

§5.4—General logarithmic and exponential functions

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Outline

- 1 General exponential functions
- 2 The derivative
- 3 The logarithm to the base a .
- 4 Solving equations
- 5 e as a limit

Definition

Let $a > 0$ be a real number (this will be the base). Notice that if r is rational, then

$$a^r = \exp(\ln(a^r)) = \exp(r \ln(a)).$$

It makes sense, then, to define

$$a^x = \exp(x \ln(a)).$$

for all $x \in \mathbb{R}$.

The laws of exponents

The *laws of exponents* are inherited from exp:

$$a^{x+y} = a^x a^y \quad a^{x-y} = \frac{a^x}{a^y} \quad (a^x)^y = a^{xy}.$$

In addition, we have the following two identities:

$$(ab)^x = a^x b^x \quad (a/b)^x = a^x / b^x.$$

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Graphs

The graph of $y = a^x$ depends on whether $a = 1$, $a > 1$ or $0 < a < 1$.

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- $\int x^2 3^{x^3} dx$
- $\int \frac{1}{1 + 2^{-x}} dx$

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$$\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)}$$

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- $y = x^x$

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- $2^x - 35 \cdot 2^{-x} = 2$

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(Hint: What is $\frac{d}{dx} \ln x \big|_{x=1}$?)

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Problem

Evaluate

$$\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n$$