

§8.3–Trigonometric Substitution

Mark Woodard

Furman U

Fall 2010

Outline

- 1 Trigonometric substitution

Example

We will learn techniques for solving integrals that contain roots of quadratic functions of x . Here are some examples of the types of integrals we will encounter.

Example

We will learn techniques for solving integrals that contain roots of quadratic functions of x . Here are some examples of the types of integrals we will encounter.

- $\int (9 - x^2)^{1/2} dx$

Example

We will learn techniques for solving integrals that contain roots of quadratic functions of x . Here are some examples of the types of integrals we will encounter.

- $\int (9 - x^2)^{1/2} dx$
- $\int_0^1 \frac{1}{(1 + x^2)^2} dx$

Example

We will learn techniques for solving integrals that contain roots of quadratic functions of x . Here are some examples of the types of integrals we will encounter.

- $\int (9 - x^2)^{1/2} dx$
- $\int_0^1 \frac{1}{(1 + x^2)^2} dx$
- $\int \frac{1}{\sqrt{x^2 - 16}} dx$

Sine substitutions

Sine substitutions

- For integrals containing the form $\sqrt{a^2 - x^2}$, make the substitution

$$x = a \sin(\theta), \quad -\pi/2 \leq \theta \leq \pi/2.$$

Sine substitutions

- For integrals containing the form $\sqrt{a^2 - x^2}$, make the substitution

$$x = a \sin(\theta), \quad -\pi/2 \leq \theta \leq \pi/2.$$

- Under this substitution, $dx = a \cos(\theta)d\theta$ and

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2(\theta)} \\ &= \sqrt{a^2(1 - \sin^2(\theta))} \\ &= a \cos(\theta).\end{aligned}$$

Problem

Evaluate the following integrals:

Problem

Evaluate the following integrals:

- $\int (9 - x^2)^{1/2} dx$

Problem

Evaluate the following integrals:

- $\int (9 - x^2)^{1/2} dx$
- $\int_0^{1/2} x^3 \sqrt{1 - 4x^2} dx$

Problem

Evaluate the following integrals:

- $\int (9 - x^2)^{1/2} dx$
- $\int_0^{1/2} x^3 \sqrt{1 - 4x^2} dx$
- $\int \frac{x^2}{(3 + 4x - 4x^2)^{3/2}} dx$

Problem

Evaluate the following integrals:

- $\int (9 - x^2)^{1/2} dx$
- $\int_0^{1/2} x^3 \sqrt{1 - 4x^2} dx$
- $\int \frac{x^2}{(3 + 4x - 4x^2)^{3/2}} dx$

Problem

Find the area enclosed by the ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Tangent substitutions

Tangent substitutions

- For integrals containing the form $\sqrt{a^2 + x^2}$, make the substitution

$$x = a \tan(\theta), \quad -\pi/2 \leq \theta \leq \pi/2.$$

Tangent substitutions

- For integrals containing the form $\sqrt{a^2 + x^2}$, make the substitution

$$x = a \tan(\theta), \quad -\pi/2 \leq \theta \leq \pi/2.$$

- Under this substitution, $dx = a \sec^2(\theta)d\theta$ and

$$\begin{aligned}\sqrt{a^2 + x^2} &= \sqrt{a^2 + a^2 \tan^2(\theta)} \\ &= \sqrt{a^2(1 + \tan^2(\theta))} \\ &= a \sec(\theta).\end{aligned}$$

Problem

Solve the following integrals:

Problem

Solve the following integrals:

- $\int \frac{1}{\sqrt{x^2 + 4}} dx$

Problem

Solve the following integrals:

- $\int \frac{1}{\sqrt{x^2 + 4}} dx$

- $\int_0^1 \frac{1}{(1 + x^2)^2} dx$

Secant substitutions

Secant substitutions

- For integrals containing the form $\sqrt{x^2 - a^2}$, make the substitution

$$x = a \sec(\theta), \quad 0 < \theta < \pi/2 \quad \text{or} \quad \pi < \theta < 3\pi/2.$$

Secant substitutions

- For integrals containing the form $\sqrt{x^2 - a^2}$, make the substitution

$$x = a \sec(\theta), \quad 0 < \theta < \pi/2 \quad \text{or} \quad \pi < \theta < 3\pi/2.$$

- Under this substitution, $dx = a \sec(\theta) \tan(\theta) d\theta$ and

$$\begin{aligned}\sqrt{x^2 - a^2} &= \sqrt{a^2 \sec^2(\theta) - a^2} \\ &= \sqrt{a^2(\sec^2(\theta) - 1)} \\ &= a \tan(\theta).\end{aligned}$$

Problem

Solve the integrals:

Problem

Solve the integrals:

- $\int \frac{1}{\sqrt{x^2 - 16}} dx$

Problem

Solve the integrals:

- $\int \frac{1}{\sqrt{x^2 - 16}} dx$

- $\int_3^6 \frac{\sqrt{x^2 - 9}}{x} dx$