1. The Method

   Newton's method is a method for finding roots of equations. It uses the derivative, so can be thought of as a practical application of the derivative. The idea is this: if one starts with an initial guess x_0 reasonably near the root, then draws the tangent line at the point $(x_0, f(x_0)$, then investigates the where this tangent line hits the curve, it is often the case that this new point, which we will call x_1, is closer to the root we are seeking than the original guess. Then we repeat this procedure to get a point x_2, and so on. The sequence $x_0, x_1, \ldots$ is approaching the root.

2. The Iterative Formula

   Let's see how to get x_1 from x_0. The tangent line at x_0 has the form

   $$y - f(x_0) = f'(x_0)(x - x_0).$$

   If we want to see where this hits the x axis, we should let $y = 0$ and let the corresponding x value be x_1. Thus we get the equation

   $$0 - f(x_0) = f'(x_0)(x_1 - x_0).$$

   If we solve this for x_1, we get $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$. In general, this leads to the formula

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. Example

   Suppose $f(x) = x^2 - k$ for some integer k. Note that finding where this hits the x axis is the same thing as finding $\sqrt{k}$, so we will get a nice iterative formula for $\sqrt{k}$. In this case, $f'(x) = 2x$, so we get

   $$x_{n+1} = x_n - \frac{x_n^2 - k}{2x_n} = \frac{x_n^2 + k}{2x_n}.$$

   For example, if $k = 2$ and an initial guess of $x_0 = 1$, we generate the sequence

   $$1, 1.5, 1.41667, 1.41422, \ldots$$

   where these are being rounded to six decimal places. Hopefully you can see that these are converging to the square root of 2.

4. Assignment

   Try problems 1 - 11 odd.