§7.1–Inverse Functions

Mark Woodard

Furman U.

Fall 2010

Outline

Example

Consider the following two examples of functional relationships among the ordered pairs:

$$f : (1, -1), (2, 1), (3, 2), (4, 0)$$

and

$$g : (1, 1), (2, 3), (3, 1), (4, 2)$$

We can easily "invert" these relations.

Example

Consider the following two examples of functional relationships among the ordered pairs:

$$f : (1, -1), (2, 1), (3, 2), (4, 0)$$

and

$$g : (1, 1), (2, 3), (3, 1), (4, 2)$$

We can easily "invert" these relations.

- Are the resulting inverse relations functions?

Example

Consider the following two examples of functional relationships among the ordered pairs:

$$f : (1, -1), (2, 1), (3, 2), (4, 0)$$

and

$$g : (1, 1), (2, 3), (3, 1), (4, 2)$$

We can easily "invert" these relations.

- Are the resulting inverse relations functions?
- How are the domains and ranges of the functions and their inverse relations related?

Example

Consider the following two examples of functional relationships among the ordered pairs:

$$f : (1, -1), (2, 1), (3, 2), (4, 0)$$

and

$$g : (1, 1), (2, 3), (3, 1), (4, 2)$$

We can easily "invert" these relations.

- Are the resulting inverse relations functions?
- How are the domains and ranges of the functions and their inverse relations related?
- By what tests can we tell whether a function will have an inverse function?

Definition (One-to-one)

Definition (One-to-one)

- A function f with domain D is called *one-to-one* if distinct elements of D have distinct images. In other words,

$$f(s) = f(t) \quad \text{if and only if} \quad s = t.$$

Definition (One-to-one)

- A function f with domain D is called *one-to-one* if distinct elements of D have distinct images. In other words,

$$f(s) = f(t) \quad \text{if and only if} \quad s = t.$$

- Said another way, a function is called one-to-one if it never takes on the same value more than once.

Definition (One-to-one)

- A function f with domain D is called *one-to-one* if distinct elements of D have distinct images. In other words,

$$f(s) = f(t) \quad \text{if and only if} \quad s = t.$$

- Said another way, a function is called one-to-one if it never takes on the same value more than once.

Example

The function $f(x) = x^2$ is not one-to-one since $f(1) = 1 = f(-1)$. In other words, two different input values produce the same output value, in violation of the definition of one-to-one.

Theorem (Horizontal line test)

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Theorem (Horizontal line test)

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Theorem (Increasing and decreasing)

If a function is either strictly increasing or strictly decreasing on an interval domain, then it is one-to-one.

Problem

Show that $f(x) = x^3$ *on* $(-\infty, \infty)$ *is one-to-one by*

Problem

Show that $f(x) = x^3$ *on* $(-\infty, \infty)$ *is one-to-one by*

- *the horizontal line test and by*

Problem

Show that $f(x) = x^3$ *on* $(-\infty, \infty)$ *is one-to-one by*

- *the horizontal line test and by*
- *showing that it is increasing.*

Problem

Show that $f(x) = x^3$ *on* $(-\infty, \infty)$ *is one-to-one by*

- *the horizontal line test and by*
- *showing that it is increasing.*

Problem

Show that $f(x) = 2x + \sin(x)$ *is one-to-one on* $(-\infty, \infty)$.

Problem

Show that $f(x) = x^3$ *on* $(-\infty, \infty)$ *is one-to-one by*

- *the horizontal line test and by*
- *showing that it is increasing.*

Problem

Show that $f(x) = 2x + \sin(x)$ *is one-to-one on* $(-\infty, \infty)$.

Problem

Explain how to restrict the domain of the function $f(x) = x^2$ *to make it one-to-one.*

Definition (Inverse function)

Definition (Inverse function)

- Let f be one-to-one with domain A and range B. The inverse function of f, denoted by f^{-1}, has domain B and range A.

Definition (Inverse function)

- Let f be one-to-one with domain A and range B. The inverse function of f, denoted by f^{-1}, has domain B and range A.
- f^{-1} maps y to x if and only if f maps x to y.

Definition (Inverse function)

- Let f be one-to-one with domain A and range B. The inverse function of f, denoted by f^{-1}, has domain B and range A.
- f^{-1} maps y to x if and only if f maps x to y.
- Equivalently, for any $y \in B$,

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y$$

Definition (Inverse function)

- Let f be one-to-one with domain A and range B. The inverse function of f, denoted by f^{-1}, has domain B and range A.
- f^{-1} maps y to x if and only if f maps x to y.
- Equivalently, for any $y \in B$,

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y$$

Theorem (Cancellation equations)

Let f be one-to-one with domain A and range B.

Definition (Inverse function)

- Let f be one-to-one with domain A and range B. The inverse function of f, denoted by f^{-1}, has domain B and range A.
- f^{-1} maps y to x if and only if f maps x to y.
- Equivalently, for any $y \in B$,

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y$$

Theorem (Cancellation equations)

Let f be one-to-one with domain A and range B.

- $f^{-1}(f(x)) = x$ *for each* $x \in A$.

Definition (Inverse function)

- Let f be one-to-one with domain A and range B. The inverse function of f, denoted by f^{-1}, has domain B and range A.
- f^{-1} maps y to x if and only if f maps x to y.
- Equivalently, for any $y \in B$,

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y$$

Theorem (Cancellation equations)

Let f be one-to-one with domain A and range B.

- $f^{-1}(f(x)) = x$ *for each* $x \in A$.
- $f(f^{-1}(y)) = y$ *for each* $y \in B$.

Problem

Let $f(x) = \sqrt{x-4}$ *on the interval* $[4, \infty)$. *Find* $f^{-1}(x)$.

Problem

Let $f(x) = \sqrt{x-4}$ *on the interval* $[4, \infty)$. *Find* $f^{-1}(x)$.

Problem

Let $f(x) = x^2 + 1$ *on the interval* $[0, \infty)$. *Show that* f *is invertible and find* f^{-1}.

Problem

Let $f(x) = \sqrt{x-4}$ on the interval $[4, \infty)$. Find $f^{-1}(x)$.

Problem

Let $f(x) = x^2 + 1$ on the interval $[0, \infty)$. Show that f is invertible and find f^{-1}.

Problem

Let $h(x) = x^3 + x + 5$. Show that f is one-to-one and find $h^{-1}(5)$.

Theorem

If f is invertible and continuous, then f^{-1} is continuous as well.

Theorem

If f is invertible and continuous, then f^{-1} is continuous as well.

Theorem

If f is a one-to-one differentiable function with inverse function f^{-1}, (a, b) is on the graph of f, and $f'(a) \neq 0$, then f^{-1} is differentiable at b and

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

Theorem

If f is invertible and continuous, then f^{-1} is continuous as well.

Theorem

If f is a one-to-one differentiable function with inverse function f^{-1}, (a, b) is on the graph of f, and $f'(a) \neq 0$, then f^{-1} is differentiable at b and

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}$$

Problem

Let $f(x) = x^3 + 3x$ and find $(f^{-1})'(4)$.