§5.5 Exponential Growth and Decay

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Outline

1. The general model
2. Examples
3. Doubling-time and half-life
4. Compound interest

The law of natural growth

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  where k is the constant of proportionality.
- If $k > 0$, then this is a model for population growth; if $k < 0$, then this is a model for population decay.

Theorem

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Proof.

See board. □

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Solution

The growth constant is

$$k = (.5 \ln 8/5) \approx .235$$

Thus $y(t) = 500e^{(.235)t}$ *and*

$$y(5) = 500e^{(.235)5} = 1619.$$

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We can derive formulas for the *doubling time* and *half life.*

- If $k > 0$, then $T = \ln(2)/k$. (doubling-time formula)
- if $k < 0$, then $T = \ln(.5)/k$. (half-life formula)

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Solution

We know that

$$y(t) = Ae^{kt} = 2e^{kt}.$$

where k is to be determined. We can find the constant k from our half-life formula.

$$k = \frac{\ln(.5)}{1590} \approx -.0004359.$$

Thus

$$y(t) = 2e^{-.0004359t} \quad \textit{and} \quad y(5000) = 2e^{-.0004359(5000)} \approx .226g$$

Compound interest formula

Suppose that you invest amount A_0 dollars into an account with annual rate r which compounds interest n times per year. Then after t years you will have

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt} \quad \text{dollars}$$

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Solution

The answer is $500(1 + .08)^{12} = \$1259.09$

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Solution

$A(7) = 150(1.005)^{84} = \$228.06.$

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If we let $n \to \infty$, *then the account is said to pay interest continuously. Show that* $A(t) = A_0 e^{rt}$ *in this case.*

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Solution

$$A(t) = \lim_{n\to\infty} A_0 \left(1 + \frac{r}{n}\right)^{nt}$$
$$= \lim_{n\to\infty} A_0 \Bigg(\underbrace{\left(1 + \frac{1}{n/r}\right)^{n/r}}_{\text{converges to } e} \Bigg)^{rt} = A_0 e^{rt}$$

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Solution

$A(7) = 150e^{7(.06)} = \$228.29.$