§7.4*–General logarithmic and exponential functions

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Outline

1. General exponential functions
2. The derivative
3. The logarithm to the base a.
4. Solving equations
5. e as a limit

Definition

Let $a > 0$ be a real number (this will be the base). Notice that if r is rational, then

$$a^r = \exp\left(\ln(a^r)\right) = \exp\left(r\ln(a)\right).$$

It makes sense, then, to define

$$a^x = \exp\left(x\ln(a)\right).$$

for all $x \in \mathbb{R}$.

The laws of exponents

The *laws of exponents* are inherited from exp:

$$a^{x+y} = a^x\, a^y \qquad a^{x-y} = \frac{a^x}{a^y} \qquad (a^x)^y = a^{xy}.$$

In addition, we have the following two identities:

$$(ab)^x = a^x b^x \qquad (a/b)^x = a^x/b^x.$$

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Graphs

The graph of $y = a^x$ depends on whether $a = 1$, $a > 1$ or $0 < a < 1$.

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- $\displaystyle\int x^2 3^{x^3} dx$
- $\displaystyle\int \frac{1}{1+2^{-x}}\, dx$

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$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

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- $2^x - 35 \cdot 2^{-x} = 2$

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(Hint: What is $\frac{d}{dx} \ln x\Big|_{x=1}$*?)*

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Problem

Evaluate

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$$