§7.1–Inverse Functions

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Outline

1. One-to-one functions
2. Inverse functions
3. Finding inverse functions
4. The calculus of inverse functions
Example

Consider the following two examples of functional relationships among the ordered pairs:

\[ f : (1, -1), (2, 1), (3, 2), (4, 0) \]

and

\[ g : (1, 1), (2, 3), (3, 1), (4, 2) \]

We can easily “invert” these relations.
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- How are the domains and ranges of the functions and their inverse relations related?
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We can easily “invert” these relations.

- Are the resulting inverse relations functions?
- How are the domains and ranges of the functions and their inverse relations related?
- By what tests can we tell whether a function will have an inverse function?
Definition (One-to-one)

A function \( f \) with domain \( D \) is called one-to-one if distinct elements of \( D \) have distinct images. In other words, \( f(s) = f(t) \) if and only if \( s = t \).

Said another way, a function is called one-to-one if it never takes on the same value more than once.

Example

The function \( f(x) = x^2 \) is not one-to-one since \( f(1) = 1 = f(-1) \). In other words, two different input values produce the same output value, in violation of the definition of one-to-one.
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Theorem (Horizontal line test)

A function is one-to-one if and only if no horizontal line intersects its graph more than once.
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Theorem (Increasing and decreasing)

If a function is either strictly increasing or strictly decreasing on an interval domain, then it is one-to-one.
Problem

Show that $f(x) = x^3$ on $(-\infty, \infty)$ is one-to-one by
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Show that \( f(x) = 2x + \sin(x) \) is one-to-one on \((-\infty, \infty)\).
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Problem

Explain how to restrict the domain of the function \( f(x) = x^2 \) to make it one-to-one.
Definition (Inverse function)

Let $f$ be one-to-one with domain $A$ and range $B$. The inverse function of $f$, denoted by $f^{-1}$, has domain $B$ and range $A$. $f^{-1}$ maps $y$ to $x$ if and only if $f$ maps $x$ to $y$. Equivalently, for any $y \in B$, $f^{-1}(y) = x$ if and only if $f(x) = y$.

Theorem (Cancellation equations)

Let $f$ be one-to-one with domain $A$ and range $B$. $f^{-1}(f(x)) = x$ for each $x \in A$. $f(f^{-1}(y)) = y$ for each $y \in B$. 
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$$f(f^{-1}(y)) = y \text{ for each } y \in B.$$
Definition (Inverse function)

Let \( f \) be one-to-one with domain \( A \) and range \( B \). The inverse function of \( f \), denoted by \( f^{-1} \), has domain \( B \) and range \( A \).

- \( f^{-1} \) maps \( y \) to \( x \) if and only if \( f \) maps \( x \) to \( y \).
- Equivalently, for any \( y \in B \),
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  f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y
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- $f^{-1}$ maps $y$ to $x$ if and only if $f$ maps $x$ to $y$.
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  \[ f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y \]

Theorem (Cancellation equations)

Let $f$ be one-to-one with domain $A$ and range $B$.

- $f^{-1}(f(x)) = x$ for each $x \in A$.
- $f(f^{-1}(y)) = y$ for each $y \in B$. 
Problem

Let \( f(x) = \sqrt{x - 4} \) on the interval \([4, \infty)\). Find \( f^{-1}(x) \).
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Let \( f(x) = \sqrt{x - 4} \) on the interval \([4, \infty)\). Find \( f^{-1}(x) \).

Problem

Let \( f(x) = x^2 + 1 \) on the interval \([0, \infty)\). Show that \( f \) is invertible and find \( f^{-1} \).
Problem

Let $f(x) = \sqrt{x - 4}$ on the interval $[4, \infty)$. Find $f^{-1}(x)$.

Problem

Let $f(x) = x^2 + 1$ on the interval $[0, \infty)$. Show that $f$ is invertible and find $f^{-1}$.

Problem

Let $h(x) = x^3 + x + 5$. Show that $f$ is one-to-one and find $h^{-1}(5)$.
The calculus of inverse functions

**Theorem**

*If f is invertible and continuous, then $f^{-1}$ is continuous as well.*
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**Theorem**

*If* $f$ *is invertible and continuous, then* $f^{-1}$ *is continuous as well.*

**Theorem**

*If* $f$ *is a one-to-one differentiable function with inverse function* $f^{-1}$, *$(a, b)$ is on the graph of* $f$, *and* $f'(a) \neq 0$, *then* $f^{-1}$ *is differentiable at* $b$ *and*

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

Problem

Let $f(x) = x^3 + 3x$ and find $(f^{-1})'(4)$. 

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The calculus of inverse functions

Theorem
If $f$ is invertible and continuous, then $f^{-1}$ is continuous as well.

Theorem
If $f$ is a one-to-one differentiable function with inverse function $f^{-1}$, $(a, b)$ is on the graph of $f$, and $f'(a) \neq 0$, then $f^{-1}$ is differentiable at $b$ and

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

Problem
Let $f(x) = x^3 + 3x$ and find $(f^{-1})'(4)$.  

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