

§6.3–Partial Fractions

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Outline

- 1 The method illustrated
- 2 Terminology
- 3 Factoring Polynomials
- 4 Partial fraction decompositions
- 5 Further examples

Example

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$$f(x) = \frac{x+1}{2x^2+7x+6} = \frac{1}{x+2} - \frac{1}{2x+3}.$$

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$$\begin{aligned} \int \frac{x+1}{2x^2+7x+6} dx &= \int \frac{1}{x+2} dx - \int \frac{1}{2x+3} dx \\ &= \ln|x+2| - \frac{1}{2} \ln|2x+3| + C \end{aligned}$$

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- Does this suggest a general method for solving integrals of the form $\int P(x)/Q(x)dx$, where P and Q are polynomials?
- If yes, how do we decompose $P(x)/Q(x)$ into partial fractions?

Definition

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- $f(x) = \frac{x + 1}{2x^2 + 7x + 6}$ is proper.
- $g(x) = \frac{x^3 + 2x + 4}{x^2 - 1}$ is not proper.

Long division

An improper rational function can be written as a sum of a polynomial and a proper rational function by long division. Thus, in the example above,

$$g(x) = \frac{x^3 + 2x + 4}{x^2 - 1} = x + \frac{3x + 4}{x^2 - 1}.$$

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- Note that

$$P(x) = x^3 + 2x^2 - 4x - 8 = x^2(x + 2) - 4(x + 2) = (x - 2)(x + 2)^2.$$

The factors of P are $(x - 2)$ and $(x + 2)$, and both factors are *linear*. We say that $(x + 2)$ has *multiplicity 2*.

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- Note that

$$Q(x) = x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1).$$

The factor $(x^2 + 1)$ cannot be further factored (over the real numbers). Such a factor is called an *irreducible quadratic*. The quadratic

$$ax^2 + bx + c$$

is irreducible when it has no real roots; thus, whenever $b^2 - 4ac < 0$.

Theorem (The Fundamental Theorem of Algebra)

Every polynomial can be expressed as a product of powers of linear factors $(ax + b)^m$ and powers of irreducible quadratic factors $(ax^2 + bx + c)^n$.

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$$\frac{D_1}{(ax + b)^1} + \cdots + \frac{D_m}{(ax + b)^m}.$$

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- To each irreducible quadratic factor $(ax^2 + bx + c)^n$ of Q , the decomposition will contain the terms

$$\frac{E_1x + F_1}{(ax^2 + bx + c)^1} + \cdots + \frac{E_nx + F_n}{(ax^2 + bx + c)^n}.$$

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Solution

The answer is

$$f(x) = \frac{1}{x + 2} - \frac{1}{2x + 3}.$$

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Solution

The answer is

$$f(x) = \frac{2}{(x+2)} + \frac{-1}{(x+2)^2} + \frac{1}{(x-2)}.$$

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Evaluate $\int \frac{x^6 + 2x^4 + x^3 - 2x^2 - x - 5}{x^4 - 1} dx.$

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Solution

The integrand becomes

$$(x^2 + 2) + \frac{1}{x + 1} + \frac{-1}{x - 1} + \frac{x + 1}{x^2 + 1}.$$

This can now be integrated, yielding

$$\frac{x^3}{3} + 2x + \ln|x + 1| - \ln|x - 1| + \frac{1}{2} \ln|x^2 + 1| + \tan^{-1}(x) + C.$$