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THE STRONG SHADOWING PROPERTY ON THE UNIT INTERVAL

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ABSTRACT. We study continuous maps of the unit interval into itself. We determine necessary and sufficient conditions so that all pseudo-orbits can be approximated by orbits with the same initial point.

1. INTRODUCTION

For a metric space X, a function $f: X \to X$, and $x \in X$, the sequence $\{f^n(x)\}_{n=0}^{\infty}$ is called the *orbit* of x, where f^n is the n-fold composition of f. Computers simplify the task of calculating orbits, but round-off error will create a "sloppy orbit" called a pseudo-orbit. These pseudo-orbits may or may not stay close to their actual orbits. If every pseudo-orbit stays close to *some* actual orbit, f is said to have the shadowing property.

To be precise, for $\delta > 0$, a *δ-pseudo-orbit* of f is defined as a sequence $\{x_n\}_{n=0}^{\infty}$ such that $d(x_{n+1}, f(x_n)) \leq \delta$ for all $n \in \mathbb{N}$. Furthermore, a function f has the *shadowing property* if for $\varepsilon > 0$, there exists $\delta > 0$ such that given a δ-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ where $d(x_n, f^n(x)) < \varepsilon$ for all $n \in \mathbb{N}$.

Notice it may be the case that an actual orbit must start at a different point than a given pseudo-orbit in order to shadow it. For example, a continuous increasing function $f : [0,1] \to [0,1]$ with $f(0) = 0$, $f(1) = 1$, and $f(x) > x$ on $(0,1)$ has the shadowing property by [3]. However, any δ-pseudo-orbit which starts at $x_0 = 0$ and climbs toward 1 cannot be shadowed by the orbit fixed at 0.

Even though the pseudo-orbit may be close to *some* orbit, this is not always the desired result. Instead, since pseudo-orbits are typically generated when trying to generate an orbit, one may be interested in whether or not an orbit and pseudo-orbit beginning at the same point will stay close to each other. This leads us to make the definition: f has the *strong shadowing property* if for $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$, $d(x_n, f^n(x_0)) < \varepsilon$ for all $n \in \mathbb{N}$. The purpose of this paper is to characterize all continuous functions on $[0,1]$ which have the strong shadowing property.

Given a continuous function $f : [0,1] \to [0,1]$, let $gr(f)$ denote the graph of f and $gr^{-1}(f)$ denote the reflection of $gr(f)$ about the diagonal $i(x) = x$. Also, f has a *k-cycle* ($k \geq 2$) if there exists a set of k distinct points, $\{x_0, x_1, \ldots, x_{k-1}\}$, such that $f(x_j) = x_{j+1} \pmod{k}$. We show that the following three conditions are equivalent:

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1. f has the strong shadowing property
2. f has no cycles and only one fixed point
3. $gr(f)$ and $gr^{-1}(f)$ have exactly one point in common.

The equivalence of (2) and (3) is easily seen. If $gr(f)$ and $gr^{-1}(f)$ have more than one point in common, then either f has a 2-cycle or f has more than one fixed point. Conversely, if $gr(f)$ and $gr^{-1}(f)$ have exactly one point in common, f cannot have any 2-cycles. The Sarkovskii ordering of cycles [1, p.62] then guarantees that f has no cycles. Furthermore, any fixed point of f will be in the intersection of $gr(f)$ and $gr^{-1}(f)$, so f has only one fixed point.

2. Preliminaries

Throughout this paper, we take $I\!N = \{0, 1, 2, \ldots\}$, "$\subset$" denotes strict containment, $f^0(x) = x$, and all intervals are intersected with $[0, 1]$.

We begin with three lemmas first proved by Sarkovskii [4]. The original proofs were not accessible, so we provide the following arguments.

Lemma 1. Let $f : [0,1] \to [0,1]$ be a continuous function with no cycles. For $x \in [0,1]$, if $f(x) > x$, then $f(y) > x$ for all $y \in [x, f(x)]$ (if $f(x) < x$, then $f(y) < x$ for all $y \in [f(x), x]$).

Proof: We prove this lemma by contradiction. Assume $x \in [0,1]$ with $f(x) > x$, and suppose there exists $z \in [x, f(x)]$ such that $f(z) \leq x$. Let $y = \max\{t \in [x, f(x)] : f(t) = x\}$. Since $f(x) > x$ and $f(y) = x < y$, f must intersect the diagonal $i(t) = t$ in $[x, y]$. Thus, there exists at least one fixed point in $[x, y]$. Let $p_L = \max\{t \in [x, y] : f(t) = t\}$. There are two cases to be considered.

Case 1: f has at least one fixed point $p \in [y, 1]$. Let $p_R = \min\{t \in [y, 1] : f(t) = t\}$. Since $f(y) = x < p_L$ and $f(p_R) = p_R > p_L$, there exists $s_1 \in (y, p_R)$ such that $f(s_1) = p_L$ by the Intermediate Value Theorem. Similarly, there exists $s_2 \in (y, p_R)$ such that $f(s_2) = s_1$. Since $f^2(y) = f(x) > y$ and $f^2(s_2) = p_L < s_2$ with $y, s_2 \in (p_L, p_R)$, f^2 must intersect the diagonal $i(t) = t$ in the interval (p_L, p_R). Hence there exists $p \in (p_L, p_R)$ such that $f^2(p) = p$ and $f(p) \neq p$, so f has at least one 2-cycle.

Case 2: f has no fixed points in $[y, 1]$. The point y was chosen so $f(y) = x$, $f^2(y) = f(x) > y$. Since the range of f^2 is contained in $[0,1]$, $f^2(1) < 1$. So f^2 must intersect the diagonal $i(t) = t$ in the interval $[y, 1]$. Hence there exists $p \in [y, 1]$ such that $f^2(p) = p$ and $f(p) \neq p$, so f has at least one 2-cycle.

Each of these cases contradicted that f has no cycles, so if $f(x) > x$, then $f(y) > x$ for all $y \in [x, f(x)]$. Similar arguments prove that if $f(x) < x$, then $f(y) < x$ for all $y \in [f(x), x]$. $\square$

Lemma 2. Let $f : [0,1] \to [0,1]$ be a continuous function with no cycles. For all $n_1, n_2, n_3 \in I\!N$ $(n_1 < n_2 < n_3)$ and $x \in [0,1]$, if $f^{n_2}(x) \neq f^{n_3}(x)$, then $f^{n_1}(x)$ is not in the interval with endpoints $f^{n_2}(x)$ and $f^{n_3}(x)$.

Proof: Choose $x \in [0,1]$ and $n_1, n_2, n_3 \in I\!N$ $(n_1 < n_2 < n_3)$ such that $f^{n_2}(x) \neq f^{n_3}(x)$. Since $f^{n_2}(x) \neq f^{n_3}(x)$, $f^{n_1}(x)$ cannot be a fixed point. Without loss of generality, assume $f^{n_1}(x) < f^{n_1+1}(x)$. We prove this lemma by using strong induction to show that $f^n(x) > f^{n_1}(x)$ for $n = n_1 + 1, \ldots, n_3$.

Assume $f^k(x) > f^{n_1}(x)$ for $k = n_1+1, \ldots, n$ where $n < n_3$. If $f^n(x) = f^{n-1}(x)$, then $f^{n+1}(x) = f^{n-1}(x) > f^{n_1}(x)$ by the induction hypothesis. If $f^n(x) > f^{n-1}(x)$, then $f(f^n(x)) > f^{n-1}(x)$ by Lemma 1. Thus, $f^{n+1}(x) > f^{n-1}(x) \geq f^{n_1}(x)$ by the induction hypothesis. If $f^n(x) < f^{n-1}(x)$, let $s = \max\{k \in I\!N, k < n : f^k(x) < f^{k+1}(x)\}$. We know s exists since $f^{n_1}(x) < f^{n_1+1}(x)$.

We claim that $f^s(x) < f^n(x) < f^{s+1}(x)$, for suppose $f^n(x) < f^s(x)$. By the definition of s, we have the inequalities $f^{s+1}(x) > f^{s+2}(x) > \cdots > f^n(x)$. Hence, there exists p, $s+1 \leq p \leq n-1$, such that $f^p(x) \in [f^s(x), f^{s+1}(x)]$ and $f^{p+1}(x) \notin [f^s(x), f^{s+1}(x)]$. This contradicts Lemma 1, so $f^s(x) < f^n(x)$. Furthermore, since $f^k(x) < f^{k-1}(x)$ for $k = s+2, \ldots, n$, $f^n(x) < f^{s+1}(x)$.

By Lemma 1, $f(f^n(x)) = f^{n+1}(x) > f^s(x) \geq f^{n_1}(x)$. Thus by induction, $f^n(x) > f^{n_1}(x)$ for $n = n_1+1, \ldots, n_3$. Therefore, $f^{n_1}(x)$ is not in the interval with endpoints $f^{n_2}(x)$ and $f^{n_3}(x)$. □

Lemma 3. Let $f : [0,1] \to [0,1]$ be a continuous function. If f has no cycles, then for any $x \in [0,1]$, the orbit $\{f^n(x)\}_{n=0}^{\infty}$ will converge to a fixed point.

PROOF: First notice that $f^n(x) \to z$ implies z is a fixed point of f. Let $x \in [0,1]$. Since f has no cycles, $m \neq n$ implies $f^m(x) \neq f^n(x)$ or $f^m(x)$ is a fixed point. We can assume that $f^m(x) \neq f^n(x)$ for $m \neq n$. By the Bolzano-Weierstrass Theorem, $\{f^n(x)\}_{n=0}^{\infty}$ has a cluster point, z. We will show that z is unique.

Suppose that z' is another cluster point of $\{f^n(x)\}_{n=0}^{\infty}$. Without loss of generality, let $z < z'$. By Lemma 2, (z, z') cannot contain any points from the orbit of x. Moreover, $\{f^n(x)\}_{n=0}^{\infty}$ is the union of two disjoint subsequences: $\{f^{n_k}(x)\} \uparrow z$ and $\{f^{m_k}(x)\} \downarrow z'$. Hence, there exists a subsequence $\{a_j\}$ of $\{f^{n_k}(x)\}_{n=0}^{\infty}$ such that $f(a_j) \to z'$. Thus, $f(z) = z'$ by continuity. Similarly, $f(z') = z$, which contradicts that f has no cycles. Therefore, the sequence $\{f^n(x)\}_{n=0}^{\infty}$ must converge to z, and z must be a fixed point of f by continuity. □

Our initial approach to proving that a function with one fixed point and no cycles has the strong shadowing property was to determine two arbitrarily small intervals containing the fixed point with one interval contained in the other. These intervals would have the property that the image of the larger interval would be contained within the smaller interval. Then we could determine δ such that all δ-pseudo-orbits would eventually be contained in the larger interval, so f would have the strong shadowing property.

However, we cannot guarantee such intervals for f. For example, a function with a fixed point which attracts on one side of the fixed point and repels on the other side cannot have these desired intervals. Nonetheless, we can prove the existence of these intervals for f^2.

Lemma 4. Let $f : [0,1] \to [0,1]$ be a continuous function with no cycles and one fixed point p. For $\varepsilon > 0$, there exists $\eta > 0$, $I = (a,b)$, and $I_\eta = (a-\eta, b+\eta)$ such that $I \subset I_\eta \subseteq B_\varepsilon(p)$ with $p \in I$ and $f^2(I_\eta) \subseteq I$.

PROOF: Let $\varepsilon > 0$ be given. By Lemma 3, the fixed point p attracts the entire interval $[0,1]$. Since p is the only fixed point of f, then $f(x) > x$ on $[0,p)$ and $f(x) < x$ on $(p,1]$. We begin by finding an interval $I_1 \subseteq B_\varepsilon(p)$ such that $f(I_1) \subseteq I_1$. There are two possible cases.

Case 1: $f(B_\varepsilon(p)) \subseteq B_\varepsilon(p)$. Then define $I_1 = B_\varepsilon(p)$ and $\varepsilon' = \varepsilon/2$.

Case 2: $\exists z \in B_\varepsilon(p)$ such that $f(z) \notin B_\varepsilon(p)$. Without loss of generality, assume $z \in (p, p+\varepsilon)$. Since $f(x) < x$ on $(p, 1]$, $f(z) < p - \varepsilon$. Let $m_1 = \min\{x \in [p, z] : f(x) = p - \varepsilon\}$, and define $I_1 = (p - \varepsilon, m_1)$, $\varepsilon' = m_1 - p$. We claim that $f(I_1) \subseteq I_1$. If $x \in (p, m_1)$, then $f(x) > p - \varepsilon$. Furthermore, $f(x) < x < m_1$, so $f(x) \in (p - \varepsilon, m_1)$. If $x \in (p - \varepsilon, p)$, then there exists $x_1 \in (p, m_1)$ such that $f(x_1) = x$ by the Intermediate Value Theorem. Since $f(x) > x$, Lemma 2 implies $f(x) < x_1 < m_1$. Thus, $f(x) \in (x, x_1) \subseteq (p - \varepsilon, m_1)$, so $f(I_1) \subseteq I_1$.

Construct I_2 for $B_{\varepsilon'}(p)$ as I_1 was constructed for $B_\varepsilon(p)$. Without loss of generality we have $I_2 = (a, b)$ such that $f(I_2) \subseteq I_2$. Let $J = (c, d)$ denote an open interval of $f^{-1}(I_2)$ such that $I_2 \subset J \subset I_1$. There are two cases to be considered.

Case 1: $c < a$, $b < d$. Taking $\eta = \min\{a - c, d - b\}$, we have $f(I_\eta) \subseteq I_2$, so $f^2(I_\eta) \subseteq I_2$.

Case 2: $c = a$, $b < d$ ($c < a$, $b = d$ done similarly). Then there exists $e < c$ such that $f((e, d)) \subseteq (c, d)$. We also know that $f((c, d)) \subseteq (a, b) = I_2$. By defining $\eta = \min\{a - e, d - b\}$, $f^2(I_\eta) \subseteq I_2$.

In each of these cases $I = I_2$ and I_η are the required intervals. □

Using this lemma, we will prove that f^2 has the strong shadowing property, where f has no cycles and one fixed point p. To show that f itself has the strong shadowing property, we first establish the uniform convergence of $\{f^k\}_{k=0}^\infty$ to $c(x) \equiv p$ on $[0, 1]$. Given $\varepsilon > 0$, by the proof of Lemma 4, there exists an open interval $I \subseteq B_\varepsilon(p)$ such that $p \in I$ and $f(I) \subseteq I$. We have pointwise convergence of $\{f^k\}$ by Lemma 3, so for each $x \in [0, 1]$ there exists $N_x \in \mathbb{N}$ such that $f^n(x) \in I$ if $n \geq N_x$. Therefore, $\bigcup_{n \in \mathbb{N}} f^{-n}(I)$ is an open cover of $[0, 1]$. By the compactness of $[0, 1]$, we can find $N \in \mathbb{N}$ such that $f^N(x) \in I$ for all $x \in [0, 1]$. Thus, $|f^n(x) - p| < \varepsilon$ for all $x \in [0, 1]$ and $n \geq N$, so $\{f^k\}_{k=0}^\infty$ must converge uniformly to $c(x) \equiv p$.

Finally we show that if f^2 has the strong shadowing property, then f has the strong shadowing property.

Lemma 5. Let $f : [0, 1] \to [0, 1]$ be a continuous function with no cycles and one fixed point p. If f^k has the strong shadowing property for some $k \in \mathbb{N}$, then f^n has the strong shadowing property for all $n \in \mathbb{N}$.

Proof: First we show that if f^k has the strong shadowing property, then f has the strong shadowing property. For arbitrary $\varepsilon > 0$, f^k having the strong shadowing property implies there exists δ', $0 < \delta' < \frac{\varepsilon}{2}$, such that all δ'-pseudo-orbits of f^k are $\frac{\varepsilon}{2}$ strong shadowed. Moreover, we know from the uniform convergence of $\{f^n\}$ that there exists $N \in \mathbb{N}$ such that if $n \geq N$, $|f^n(x) - p| < \frac{\varepsilon}{4}$ for $x \in [0, 1]$. Let $K = \max\{N, k\}$. From uniform continuity, we have the existence of $0 < \delta_1 < \delta_2 < \cdots < \delta_K < \frac{\delta'}{K}$ such that $\mathrm{d}(x, t) < \delta_i \Rightarrow \mathrm{d}(f(x), f(t)) < \delta_{i+1}$ for $i = 1, \ldots, K - 1$.

We constructed this δ-chain so that for any δ_1-pseudo-orbit $\{x_m\}_{m=0}^\infty$ of f, the sequence $\{x_{j+nk}\}_{n=0}^\infty$ is a δ'-pseudo-orbit of f^k, where $j \in \mathbb{N}$. To see this, take an arbitrary $j \in \mathbb{N}$. Then $\mathrm{d}(x_{j+1}, f(x_j)) < \delta_1 \Rightarrow \mathrm{d}(f(x_{j+1}), f^2(x_j)) < \delta_2 \Rightarrow \cdots \Rightarrow \mathrm{d}(f^{k-1}(x_{j+1}), f^k(x_j)) < \delta_k$. Similarly, $\mathrm{d}(f^{k-2}(x_{j+2}), f^{k-1}(x_{j+1})) < \delta_{k-1} < \delta_k$, $\mathrm{d}(f^{k-3}(x_{j+3}), f^{k-2}(x_{j+2})) < \delta_{k-2} < \delta_k$, $\ldots$, $\mathrm{d}(x_{j+k}, f(x_{j+k-1})) < \delta_1 < \delta_k$. Applying the triangle inequality, $\mathrm{d}(x_{j+k}, f^k(x_j)) < k \cdot \delta_k < K \cdot \frac{\delta'}{K} = \delta'$. Therefore $\{x_{j+nk}\}_{n=0}^\infty$ is a δ'-pseudo-orbit of f^k.

By the construction of the δ-chain, $\mathrm{d}(x_n, f^n(x_0)) < \delta_n < \frac{\varepsilon}{2} < \varepsilon$ for $n = 1, \ldots, K$. If $n > K$, let $r \equiv n \bmod K$, $0 \leq r < K$. If $r = 0$, then $\mathrm{d}(x_n, f^n(x_0)) =$

$d(x_{mk}, f^{mk}(x_0)) < \delta' < \varepsilon$ since f^k has the strong shadowing property. Otherwise, $n = mk + r$ where $r \neq 0$, so $d(x_{mk+r}, f^{mk}(x_r)) < \frac{\varepsilon}{2}$. Since $n > K$, then $mk \geq K$, which implies $mk \geq N$. Therefore, $d(f^{mk+r}(x_0), p) < \frac{\varepsilon}{4}$ and $d(f^{mk}(x_r), p) < \frac{\varepsilon}{4}$, so $d(f^{mk+r}(x_0), f^{mk}(x_r)) < \frac{\varepsilon}{2}$. Applying the triangle inequality again, we have $d(f^{mk+r}(x_0), x_{mk+r}) = d(f^n(x_0), x_n) < \varepsilon$. Thus, $d(x_0, f^n(x_0)) < \varepsilon$ for all $n \in \mathbb{N}$, so f has the strong shadowing property.

Finally, if f has the strong shadowing property, then so does f^n for all $n \in \mathbb{N}$. This follows because the δ-pseudo-orbits for f with $d(x_k, f(x_{k-1})) = 0$ for all $k \not\equiv 0$ mod n are simply the δ-pseudo-orbits for f^n. □

Notice that the last statement is independent of the condition that f has no cycles and one fixed point. We use this fact in the proof of the main theorem.

3. Main Result - The Strong Shadowing Property

Theorem 1. Given a continuous function $f : [0, 1] \longrightarrow [0, 1]$ the following are equivalent:

1. f has the strong shadowing property
2. f has no cycles and one fixed point
3. $gr(f)$ and $gr^{-1}(f)$ have exactly one point in common.

Proof:

($1 \Rightarrow 2$) Suppose f has more than one fixed point. Notice that f cannot have an interval of fixed points, otherwise the pseudo-orbit could move about the whole interval while the actual orbit remains fixed. Moreover, since f is continuous, the fixed points of f cannot even be dense in an interval by the same argument. Thus, there must exist fixed points $p_1 < p_2$ with no fixed points in the interval (p_1, p_2).

Without loss of generality we can assume $f(x) > x$ on (p_1, p_2). Given $\delta > 0$, consider the δ-pseudo-orbit $x_0 = p_1$, $x_1 = x_0 + \frac{\delta}{2}$, $x_n = f(x_{n-1})$ for $n \geq 2$. Since $f(x) > x$ on $[p_1 + \frac{\delta}{2}, p_2 - \frac{\delta}{2}]$, the pseudo-orbit $\{x_n\}_{n=0}^{\infty}$ must move at least the distance $\frac{p_2 - p_1}{2}$ from p_1. Thus, f does not have the strong shadowing property.

We know by Lemma 5 that f having the strong shadowing property implies f^n has the strong shadowing property for all $n \in \mathbb{N}$. By the above, f^n must have one fixed point for all $n \in \mathbb{N}$. Therefore, f can have no cycles.

($2 \Rightarrow 1$) Let $g = f^2$, then g has one fixed point p and no cycles. For arbitrary $\varepsilon > 0$, Lemma 4 implies there exist $I = (a, b)$ and $I_\eta \subseteq (p - \frac{\varepsilon}{2}, p + \frac{\varepsilon}{2})$ with $p \in I$ and $g(I_\eta) \subseteq I$. Furthermore, the uniform convergence of $\{g^n\}_{n=0}^{\infty}$ to $c(x) \equiv p$ guarantees the existence of $N \in \mathbb{N}$ such that $g^N(x) \in I$ for all $x \in [0, 1]$.

Since $I_\eta \subseteq (p - \frac{\varepsilon}{2}, p + \frac{\varepsilon}{2})$, $\eta < \frac{\varepsilon}{2}$. From the uniform continuity of g, we have the existence of $0 < \delta_1 < \delta_2 < \ldots < \delta_N < \frac{\eta}{N}$ such that $d(x, t) < \delta_i \Rightarrow d(g(x), g(t)) < \delta_{i+1}$ for $i = 1, \ldots, N$. Consider a δ_1-pseudo-orbit of g, $\{x_n\}_{n=0}^{\infty}$. Thus, $d(x_1, g(x_0)) < \delta_1 \Rightarrow d(g(x_1), g^2(x_0)) < \delta_2 \Rightarrow \cdots \Rightarrow d(g^{N-1}(x_1), g^N(x_0)) < \delta_N$. Similarly, $d(g^{N-2}(x_2), g^{N-1}(x_1)) < \delta_{N-1} < \delta_N$, $d(g^{N-3}(x_3), g^{N-2}(x_2)) < \delta_{N-2} < \delta_N, \ldots,$ $d(x_N, g(x_{N-1})) < \delta_1 < \delta_N$. By applying the triangle inequality, $d(x_N, g^N(x_0)) < N \cdot \delta_N < N \cdot \frac{\eta}{N} \leq \eta < \varepsilon$. Repeating this procedure, we get $d(x_n, g^n(x_0)) < \varepsilon$ for $n = 1, \ldots, N$.

Since N iterations have occurred, $g^k(x) \in I \subseteq I_\eta$ for all $k > N$. Furthermore, the δ_1-pseudo-orbit $\{x_{N+k}\}_{k=1}^{\infty}$ cannot leave I_η since $\delta < \eta$. Since diam $I_\eta < \varepsilon$, $d(x_k, g^k(x_0)) < \varepsilon$ for all $k > N$. Therefore, $g = f^2$ has the strong shadowing property, so f has the strong shadowing property by Lemma 5. □

A corollary to this theorem is that surjective functions on the unit interval do not have the strong shadowing property. By the theorem, we need only consider surjective functions with one fixed point. Let g be such a function with fixed point p. Therefore, $g(x) > x$ on $[0, p)$ and $g(x) < x$ on $(p, 1]$. In particular, there exists $a \in (p, 1]$ so that $g(a) = 0$, and there exists $b \in [0, p)$ such that $f(b) = a$. Therefore, $g^2(x) < x$ on some interval of $[0, p)$, so g has at least one 2-cycle. Then g does not have the strong shadowing property.

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