§7.8–Indeterminate forms and L'Hôpital's rule

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Outline

1. The forms "$0/0$" and "∞/∞"
2. Examples
3. Indeterminate products: "$0 \cdot \infty$"
4. Indeterminate differences: "$\infty - \infty$"
5. Indeterminate powers: "0^0", "∞^0" and "1^∞"

Definition

We say that the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

is of the form "0/0" or "∞/∞" if

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \quad \text{or} \quad \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty,$$

respectively. These are called *indeterminate.* In this context, a can be any of $a^+, a^-, \pm\infty$.

Theorem (L'Hôpital's Rule)

If $\lim_{x\to a} f(x)/g(x)$ *is an ideterminate form of type "0/0" or "∞/∞" and if*

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = L,$$

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Remark

When facing an indeterminate form, students will often write:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

which is, strictly speaking, wrong. L'Hôpital's rule states that these are equal **only when the limit on the right exists**.

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Evaluate the following limits:

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- $\lim_{x\to 0} \dfrac{\sin(x)}{x}$

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- $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$
- $\lim_{x\to\infty} \dfrac{x^4}{e^x}$.

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Strategy for the "$0 \cdot \infty$" form

Suppose that the limit $\lim_{x\to a} f(x)g(x)$ is of the form $0 \cdot \infty$. This form can be converted into either a "$0/0$" or an "∞/∞" form by algebra:

$$f(x)g(x) = \frac{g(x)}{1/f(x)} \quad \text{or} \quad f(x)g(x) = \frac{f(x)}{1/g(x)}.$$

Now the limit can be attacked by the previous methods.

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- *Find* $\lim_{x\to 0^+} x\ln(x)$.
- *Find* $\lim_{x\to\infty} xe^{-x}$.
- *Find* $\lim_{x\to\infty} x\big(\pi/2 - \tan^{-1}(x)\big)$.

The basic strategy for indeterminate differences

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- An *indeterminate difference* is any limit of the form

$$\lim_{x \longrightarrow a} \left(f(x) - g(x)\right)$$

in which f and g simultaneously approach $+\infty$ or $-\infty$.

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- An *indeterminate difference* is any limit of the form

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  in which f and g simultaneously approach $+\infty$ or $-\infty$.
- To handle an "$\infty - \infty$"' form, use algebra to convert this form into one of the other forms.

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Find $\lim_{u \to 0^+} \left(\frac{1}{1 - e^{-u}} - \frac{1}{u} \right)$.

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Find $\displaystyle\lim_{u\to 0^+}\left(\frac{1}{1-e^{-u}}-\frac{1}{u}\right)$.

Solution

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- *This is called an "$\infty - \infty$" form, since both terms are approaching* $+\infty$*.*

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- *We can force a common denominator:*

$$\frac{1}{1 - e^{-u}} - \frac{1}{u} = \frac{u - 1 + e^{-u}}{u(1 - e^{-u})}.$$

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- *As* $u \to 0^+$, *the right-hand-side is now a "*$0/0$*" form and can be treated using l'Hôpital's rule. The answer is* $1/2$.

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- An *indeterminate power* is any limit of the form

$$\lim_{x \longrightarrow a} f(x)^{g(x)}$$

resulting in "0^0", "∞^0" and "1^∞".

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resulting in "0^0", "∞^0" and "1^∞".

- In each of these cases, first write

$$f(x)^{g(x)} = \exp\left(g(x)\ln f(x)\right).$$

The exponent, $g(x)\ln f(x)$, will be in one the preceding forms and can be handled by those methods.

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- *Find* $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x$.

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- *Find* $\lim_{x\to\infty} x^{1/x}$.
- *Find* $\lim_{x\to 0^+} x^{\sin(x)}$.