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COMPACTIFICATIONS OF TOPOLOGICAL SPACES

JAY BLANKESPOOR AND JOHN KRUEGER

Abstract. Given a locally compact, Hausdorff space, X, there are several ways to compactify it. We examine two of these compactifications – the one-point compactification, $\hat{X}$, and the Stone-Čech compactification, $\beta(X)$ – and give conditions which guarantee that $\hat{X} \neq \beta(X)$. We also give new examples of topological spaces for which $\hat{X} = \beta(X)$.

1. Introduction

Every calculus student knows that a continuous, real-valued function of a real variable is guaranteed of attaining a maximum or a minimum only on a closed and bounded set. Students of topology and analysis learn that continuous functions behave very nicely on compact sets. But when first presented with the formal definition of compact set, it is unlikely that they have much appreciation for the fact that the definition evolved as the answer to the question: "What is it about closed and bounded subsets of the real line that makes continuous functions defined on them behave so well?" It is important to recognize that it is precisely this "open cover" condition that forces the "nice" behavior of continuous functions defined on such spaces.

If a topological space is not compact, natural questions to ask are: "How 'close' is it to being compact?," or "Can we 'add' anything to it to make it compact?" If the space is locally compact and Hausdorff, the answer to the second question is "yes". (See, e.g., ([1] p. 183)). Generally, in fact, there are several different ways to compactify a locally compact Hausdorff space. In this paper we will investigate the "smallest" (one-point) compactification of X and the "largest" (Stone-Čech) compactification of X. We will give examples of spaces for which these two compactifications are the same (and thus X has only one compactification) and give conditions which guarantee that these two compactifications are different.

2. Compactification of a space, X

Formally, we say that a compact Hausdorff space Y is a compactification of X if X is a subspace of Y such that X is dense in Y. Two compactifications Y_1 and Y_2 of X are called equivalent if there is a homeomorphism $h : Y_1 \longrightarrow Y_2$ such that $h(x) = x$ for all $x \in X$. Does every locally compact Hausdorff space have a compactification? We answer this question affirmatively in the following sections.

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3. The one-point compactification of X

The "easiest" or "smallest" compactification of a locally compact Hausdorff space is the one-point compactification which we denote by $\hat{X}$. $\hat{X}$ is constructed as follows: let the symbol ∞ (called the "point at infinity") denote any object outside of X and let $\hat{X} = X \cup \{\infty\}$. Define $U \subseteq \hat{X}$ to be open in $\hat{X}$ if and only if either $U \subseteq X$ and U is open in X or $U = \hat{X} - C$ where C is a compact subset of X. It is easy to check that this gives a topology for X and that $\hat{X}$ with this topology is a compactification of X. Some easy examples are:

1. If $X = (0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$, then $\hat{X} = [0, 1]$.
2. If $X = \mathbb{R}$, which is homeomorphic to the open interval $(0, 1)$, then $\hat{X}$ is homeomorphic to $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
3. If X is an infinite set with the discrete topology, then every neighborhood of ∞ in $\hat{X}$ contains all but a finite number of points of X.

4. The Stone-Čech compactification of X

A topological space X is completely regular if single point sets are closed in X and if for each point x_0 in X and each closed subset A of X which does not contain x_0, there is a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$. It is well-known that a topological space X has a compactification if and only if X is completely regular. (See, e.g., ([1], corollary 2.3, p. 237)). It should be pointed out here that a locally compact Hausdorff space is completely regular because its one-point compactification, $\hat{X}$, is compact and Hausdorff, hence $\hat{X}$ is normal, hence completely regular, and every subspace of a completely regular space is completely regular.

A basic question which arises in studying compactifications of a space X is the following: *If Y is a compactification of X, under what conditions can a continuous real-valued function f defined on X be continuously extended to Y?* Obviously f must be bounded, because f will carry the compact space Y into $\mathbb{R}$, and compact subsets of $\mathbb{R}$ must be bounded. But being bounded is not enough. A standard example is $f(x) = sin(\frac{1}{x})$ defined on $(0, 1]$. Historically, this problem of continuously extending any bounded, continuous, real-valued function defined on X motivated the development of the Stone-Čech compactification of X, denoted $\beta(X)$, which we now describe.

Let X be a completely regular space and let $\{f_\alpha\}_{\alpha \in J}$ be the collection of *all* bounded, continuous, real-valued functions defined on X, indexed by some set J. For each $\alpha \in J$, choose the closed interval I_α in $\mathbb{R}$ to be

$$I_\alpha = [glb\{f_\alpha(X)\}, lub\{f_\alpha(X)\}].$$

Then define $h : X \to \Pi_{\alpha \in J} I_\alpha$ by $h(x) = (f_\alpha(x))_{\alpha \in J}$. By the Tychonoff theorem, ΠI_α is compact. Since X is completely regular, the collection $\{f_\alpha\}_{\alpha \in J}$ separates points from closed sets in X. Thus by ([1], theorem 4.2, p. 220), h is an imbedding. We define $\beta(X)$, the Stone-Čech compactification of X, to be the closure of $h(X)$ in ΠI_α. See ([1], theorem 3.1, p. 241) for a proof that every bounded, continuous, real-valued function on X is uniquely extendable to $\beta(X)$.

To show that $\beta(X) \neq \hat{X}$ for a particular space X, we need only exhibit one bounded, continuous, real-valued function f on X which cannot be continuously extended to $\hat{X}$. If $X = (0, 1]$ or $(0, 1)$, let $f(x) = sin(\frac{1}{x})$. If X is an infinite set with the discrete topology, let f be any function which takes on different values on two different infinite subsets of X. But there are spaces for which $\beta(X) = \hat{X}$, and we introduce such an example in the next section.

5. S_Ω and $\bar{S}_\Omega$

Let X be an ordered set and $\alpha \in X$. The subset $S_\alpha = \{x \in X : x < \alpha\}$ is called a section of X by α. See ([1], corollary, p. 66) for a proof that there exists an uncountable well-ordered set, Y, at least one section of which is uncountable. Consider the subset of Y consisting of those $\alpha \in Y$ for which the section S_α is uncountable and let Ω denote the smallest such element of Y. Then S_Ω is a well-ordered set which is uncountable, but every section of S_Ω is countable. Munkres also proves ([1] corollary 10.3, p. 67) that every countable subset of S_Ω has an upper bound (and therefore a least upper bound) in S_Ω.

If α_0 denotes the smallest element of S_Ω, then S_Ω is often written as $[\alpha_0, \Omega)$, and $S_\Omega \cup \{\Omega\}$ is written $[\alpha_0, \Omega]$ and is denoted by $\bar{S}_\Omega$. We next show that $\hat{S}_\Omega = \beta(S_\Omega) = \bar{S}_\Omega$ by proving the following theorems:

Theorem 1. $\hat{S}_\Omega = \bar{S}_\Omega$.

Proof: First we must show that $\bar{S}_\Omega$ is compact. Note that $\{\alpha_0\}$ is a nonempty compact subset of S_Ω. Let $\alpha = lub\{x \in \bar{S}_\Omega : S_x \text{ is compact }\}$. Certainly α cannot be the immediate predecessor of an element $\beta \in \bar{S}_\Omega$. But the only element of $\bar{S}_\Omega$ with no immediate successor is Ω. Thus $\alpha = \Omega$ and $\bar{S}_\Omega$ is compact. Since $\bar{S}_\Omega - S_\Omega = \{\Omega\}$, $\bar{S}_\Omega = \hat{S}_\Omega$.

Theorem 2. Every continuous, real-valued function defined on S_Ω is eventually constant.

Proof: Suppose there is an $\epsilon > 0$ such that for every $\alpha \in S_\Omega$ there is a $\beta > \alpha$ such that $|f(\beta) - f(\alpha)| \geq \epsilon$. Then for every positive integer n there is an $\alpha_n \in S_\Omega$ so that $\alpha_n > \alpha_{n-1}$ and $|f(\alpha_n) - f(\alpha_{n-1})| \geq \epsilon$. Let $A = \{\alpha_1, \alpha_2, \cdots\} \subseteq S_\Omega$ and let $\alpha = lub(A) \in S_\Omega$. Since f is continuous at α, $f(\alpha)$ should equal $lim_{n\to\infty} f(\alpha_n)$. But $lim_{n\to\infty} f(\alpha_n)$ does not exist, so there cannot be such an ϵ. Hence, for every $\epsilon > 0$, there is an $\alpha \in S_\Omega$ such that $|f(\beta) - f(\alpha)| < \epsilon$ for all $\beta > \alpha$.

Now, for every positive integer n, choose $\gamma_n \in S_\Omega$ so that for every $\beta > \gamma_n$, $|f(\beta) - f(\gamma_n)| < \frac{1}{n}$. Let $B = \{\gamma_1, \gamma_2, \cdots\} \subseteq S_\Omega$ and let $\gamma = lub(B)$. It is easy to see that if $\beta > \gamma$, then $|f(\beta) - f(\gamma)| < \frac{1}{n}$ for every n, so $f(\beta) = f(\gamma)$; i.e., f is constant after β.

It follows immediately from Theorem 2 that $\beta(S_\Omega) = \bar{S}_\Omega = \hat{S}_\Omega$. It should also be noted that essentially the same proof yields the following generalization of the previous theorem:

Theorem 3. If (Y, d) is any metric space, then every real-valued function $f : S_\Omega \to Y$ is eventually constant.

6. Comparing the Stone-Čech and one-point compactifications

Clearly if X is a locally compact Hausdorff space satisfying that every continuous, real-valued function defined on X is eventually constant (i.e., constant off of some compact subset of X), then $\beta(X) = \hat{X}$. It is natural to ask if there are conditions on X which guarantee that $\beta(X) \neq \hat{X}$. An answer is given in the following theorem, which we then generalize in theorem 5. [Note that $Cl_{\hat{X}}(A)$ denotes the closure of A in $\hat{X}$.]

Theorem 4. If X is locally compact, Hausdorff and normal and if there is a countable subset $A = \{x_1, x_2, \dots\}$ of X such that $Cl_{\hat{X}}(A) = A \cup \{\infty\}$, then $\beta(X) \neq \hat{X}$.

PROOF: By assumption, the point at infinity is the only limit point of A in $\hat{X}$ and therefore A is closed in X. Write $A = B \cup C$, where $B = \{x_1, x_3, x_5, \dots\}$ and $C = \{x_2, x_4, x_6, \dots\}$. Since X is normal and B and C are disjoint closed subsets of X, the Urysohn lemma implies there is a continuous $f : X \to [0,1]$ such that $f(B) = \{0\}$ and $f(C) = \{1\}$. This f is clearly not continuously extendable to $\hat{X}$ and thus $\beta(X) \neq \hat{X}$.

An immediate consequence of the previous theorem is that if $\hat{X}$ is metrizable, or if X can be written as the countable union of compact sets (e.g., $\mathbb{R} = \cup_{n=1}^{\infty}[-n, n]$), then $\beta(X) \neq \hat{X}$. It should be pointed out that the converse of the theorem is false. For example, if X is the disjoint union of two copies of S_Ω, then the "point at infinity" in $\hat{X}$ will be a point (denoted Ω) joining both copies of S_Ω at their "ends". If f is defined from X into the reals by $f(x) = 0$ for all x in one of the copies of S_Ω, and $f(x) = 1$ for all x in the other, then certainly f cannot be continuously extended to $\hat{X}$, and hence $\beta(X) \neq \hat{X}$. But there is no (countable) sequence of points in $\hat{X}$ which can converge to Ω.

The requirement that X be normal is necessary in order to apply the Urysohn lemma. The following lemma allows us to "weaken" that hypothesis to requiring that X be Lindelöf. [Note that a space X is Lindelöf if every open covering of X has a countable subcovering.]

Lemma 1. If X is locally compact, Hausdorff and Lindelöf, then X is normal.

PROOF: It is a standard sequence of exercises (see [1], p. 205, exercises 6,7) to show that every locally compact, Hausdorff space is regular and that every regular, Lindelöf space is normal.

Theorem 5. If X is locally compact, Hausdorff and Lindelöf, then $\beta(X) \neq \hat{X}$.

PROOF: By theorem 4, we need only show that there is a countable subset A of X such that $Cl_{\hat{X}}(A) = A \cup \{\infty\}$. Since X is locally compact, for each $x \in X$, there is an open neighborhood U_x of x such that $\bar{U}_x = Cl_X(U_x)$ is compact. Then $\mathcal{U} = \{U_x : x \in X\}$ is an open cover of X. Since X is Lindelöf, there is a countable subcover, say $\{U_{x_1}, U_{x_2}, \dots\}$. For each positive integer n, let $V_n = \cup_{i=1}^{n} U_{x_i}$ (hence, of course, $\bar{V}_n = Cl_X(V_n)$, which is compact) and choose $y_n \in X - \bar{V}_n$. Let $A = \{y_1, y_2, \dots\}$. If $x \in X - A$, then $x \in U_{x_n}$ for some n, so U_{x_n} is a neighborhood of x which does not contain any of $y_n, y_{n+1}, y_{n+2}, \dots$. Since X is Hausdorff, we can find a neighborhood of x disjoint from A. Thus $x \notin Cl_X(A)$. Now, if C is a compact subset of X, (so that $X - C$ is a neighborhood of ∞ in $\hat{X}$), then there is a V_n such that $C \subseteq V_n$. Thus $\{y_n, y_{n+1}, y_{n+2}, \dots\} \subseteq (\hat{X} - C)$; i.e., every neighborhood of ∞ in $\hat{X}$, contains points of A. Hence $\infty \in Cl_{\hat{X}}(A)$, or $Cl_{\hat{X}}(A) = A \cup \{\infty\}$.

In general, trying to visualize or understand the Stone-Čech compactification of a topological space can be mind-boggling if not impossible. We have seen that this is not the case for S_Ω. Next we calculate $\beta(S_\Omega \times X)$ where X is a compact space. First consider $\mathcal{C}(X, I\!\!R)$ = the collection of all continuous $f : X \to I\!\!R$.

Lemma 2. $d : \mathcal{C}(X, I\!\!R) \times \mathcal{C}(X, I\!\!R) \to I\!\!R$ defined by $d(f, g) = lub_{x \in X}\{|f(x) - g(x)|\}$ defines a metric on $\mathcal{C}(X, I\!\!R)$.

Proof: Clearly $d(f, g) \geq 0$ and $d(f, g) = 0$ if and only if $f = g$. Also, it is clear that $d(f, g) = d(g, f)$. Suppose $f, g, h \in \mathcal{C}(X, I\!\!R)$. For any $x \in X$, $|f(x) - g(x)| = |f(x) - h(x) + h(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)| \leq lub\{|f(x) - h(x)|\} + lub\{|h(x) - g(x)|\} = d(f, h) + d(h, g)$. Since this holds for *any* $x \in X$, we have $lub\{|f(x) - g(x)|\} \leq d(f, h) + d(h, g)$; or, $d(f, g) \leq d(f, h) + d(h, g)$.

Theorem 6. If X is compact, then $\beta(S_\Omega \times X) = \bar{S}_\Omega \times X$.

Proof: Let $f : S_\Omega \times X \to I\!\!R$ be a bounded, continuous function. Define $\hat{f} : S_\Omega \to \mathcal{C}(X, I\!\!R)$ by $\hat{f}(\alpha)(x) = f(\alpha, x)$. By ([1], corollary 5.4, p. 287), $\hat{f}$ is continuous. Since $\mathcal{C}(X, I\!\!R)$ is a metric space, theorem 3 implies that $\hat{f}$ is eventually constant and hence continuously extendable to $\hat{F} : \bar{S}_\Omega \to \mathcal{C}(X, I\!\!R)$. Hence $F : \bar{S}_\Omega \times X \to I\!\!R$ defined by $F(\alpha, x) = \hat{F}(\alpha)(x)$ is a continuous extension of f, and therefore $\beta(S_\Omega \times X) = \bar{S}_\Omega \times X$.

Note that $\beta(S_\Omega \times X) = \bar{S}_\Omega \times X$ is constructed from $S_\Omega \times X$ by "adding" an entire copy of X, not just a single point. Thus the one-point compactification of $S_\Omega \times X$ is not $\beta(S_\Omega \times X)$.

Next we describe a new example of a space X for which $\hat{X} = \beta(X)$. Let $I\!\!R$ denote the reals, $\beta(I\!\!R)$ the Stone-Čech compactification of $I\!\!R$, and choose $x \in \beta(I\!\!R) - I\!\!R$. Let $X = \beta(I\!\!R) - \{x\}$. If f is any bounded, continuous, real-valued function on X, then $f|_{I\!\!R}$, the restriction of f to $I\!\!R$, has a unique extension to $\beta(I\!\!R)$, say $\tilde{f}$. But since $Cl_X(I\!\!R) = X$, $f|_{I\!\!R}$ has at most one continuous extension to X (see [1], Lemma 3.2, p. 241). Since both f and $\tilde{f}|_X$ are continuous extensions of $f|_{I\!\!R}$, they must be identical. Thus $f : X \to I\!\!R$ has a continuous extension, $\tilde{f}$, to $\beta(I\!\!R)$ and so $\beta(X) = \beta(I\!\!R)$. But clearly $\hat{X} = X \cup \{x\} = \beta(I\!\!R)$, so $\hat{X} = \beta(X)$. This construction can be generalized as:

Theorem 7. Let X be a locally compact Hausdorff space, $x \in \beta(X) - X$, and $Y = \beta(X) - \{x\}$. Then $\hat{Y} = \beta(Y)$.

References

[1] Munkres, James R., *Topology, A First Course*, Prentice-Hall, 1975.

Jay Blankespoor: Undergraduate at Calvin College
E-mail address: jblank48@calvin.edu

John Krueger: Undergraduate at Hope College
E-mail address: krueger@cs.hope.edu

Sponsor: John Richard Stoughton, Department of Mathematics, Hope College, Holland, MI 49423
E-mail address: stoughton@math.hope.edu