§8.4–Partial Fractions

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Outline

1. The method illustrated
2. Terminology
3. Factoring Polynomials
4. Partial fraction decompositions
5. Further examples
The method illustrated

Example

Notice that \( f(x) = x^2 + 7x + 6 = x^2 - 1\). The terms on the right form the partial fraction decomposition of the rational function \( f \).

Thus \( \int \frac{x+1}{x^2+7x+6} \, dx = \int \frac{1}{x^2+7} \, dx - \int \frac{1}{2x+3} \, dx = \ln |x^2+7| - \frac{1}{2} \ln |2x+3| + C \).

Does this suggest a general method for solving integrals of the form \( \int \frac{P(x)}{Q(x)} \, dx \), where \( P \) and \( Q \) are polynomials? If yes, how do we decompose \( \frac{P(x)}{Q(x)} \) into partial fractions?

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\[ f(x) = \frac{x + 1}{2x^2 + 7x + 6} = \frac{1}{x + 2} - \frac{1}{2x + 3}. \]

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- Does this suggest a general method for solving integrals of the form \( \int P(x)/Q(x) \, dx \), where \( P \) and \( Q \) are polynomials?
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- If yes, how do we decompose \( P(x)/Q(x) \) into partial fractions?
**Definition**

A *rational function* is any function of the form $f(x) = P(x)/Q(x)$, where $P$ and $Q$ are polynomials. The rational function $f$ is said to be *proper* if \( \deg P < \deg Q \).
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**Example**

Here are two examples:

- \( f(x) = \frac{x + 1}{2x^2 + 7x + 6} \) is proper.
- \( g(x) = \frac{x^3 + 2x + 4}{x^2 - 1} \) is not proper.
Long division

An improper rational function can be written as a sum of a polynomial and a proper rational function by long division. Thus, in the example above,

\[
g(x) = \frac{x^3 + 2x + 4}{x^2 - 1} = x + \frac{3x + 4}{x^2 - 1}.
\]
Example

Note that \( P(x) = x^3 + 2x^2 - 4x - 8 = x^2(x + 2) - 4(x + 2) = (x - 2)(x + 2)^2. \)

The factors of \( P \) are \((x - 2)\) and \((x + 2)\), and both factors are linear.

We say that \((x + 2)\) has multiplicity 2.

Note that \( Q(x) = x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1). \)

The factor \((x^2 + 1)\) cannot be further factored (over the real numbers). Such a factor is called an irreducible quadratic.

The quadratic \( ax^2 + bx + c \) is irreducible when it has no real roots; thus, whenever \( b^2 - 4ac < 0 \).
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Theorem (The Fundamental Theorem of Algebra)

Every polynomial can be expressed as a product of powers of linear factors \((ax + b)^m\) and powers of irreducible quadratic factors \((ax^2 + bx + c)^n\).
Theorem (Partial Fraction Decompositions)

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  \frac{E_1x + F_1}{(ax^2 + bx + c)^1} + \cdots + \frac{E_nx + F_n}{(ax^2 + bx + c)^n}.
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Further examples

Problem

Find the partial fraction decomposition of \( f(x) = \frac{x + 1}{2x^2 + 7x + 6} \).
Problem

Find the partial fraction decomposition of \( f(x) = (x + 1)/(2x^2 + 7x + 6) \).

Solution

The answer is

\[
f(x) = \frac{1}{x + 2} - \frac{1}{2x + 3}.
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Problem

Find the partial fraction decomposition of
\[ f(x) = \frac{3x^2 + 3x - 2}{x^3 + 2x^2 - 4x - 8}. \]
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\[ f(x) = \frac{3x^2 + 3x - 2}{x^3 + 2x^2 - 4x - 8}. \]

Solution

The answer is
\[ f(x) = \frac{2}{(x + 2)} + \frac{-1}{(x + 2)^2} + \frac{1}{(x - 2)}. \]
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Evaluate \( \int \frac{x^6 + 2x^4 + x^3 - 2x^2 - x - 5}{x^4 - 1} \, dx \).
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Solution
After long division, we have
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x^2 + 2 + \frac{x^3 - x^2 - x - 3}{(x^2 + 1)(x - 1)(x + 1)}
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Further examples

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The integrand becomes

\[
(x^2 + 2) + \frac{1}{x + 1} + \frac{-1}{x - 1} + \frac{x + 1}{x^2 + 1}.
\]
Further examples

Solution

This can now be integrated, yielding

\[ \frac{x^3}{3} + 2x + \ln|x+1| - \ln|x-1| + \frac{1}{2} \ln|x^2 + 1| + \tan^{-1}(x) + C. \]