Furman University
Electronic Journal of Undergraduate Mathematics
Volume 1, 1995 12-21

CONJUGACY CLASSES OF TRIPLE PRODUCTS IN FINITE GROUPS

KEVIN HUTSON AND EMILY SALVO

ABSTRACT. Let G be a finite group and let T_1 denote the number of times a triple $(x, y, z) \in G^3$ binds X, where $X = \{xyz, xzy, yxz, yzx, zxy, zyx\}$, to one conjugacy class. Let T_2 denote the number of times a triple in G^3 breaks X into two conjugacy classes. We have established the following results:
i) the probability that a triple $(x, y, z) \in D_n^3$ binds X to one conjugacy class is $\geq \frac{5}{8}$.
ii) for groups such that $2|Z(G)||G'| = |G|$, $T_2 \geq 3(|Z(G)|)^3|G'|(|G'| - 1)^2$.

1. INTRODUCTION

The motivation for our research stems from the following problem:

Let $x, y \in G$, It is a common exercise in group theory texts to have students show that the number of pairs (x, y) such that $y \in C(x)$ is $k|G|$ where k represents the number of conjugacy classes in G and $C(x)$ is the centralizer of x. For the problem addressed in this paper, note that the pair (x, y) will always produce one conjugacy class in the set $\{xy, yx\}$, since $xy = y^{-1}(yx)y$. A natural extension of this pairs problem is to next look at products of three elements of G. Let G be a finite group, $x, y, z \in G$. Let X denote the following set of triple products:

$$\{xyz, zyx, xzy, zxy, yxz, yzx\}.$$

We examine the conjugacy classes of X produced by the triple $(x, y, z) \in G^3$, where G is a finite group.

2. NOTATION AND OBSERVATIONS

X can represent, at most, two conjugacy classes since the triple products xyz, yzx and zxy are conjugate and the triple products xzy, yxz and zyx are conjugate.

Definition 1. Whenever a triple $(x, y, z) \in G^3$ produces one conjugacy class in X, we will call this "binding" X to one conjugacy class. T_1 denotes the number of times that a triple $(x, y, z) \in G^3$ will bind X to one conjugacy class.

Definition 2. Whenever a triple $(x, y, z) \in G^3$ produces two conjugacy classes in X, we will denote this as "breaking" X into two conjugacy classes. T_2 denotes the number of times that a triple will break X into two conjugacy classes.

Final version received by the editors December 5, 1995.
1991 *Mathematics Subject Classification.* 20B05.
Key words and phrases. Conjugacy classes in Finite Groups; Dihedral Groups; Derived Subgroup.
Both authors supported by NSF Grant NSF–DMS 9322338. Research done at Rose-Hulman Institute of Technology in NSF sponsored REU in Computational Finite Group Theory, under the directorship of Gary Sherman.

Fact 1. A triple $(x, y, z) \in G^3$ binds X to one conjugacy class when any of the following occur:

i) one or both of $y, z \in C(x)$
ii) $z \in C(y)$
iii) a "flip", e.g., $xyz = zyx$ or $yzx = xzy$.

Fact 2. A triple $(x, y, z) \in G^3$ breaks X into two conjugacy classes when all of the following occur:

i) $x \notin Z(G)$
ii) $y \notin C(x)$
iii) $z \notin C(x) \bigcup C(y)$.

where $Z(G)$ denotes the center of the group G.

We present our study of the problem of counting the number of times X is bound to one conjugacy class and the number of times X is broken into two conjugacy classes by the triples $(x, y, z) \in G^3$ of a finite group, G.

3. Abelian Groups

Theorem 1. $T_1 = |G|^3$ if, and only if, G is abelian.

Proof: If G is abelian, then all of the elements of G commute, so $T_1 = |G|^3$. If $T_1 = |G|^3$ then all of the elements must commute, so G is abelian. □

4. Characterizing Dihedral Groups

Let D_n denote the nth dihedral group, the group of symmetries of a regular n-gon.

Theorem 2. A triple $(x, y, z) \in D_n^3$ will bind X to one conjugacy class when one of the following cases occurs:

i) x, y, z are all rotations in D_n
ii) x, y, z are all reflections in D_n
iii) x is a reflection and y, z are rotations
iv) $x \in Z(D_n)$ and y, z are reflections
v) $x \notin Z(D_n)$ but is a rotation and y, z are reflections such that $yz = zy$

Proof: i. Let x, y, z all be rotations. Since the rotations commute, $xyz = yzx = zxy = xzy = zyx = yxz$. Thus, a triple of three rotations binds X to one conjugacy class.

ii. Let x, y, z all be reflections. All of the reflections have order two, so zyx is $(xyz)^{-1}$. Since zyx and xyz are both reflections, $zyx = xyz$. This implies that a triple comprised of three reflections will bind X to one conjugacy class.

iii. Let x be a reflection in D_n and let y, z be rotations in D_n. By the commutativity of the rotations in D_n, $xyz = xzy$. Thus, a triple of one reflection and two rotations will bind X to one conjugacy class.

iv. Let $x \in Z(D_n)$ and let y, z be reflections. Since $x \in Z(D_n)$, then $xyz = yxz$, so a triple in which one element is in $Z(D_n)$ and the other two elements are reflections will bind X to one conjugacy class.

v. Let x be a rotation such that $x \notin Z(D_n)$ and let y, z be reflections such that $yz = zy$. Since $yz = zy$, then $xyz = xzy$. Thus, a triple of this form will bind X to one conjugacy class. □

Theorem 3. A triple $(x, y, z) \in D_n^3$ will break X into two conjugacy classes if x is a rotation not in $Z(D_n)$ and y, z are reflections such that $yz \neq zy$.

PROOF: Let x be a rotation such that $x \notin Z(D_n)$. Let y, z be reflections such that $yz \neq zy$. Suppose xyz and zyx are conjugate, which would imply that the triple binds X to one conjugacy class. Since xyz is a rotation, it can have at most two elements in its conjugacy class, because the conjugacy classes of D_n are well defined. Let r indicate a rotation through $\frac{360}{n}$ degrees of the regular n-gon and let s indicate a fixed reflection. Without loss of generality, let $x = r^k$, $y = r^j s$, and $z = r^l s$ such that $0 < k < n$ and $0 \leq j, l < n$ such that $j \neq l$. Thus $xyz = r^{j+k-l}$ and $zyx = r^{l-j+k}$. Since zyx and xyz are assumed conjugate, then $zyx = r^{l-j+k}$ must equal xyz, zxy or yzx.

i) if $zyx = xyz$ then $l - j + k = j + k - l \Rightarrow j = l$, which is a contradiction.
ii) if $zyx = zxy$ then $l - j + k = l - j - k \Rightarrow k = 0$, which is a contradiction.
iii) if $zyx = yzx$ then $l - j + k = j - l + k \Rightarrow j = l$, which is a contradiction.

Hence, a triple product of this form breaks X into two conjugacy classes. □

5. Counting Triples in Dihedral Groups

The cases enumerated in Theorems 2 and 3 exhaust all possible choices for triples. Now that we have determined when a triple product will bind X to one conjugacy class and when it will break X into two conjugacy classes for the dihedral groups, we focus our attention on formulas to count the specific times that these cases occur.

It is evident that the number of total triples that exist, such that the triples are comprised of elements of D_n, is $(2n)^3$. Note the following observations:

i) There is one way to write a triple of the form (x, x, x).
ii) There are three ways to write a triple of the form (x, y, y) such that $x \neq y$.
iii) There are six ways to write a triple of the form (x, y, z) such that $x \neq y \neq z$.

Theorem 4. The number of times that a triple $(x, y, z) \in D_n^3$ will break X into two conjugacy classes is

$$T_2 = \begin{cases} 6(n-1)\binom{n}{2} & \text{if } n \text{ is even} \\ 6(n-2)[\binom{n}{2} - \frac{n}{2}] & \text{if } n \text{ is odd} \end{cases}$$

PROOF: From Theorem 3, the only time a triple will break X into two conjugacy classes is if the triple is of the form (x, y, z) such that x is a rotation, $x \notin Z(D_n)$, and y, z are reflections such that $yz \neq zy$. In D_n, this implies that $yz, zy \notin Z(D_n)$. And thus, x, y, z are all distinct. Note: if n is odd, $Z(D_n) = \{e\}$, and if n is even, $Z(D_n) = \{e, r^{\frac{n}{2}}\}$. If n is odd, there are $(n-1)$ choices for x, a rotation not equal to the identity, and there are $\binom{n}{2}$ choices for y and z, both reflections such that $y \neq z$. If n is even, there are $(n-2)$ choices for x, a rotation such that $x \notin Z(D_n)$. The choices for y and z as reflections are $(\binom{n}{2} + n)$, but we must subtract $[(n-2)(n)]$, which counts the number of times that y, z are reflections such that $y = z$ and $x \notin Z(D_n)$. From this, we also subtract the number of times that $yz = zy = r^{\frac{n}{2}}$, which occurs $(n-2)$ times, since $x \notin Z(D_n)$. These terms, finally, are multiplied

by a factor of six, because $x \neq y \neq z$. Thus if n is odd, we have the following:

$$\begin{aligned} T_2 &= 6[(n-2)(\binom{n}{2}+n)-(n)(n-2)-\frac{n}{2}(n-2)] \\ &= 6(n-2)[\binom{n}{2}-\frac{n}{2}]. \end{aligned}$$

Corollary 1. The number of times a triple $(x,y,z) \in D_n^3$ will bind X to one conjugacy class is

$$T_1 = (2n)^3 - T_2.$$

6. Probability in Dihedral Groups

We now turn our attention to the probability of picking a triple $(x,y,z) \in G^3$ that splits X into two conjugacy classes.

Definition 3. Let $Pr_2(G,X)$ denote the probability of picking a triple $(x,y,z) \in G^3$ that breaks X into two conjugacy classes and let $Pr_1(G,X)$ denote the probability of picking a triple $(x,y,z) \in G^3$ that binds X to one conjugacy class.

If n is odd, the probability of picking a triple that splits X into two conjugacy classes is

$$Pr_2(D_n,X) = \frac{3(n-1)^2}{8n^2}. \tag{1}$$

The probability of choosing x, which is not the identity, but is a rotation, is $\frac{(n-1)}{2n}$. The probability of choosing $y \neq z$ such that both y and z are reflections is $\frac{(n-1)}{4n}$. Since xyz, yzx and zxy are all conjugate, there are three ways to achieve this triple product. Thus, the probability of choosing a triple that breaks X into two conjugacy classes when n is odd is $\frac{3(n-1)^2}{8n^2}$.

If n is even, the probability of picking a triple that breaks X into two conjugacy classes is

$$Pr_2(D_n,X) = \frac{3(n-2)^2}{8n^2}. \tag{2}$$

The probability of choosing x, which is a rotation not in $Z(D_n)$, is $\frac{(n-2)}{2n}$. The probability of choosing y and z, both reflections such that $yz \neq zy \notin Z(D_n)$, is $\frac{(n-2)}{4n}$. Since xyz, yzx and zxy are all conjugate, again there are three ways to achieve a triple product of this form. Thus, the probability of choosing a triple that breaks X into two conjugacy classes when n is even is $\frac{3(n-2)^2}{8n^2}$.

It is clear that the probability of choosing a triple $(x,y,z) \in D_n^3$ that binds X to one conjugacy class when n is odd is

$$Pr_1(D_n,X) = 1 - \frac{3(n-1)^2}{8n^2}. \tag{3}$$

The probability of choosing a triple $(x,y,z) \in D_n^3$ that binds X to one conjugacy class when n is even is

$$Pr_1(D_n, X) = 1 - \frac{3(n-2)^2}{8n^2}. \tag{4}$$

This leads us to our next theorem.

Theorem 5. $Pr_1(D_n, X) \geq \frac{5}{8}$.

PROOF: From eq. (1) we see that for n odd, the probability that a triple (x, y, z) breaks X into two conjugacy classes is $\frac{3(n-1)^2}{8n^2} = \frac{(3n^2-6n+3)}{8n^2} \leq \frac{3}{8}$. Hence, from eq. (3), we see that the probability that a triple (x, y, z) binds X to one conjugacy class is $\geq \frac{5}{8}$. The same results follow from eqs. (2) and (4) for n even. □

Corollary 2. As $n \rightarrow \infty$, $Pr_1(D_n, X) = \frac{5}{8}$.

Corollary 3. If n is odd, $Pr_1(D_n, X) = Pr_1(D_{2n}, X)$.

PROOF: This follows from careful inspection of eqs. (1) and (2). □

7. Groups in which $2|G'||Z(G)| = |G|$

The dihedral groups are actually just a specific case of a more generalized situation. To fully explain the more general situation we must first examine the following normal subgroup $(G' \cdot Z(G))$ of G, where G' denotes the derived subgroup of G. In general, $|G' \cdot Z(G)| = \frac{|G'||Z(G)|}{|G' \cap Z(G)|}$.

For the dihedral groups D_n with n odd, the normal subgroup $(D_n' \cdot Z(D_n))$ is the entire subgroup of rotations. Recall that in the dihedral groups D_n, if n is even, $|G' \bigcap Z(G)| \neq 1$, while if n is odd, $|G' \bigcap Z(G)| = 1$. However, $2(|G'||Z(G)|) = |G|$ for all dihedrals. In the dihedral groups, the order of the normal subgroup of the rotations is $\frac{|G|}{2}$. It is this normal subgroup we explored in the dihedrals that leads us to a more general formula.

Theorem 6. Let G be a finite group, and let $|G' \cdot Z(G)| = \frac{|G|}{2}$. If $(x, y, z) \in G^3$ such that $x \in (G' \cdot Z(G) - Z(G))$ and $y, z \notin (G' \cdot Z(G))$, with $y \notin C(z)$, then (x, y, z) breaks X into two conjugacy classes.

PROOF: To show that this arrangement leads to X breaking into two conjugacy classes, it must satisfy the conditions for Fact 2. We have already chosen x, y such that $x \notin Z(G)$ and $y \notin C(z)$. Now we must only show that $x \notin C(y) \bigcup C(z)$. The normal subgroup $G' \cdot Z(G)$ is a union of conjugacy classes, which implies that x and y are not conjugate and x and z are not conjugate. Thus $x \notin C(y) \bigcup C(z)$. Since none of x, y or z commute, X is broken into two conjugacy classes. □

Now that we know that a triple that meets the criteria of Theorem 6 breaks X into two conjugacy classes, we wish to classify groups in which this is the only time that X is broken into two conjugacy classes. From Fact 2 cited in the proof of Theorem 6 above, in order for X to break into two conjugacy classes, the following conditions must be satisfied:

i) $x \notin Z(G)$
ii) $y \notin C(x)$
iii) $z \notin C(x) \bigcup C(y)$

Once again, we turn to the dihedrals.

Corollary 4. If n is odd, then a triple $(x, y, z) \in D_n^3$ which fits the criteria of Theorem 6 is the only arrangement that will break X into two conjugacy classes.

PROOF: In the dihedrals, every $y \in (D_n - (D_n' \cdot Z(D_n)))$ has order two. Thus suppose $x \notin Z(D_n)$. We must show that $x \in (D_n' \cdot Z(D_n) - Z(D_n))$. We wish to prove this by contradiction. Suppose $x \notin (D_n' \cdot Z(D_n) - Z(D_n))$, $y \notin C(x)$, and $z \notin C(x) \bigcup C(y)$. But in the dihedrals, the reflections can have at most two conjugacy classes. Thus either z and x, z and y, or x and y are conjugate, which contradicts our original assumption. Thus, $x \in (D_n' \cdot Z(D_n) - Z(D_n))$.

Let $y \notin C(x)$ and let $x \in (D_n' \cdot Z(D_n) - Z(D_n))$. We need show that $y \notin (D_n' \cdot Z(D_n))$. Since y does not commute with x it cannot lie in $D_n' \cdot Z(D_n)$ because $D_n' \cdot Z(D_n)$ is normal. Hence $y \notin (D_n' \cdot Z(D_n))$.

Let $z \notin C(x) \bigcup C(y)$, and let $x \in (D_n' \cdot Z(D_n) - Z(D_n))$. We need show that $z \notin (D_n' \cdot Z(D_n))$ such that $yz \neq zy$. We see that $z \notin (D_n' \cdot Z(D_n))$ for the same reason that $y \notin (D_n' \cdot Z(D_n))$. Since $y \notin C(z)$ then $yz \neq zy$. □

Corollary 5. If n is even, then a triple $(x, y, z) \in D_n^3$ which fits the criteria of Theorem 6 is the only arrangement that will break X into two conjugacy classes.

PROOF: The proof is essentially the same as the previous one except that we replace the normal subgroup $D_n' \cdot Z(D_n)$ with the normal subgroup whose order is $|D_n'||Z(D_n)|$ which again is the normal subgroup containing the rotations. □

Lemma 1. Let G be a finite group such that $G = Z_n \times D_n$. Let $(a, b, c) \in Z_n^3$ and let $(i, j, k) \in D_n^3$. The elements $(a, i), (b, j), (c, k)$ commute if, and only if, the elements i, j, k commute.

PROOF: Suppose $(a, i), (b, j), (c, k)$ commute. It is clear that a, b, c must commute under the group operation, since Z_n is cyclic. Since $(a, i), (b, j), (c, k)$ commute, then i, j, k must also commute. Now suppose i, j, k commute. Obviously, a, b, c commute since they are elements in a cyclic group, Z_n. Thus, $(a, i), (b, j), (c, k)$ commute. □

Theorem 7. Let $G = Z_n \times D_n$. A triple $(x, y, z) \in G^3$ breaks X into two conjugacy classes if
$x \in (G' \cdot Z(G) - Z(G))$ and $y, z \notin (G' \cdot Z(G))$, with $y \notin C(z)$.

PROOF: The proof follows from the proofs of Theorem 6 and Lemma 1. □

8. Dicyclic Groups

We now move away from the dihedral groups. However, we must stay within the realm of groups that have a large, cyclic subgroup of index two in the group. So we turn our attention to the dicyclic groups of order $4m$. The dicyclic groups can be defined as follows:

$$e, x, ..., x^{2m-1}, y, xy, ..., x^{2m-1}y.$$

with multiplication

$$\begin{aligned} x^a x^b &= x^{a+b}, \\ x^a(x^b y) &= x^{a+b}y, \\ x^a y x^b &= x^{a-b}y, \\ (x^a y)(x^b y) &= x^{a-b+m}, \end{aligned}$$

where $0 \leq a, b \leq 2m-1$. Note that when $m = 2$ this group is isomorphic to the quaternion group.

Let $G = Q_n$, where Q_n denotes a dicyclic group of order n. The subgroup generated by x^2, $< x^2 >$, is a normal subgroup in Q_n of index 4, so $Q_n/ < x^2 >$ must be abelian. Hence $Q_n' \subseteq < x^2 >$. But $xyx^{-1}y^{-1} = x^2$, thus $x^2 \in Q_n'$. The center $Z(Q_n)$ is always $\{1, x^{\frac{n}{2}}\}$. Thus $\frac{|Q_n|}{|Q_n' \cdot Z(Q_n)|} = 4$ since $Z(Q_n) \leq Q_n'$, but $\frac{|Q_n|}{|Z(Q_n)||Q_n'|} = 2$ which is analagous to D_n where n is even.

Theorem 8. The only arrangement of $(i, j, k) \in Q_n^3$ that will break X into two conjugacy classes occurs when $i \in (< x > -Z(Q_n))$, which is of order $|Q_n'||Z(Q_n)|$, and $j, k \notin < x >$ such that $jk \neq kj$.

Proof: We must first show that i has to be in $(< x > -Z(Q_n))$. We know that if $i \in Z(Q_n)$ then we will have one conjugacy class, thus $i \notin Z(Q_n)$. Suppose now that $i \notin < x >$ and $y, z \notin < x >$ such that $yz \notin zy$ and suppose this breaks X into two conjugacy classes. Without loss of generality, set $i = x^a y, j = x^b y$, and $k = x^c y$, for $0 \leq a, b, c \leq 2m-1$. But then $ijk = x^{a-b+m+c}y = x^{c-b+m+a}y = kji$. This implies that X is bound to one conjugacy class – a contradiction. Thus $i \in (< x > -Z(Q_n))$. Similar arguments to those that were used with the dihedral groups can again be used to show that $y, z \notin < x >$ such that $yz \neq zy$. □

Lemma 2. Let $G = Z_n \times Q_n$ and let $(a, b, c) \in Z_n^3$ and $(i, j, k) \in Q_n^3$. The elements $(a, i), (b, j), (c, k) \in G$ commute if, and only if, i, j, k commute.

Proof: This proof is essentially the same as the proof for $Z_n \times D_n$. □

Theorem 9. Let $G = Z_n \times Q_n$. The only triples $(x, y, z) \in G^3$ that break X into two conjugacy classes occur if $x \in (G' \cdot Z(G) - Z(G))$ and $y, z \notin (G' \cdot Z(G))$, with $y \notin C(z)$.

Proof: This proof follows from Lemmma 2 and the proof of Theorem 8. □

9. A counterexample: S_n

We have started classifying groups such that $2|G'||Z(G)| = |G|$ by how they break or bind X. We had hoped that all groups of this property would be limited to breaking X into two conjugacy classes only if $x \in (G' \cdot Z(G) - Z(G))$ and $y, z \notin (G' \cdot Z(G))$, with $y \notin C(z)$. However, S_n provides a frustrating counterexample. Although we should have to choose x from the coset $G' \cdot Z(G) = A_n$ in S_n, there exist three elements x, y, z in $S_n - A_n$ whose triple product breaks X into two conjugacy classes. Consider the following example in S_4:

$$x = (1234) \notin A_4, y = (12) \notin A_4, z = (1324) \notin A_4$$

$$xyz = (1432), \text{ but } yxz = (14)$$

Since xyz and yxz are not conjugate, these choices for x, y, z break X into two conjugacy classes. This is a valid counterexample, since $x, y, z \notin A_4$.

10. Equations for groups where $2(|Z(G)||G'|) = |G|$

We are finally ready to explain one of our original premises: our work with the dihedral groups is nevertheless (despite the counterexample) a corollary to a more general formula.

Theorem 10. For groups such that the subgroup $2(|Z(G)||G'|) = |G|$, $3(|Z(G)|)^3|G'|(|G'|-1)^2$ is a lower bound for T_2.

Proof: Theorem 10 uses a polished version of a more understandable formula:

$$\begin{aligned} T_2 &\geq 3(|Z(G)|)^3|G'|(|G'|-1)^2 \\ &= 3(|Z(G)||G'|-|Z(G)|)(|G|-|Z(G)||G'|)(|Z(G)||G'|-|Z(G)|) \end{aligned}$$

This formula indicates that x must be chosen from $(|Z(G)||G'|-|Z(G)|)$ number of elements and y, z must be chosen one from $(|G|-|Z(G)||G'|)$ number of elements and the other from $3(|Z(G)||G'|-|Z(G)|)$ number of elements. As shown in previous proofs, x is chosen from the first coset, such that x is not in $Z(G)$. Next, both y and z are chosen from the second coset, such that z does not commute with y. The formula is multiplied by three since although there are six ways of rewriting xyz, each product has been double counted. In the groups we have already characterized, the formula above is an equality for the triples that break X into two conjugacy classes. In a group such as S_n, however, although this formula is not an equality, it is a lower bound, because Theorem 6 holds true for all groups such that $2|Z(G)||G'| = |G|$. □

It is obvious from the previous theorem that we can simply deduce a formula for T_1 that acts as an upper bound for all groups:

$$T_1 = |G|^3 - T_2. \tag{5}$$

In addition to this formula for T_1, we devised a counting method with different parameters:

$$T_1 = \sum_{x \in G} 2(|G|-|C(x)|)(|C(x)|)+|C(x)|^2+\frac{(|G|-|C(x)|)^2|Z(G)|}{|C(x)|}$$

.

The connection between the dihedral groups and this general situation follows from this formula. For example, we know from our most recent theorem that when $2|Z(G)||G'| = |G|$, then

$$\begin{aligned} T_2 &\geq 3(|Z(G)|)^3|G'|(|G'|-1)^2 \\ &= \frac{3}{8}|G|^3\frac{(|G'|-1)^2}{|G'|^2} \end{aligned}$$

In dihedral groups, as $|G| \to \infty$, $|G'| \to \infty$, hence

$$T_2 \geq \frac{3}{8}|G|^3$$

This result is already shown in Corollary 2 of Theorem 5.

11. Observations for all Groups

After achieving such promising results with groups such that $2|Z(G)||G'| = |G|$, we hoped to be able to generalize this partitioning system to yield equations for groups such that $|G| = 3|Z(G)||G'|$, $|G| = 4|Z(G)||G'|$, $|G| = 5|Z(G)||G'|$, etc. When we worked with data from higher order groups, however, we were unable to generate equations similar to our established results for groups such that $2|Z(G)||G'| = |G|$.

We did, however, make several empirical observations about all groups that are classified under this partitioning system:

Observation 1. Define R to be the proportion of triples which bind X to one conjugacy class to all triples. All groups which have the same value of R share the same $|G'|$.

Observation 2. For any two groups G_1 and G_2 which have the same values of the above stated R and $|G'|$,

$$\frac{|G_2|}{|G_1|} = \frac{|Z(G_2)|}{|Z(G_1)|} = \frac{k_2}{k_1}$$

.

12. Questions

In section 11, Observations for all Groups, we refer to our hope that we could use a partitioning system of the index of $(Z(G) \cdot G')$ in G to create formulas for counting the number of triples that break X into two conjugacy classes. We have not yet found any substantial results to support formulas for groups such that $[G : (G' \cdot Z(G))] \geq 3$. Perhaps, we need to examine additional commutativity that exists within these groups. Also, we still believe that the subgroup $G' \cdot Z(G)$ is an interesting one to explore. In what other ways can all groups be characterized by the subgroup $G' \cdot Z(G)$?

We set the following conjecture: A finite group G is simple if, and only if, $|Z(G) \cdot G'| = 1$. Although we have found it easy to prove that if G is simple then $|Z(G) \cdot G'| = 1$, we have not yet proven that $|Z(G) \cdot G'| = 1$ implies that G is simple. However, we have also not found a counterexample in any group of order ≤ 128.

For the dihedrals, $\frac{5}{8}$ is a lower bound for the probability that a triple binds X to one conjugacy class. Does a similar lower bound exist for all groups? A potential next starting point is to examine simple groups, or possibly an alternating group. Also, does there exist an upper bound for the number of triples that bind X to one conjugacy class in non-Abelian groups? In Abelian groups this number is $|G|^3$. How close does this bound get to $|G|^3$? We conjecture that it is possible to create a sequence of groups such that this bound approaches $|G|^3$.

References

[1] Annin, S. A. and Ziebarth, J. J. 1994. Computing order classes of triple products in finite groups. *Rose-Hulman Math. Tech. Report Series.* 94-02.

[2] Armstrong, M. A. 1988. *Groups and Symmetry.* New York: Springer-Verlag.

[3] Rose, John S. 1978. *A Course on Group Theory.* London: Cambridge UP.

[4] Rusin, David J. *What is the probability that two elements of a finite group commute?* Pacific Journal of Mathematics **82** (1979), pp. 237 - 247.

Emily Salvo: Undergraduate at Villanova University, Villanova, PA
E-mail address: salvo@monet.vill.edu

Kevin Hutson: Undergraduate at Clemson University, Clemson, SC
E-mail address: khutson@clemson.edu

Sponsor: Alice A. Deanin, Department of Mathematical Sciences, Villanova University, Villanova PA