* if $\deg P < \deg Q$.

Definition

A *rational function* is any function of the form $f(x) = P(x)/Q(x)$, where P and Q are polynomials. The rational function f is said to be *proper* if $\deg P < \deg Q$.

Example

Here are two examples:

Definition

A *rational function* is any function of the form $f(x) = P(x)/Q(x)$, where P and Q are polynomials. The rational function f is said to be *proper* if $\deg P < \deg Q$.

Example

Here are two examples:

- $f(x) = \dfrac{x+1}{2x^2+7x+6}$ is proper.

Definition

A *rational function* is any function of the form $f(x) = P(x)/Q(x)$, where P and Q are polynomials. The rational function f is said to be *proper* if $\deg P < \deg Q$.

Example

Here are two examples:

- $f(x) = \dfrac{x+1}{2x^2+7x+6}$ is proper.
- $g(x) = \dfrac{x^3+2x+4}{x^2-1}$ is not proper.

Long division

An improper rational function can be written as a sum of a polynomial and a proper rational function by long division. Thus, in the example above,

$$g(x) = \frac{x^3 + 2x + 4}{x^2 - 1} = x + \frac{3x + 4}{x^2 - 1}.$$

Example

Example

- Note that

$$P(x) = x^3 + 2x^2 - 4x - 8 = x^2(x+2) - 4(x+2) = (x-2)(x+2)^2.$$

The factors of P are $(x-2)$ and $(x+2)$, and both factors are *linear*. We say that $(x+2)$ has *multiplicity* 2.

Example

- Note that

$$P(x) = x^3 + 2x^2 - 4x - 8 = x^2(x+2) - 4(x+2) = (x-2)(x+2)^2.$$

The factors of P are $(x-2)$ and $(x+2)$, and both factors are *linear*. We say that $(x+2)$ has *multiplicity* 2.

- Note that

$$Q(x) = x^4 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1).$$

The factor (x^2+1) cannot be further factored (over the real numbers). Such a factor is called an *irreducible quadratic*. The quadratic

$$ax^2 + bx + c$$

is irreducible when it has no real roots; thus, whenever $b^2 - 4ac < 0$.

Theorem (The Fundamental Theorem of Algebra)

Every polynomial can be expressed as a product of powers of linear factors $(ax+b)^m$ *and powers of irreducible quadratic factors* $(ax^2+bx+c)^n$.

Theorem (Partial Fraction Decompositions)

Assume that the rational function $\dfrac{P(x)}{Q(x)}$ *is proper.*

Theorem (Partial Fraction Decompositions)

Assume that the rational function $\dfrac{P(x)}{Q(x)}$ *is proper.*

- *Each factor of* Q *will generate terms of the partial fraction decomposition of* P/Q.

Theorem (Partial Fraction Decompositions)

Assume that the rational function $\dfrac{P(x)}{Q(x)}$ *is proper.*

- *Each factor of Q will generate terms of the partial fraction decomposition of P/Q.*
- *To each linear factor $(ax+b)^m$ of Q, the decomposition of P/Q will contain the terms*

$$\frac{D_1}{(ax+b)^1} + \cdots + \frac{D_m}{(ax+b)^m}.$$

Theorem (Partial Fraction Decompositions)

Assume that the rational function $\dfrac{P(x)}{Q(x)}$ *is proper.*

- *Each factor of* Q *will generate terms of the partial fraction decomposition of* P/Q.
- *To each linear factor* $(ax+b)^m$ *of* Q, *the decomposition of* P/Q *will contain the terms*

$$\frac{D_1}{(ax+b)^1} + \cdots + \frac{D_m}{(ax+b)^m}.$$

- *To each irreducible quadratic factor* $(ax^2+bx+c)^n$ *of* Q, *the decomposition will contain the terms*

$$\frac{E_1 x + F_1}{(ax^2+bx+c)^1} + \cdots + \frac{E_n x + F_n}{(ax^2+bx+c)^n}.$$

Problem

Find the partial fraction decomposition of $f(x) = (x+1)/(2x^2+7x+6)$.

Problem

Find the partial fraction decomposition of $f(x) = (x+1)/(2x^2+7x+6)$.

Solution

The answer is

$$f(x) = \frac{1}{x+2} - \frac{1}{2x+3}.$$

Problem

Find the partial fraction decomposition of
$f(x) = (3x^2 + 3x - 2)/(x^3 + 2x^2 - 4x - 8)$.

Problem

Find the partial fraction decomposition of $f(x) = (3x^2 + 3x - 2)/(x^3 + 2x^2 - 4x - 8)$.

Solution

The answer is

$$f(x) = \frac{2}{(x+2)} + \frac{-1}{(x+2)^2} + \frac{1}{(x-2)}.$$

Problem

Evaluate $\displaystyle\int \frac{x^6 + 2x^4 + x^3 - 2x^2 - x - 5}{x^4 - 1}\,dx.$

Problem

Evaluate $\displaystyle\int \frac{x^6+2x^4+x^3-2x^2-x-5}{x^4-1}\,dx.$

Solution

After long division, we have

$$x^2+2+\frac{x^3-x^2-x-3}{(x^2+1)(x-1)(x+1)}$$

Problem

Evaluate $\displaystyle\int \frac{x^6 + 2x^4 + x^3 - 2x^2 - x - 5}{x^4 - 1}\, dx.$

Solution

After long division, we have

$$x^2 + 2 + \frac{x^3 - x^2 - x - 3}{(x^2+1)(x-1)(x+1)}$$

Problem

Evaluate $\displaystyle\int \frac{x^6 + 2x^4 + x^3 - 2x^2 - x - 5}{x^4 - 1}\, dx.$

Solution

After long division, we have

$$x^2 + 2 + \frac{x^3 - x^2 - x - 3}{(x^2+1)(x-1)(x+1)}$$

which we write in the form

$$x^2 + 2 + \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx + D}{x^2+1}.$$

Problem

Evaluate $\displaystyle\int \frac{x^6 + 2x^4 + x^3 - 2x^2 - x - 5}{x^4 - 1}\,dx.$

Solution

After long division, we have

$$x^2 + 2 + \frac{x^3 - x^2 - x - 3}{(x^2+1)(x-1)(x+1)}$$

which we write in the form

$$x^2 + 2 + \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}.$$

The integrand becomes

$$(x^2+2) + \frac{1}{x+1} + \frac{-1}{x-1} + \frac{x+1}{x^2+1}.$$

Solution

This can now be integrated, yielding

$$\frac{x^3}{3} + 2x + \ln|x+1| - \ln|x-1| + \frac{1}{2}\ln|x^2+1| + \tan^{-1}(x) + C.$$