

§8.2–Infinite series

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Outline

- 1 Some problems to set the stage
- 2 The proper definition
- 3 The geometric series
- 4 Telescoping series
- 5 The harmonic series
- 6 A test for divergence
- 7 Some useful theorems on series

Problem

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Commentary

The preceding examples underscore the need for a proper definition of the sum of an infinite series.

Definition

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- Given a sequence $\{a_n\}$ of real numbers, we form a new sequence as follows:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\vdots$$

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The sequence $\{s_n\}$ is called the *sequence of partial sums*.

- If $s_n \rightarrow L$, then we say that the infinite series $\sum_{k=1}^{\infty} a_k$ converges and we write

$$\sum_{k=1}^{\infty} a_k = L.$$

If $\{s_n\}$ diverges, then we say that the infinite series $\sum_{k=1}^{\infty} a_k$ diverges.

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Solution

- Let s_n denote the n th partial sum of the series. It is easy to see that $s_n = 1$ if n is odd and $s_n = 0$ if n is even.
- Since the sequence $0, 1, 0, 1, \dots$ does not converge, the infinite series $\sum_{k=1}^{\infty} (-1)^{k+1}$ diverges.

Definition

Let $a, r \neq 0$. Any series of the form

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar^1 + ar^2 + ar^3 + \dots$$

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Theorem

$$\sum_{k=1}^{\infty} ar^{k-1} \begin{cases} = \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

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Evaluate the series $\sum_{k=1}^{\infty} 5(1/3)^{k+2}$.

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Solution

The series

$$5(1/3)^3 + 5(1/3)^4 + 5(1/3)^5 + \dots$$

is geometric with $a = 5/27$ and $r = 1/3$. Thus the series converges to

$$\frac{(5/27)}{1 - (1/3)} = \frac{5}{18}.$$

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Solution

The total distance is the infinite series

$$\begin{aligned}
 & 20 + \underbrace{2(.6)(20) + 2(.6)^2(20) + 2(.6)^3(20) + \dots}_{\text{geometric part}} \\
 &= 20 + \frac{24}{1 - .6} \\
 &= 20 + 60 \\
 &= 80
 \end{aligned}$$

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- Thus it can be shown that $s_n = 1 - \frac{1}{n+1}$.
- Since $s_n \rightarrow 1$ as $n \rightarrow \infty$, we conclude that the series converges to 1.

Problem

Show that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

diverges.

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- Consider s_8 and observe that

$$\begin{aligned} s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &\geq 1 + \frac{1}{2} + 2\frac{1}{4} + 4\frac{1}{8} \\ &= 1 + \frac{3}{2} \end{aligned}$$

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- In general, $s_{2^n} \geq 1 + \frac{n}{2}$, which show that $s_n \rightarrow \infty$ as $n \rightarrow \infty$.

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- Thus $s_n \rightarrow L$ for some limit L .
- It is also the case that $s_{n-1} \rightarrow L$.
- Thus $a_n = (s_n - s_{n-1}) \rightarrow L - L = 0$, as was to be shown. □

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Does the series $\sum_{k=1}^{\infty} (1 + k^{-1})^k$ converge?

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Solution

Since $(1 + k^{-1})^k \rightarrow e \neq 0$, the series diverges by the *nth Term Test*.

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- $\sum(a_n + b_n) = \sum a_n + \sum b_n$
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Solution

- We have already observed that $\sum_{k=1}^{\infty} \frac{1}{k^2+k} = 1$.

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Find the sum of the series $\sum_{k=1}^{\infty} \left(\frac{1}{k^2 + k} + 5(.5)^{k+3} \right)$.

Solution

- We have already observed that $\sum_{k=1}^{\infty} \frac{1}{k^2+k} = 1$.
- The series $\sum_{k=1}^{\infty} 5(.5)^{k+3} = 5/8$, being geometric.

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- The series $\sum_{k=1}^{\infty} 5(.5)^{k+3} = 5/8$, being geometric.
- Thus the original series converges to $1 + 5/8 = 13/8$.