Furman University
Electronic Journal of Undergraduate Mathematics
Volume 1, 1995 1-11

THE N-EXTENT OF $S^3(p,m)$

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ABSTRACT. In this paper we estimate $xt_n(S^3(p,m))$, the n-extent of various lens spaces. We give numerical evidence for extending certain results of D. G. Yang [*Duke Math. J.*, **74** (1994), 531-545.] to primes p between 11 and 37. We explain the implication of our results for the topology of 4-dimensional manifolds of positive sectional curvature with nontrivial isometric $\mathbb{Z}_p$-actions.

1. Introduction

A recent work of D. G. Yang [5] has shown that given a prime p, a uniform estimate $xt_5(S^3(p,m)) \leqslant \frac{\pi}{3}$ of the 5-extent of the lens space $S^3(p,m)$, used in conjunction with Toponogov's theorem [4], yields an estimate of the Euler characteristic of compact, simply connected, 4-dimensional Riemannian manifolds of positive sectional curvature admitting a nontrivial isometric $\mathbb{Z}_p$-action.

Definition 1. The *n-extent* of a compact metric space is the maximal average distance between n points on the given metric space.

$$xt_n(X) = \binom{n}{2}^{-1} \max\{\sum_{1\leqslant i<j\leqslant n} d(x_i,x_j) | (x_i)_{i=1}^n \subset X\}$$

Definition 2. The *extent* of a compact metric space X is

$$xt(X) = \lim_{n\to\infty} xt_n(X).$$

Definition 3. The lens space $S^3(p,m)$ is the quotient manifold of the isometric $\mathbb{Z}_p$-action

$$\Psi_{1,m} : \mathbb{Z}_p \times S^3 \longrightarrow S^3$$
$$(g,z_1,z_2) \mapsto \left(e^{ig\theta_p}z_1, e^{igm\theta_p}z_2\right)$$

on S^3, where $g \in \mathbb{Z}_p, \theta_p = \frac{2\pi}{p}, 0 < m < p,$ and $(z_1,z_2) \in S^3 \subset \mathbb{C}^2$.

By considering some special configurations, we obtain a lower bound for the n-extent of $S^3(p,m)$.

Proposition .

$$xt_n(S^3(p,m)) \geqslant \tag{1}$$
$$\binom{n}{2}^{-1}\left(\frac{\pi}{2}\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] + \frac{\pi}{p}\left[\frac{\left[\frac{n}{2}\right]}{2}\right]\left[\frac{\left[\frac{n}{2}\right]+1}{2}\right] + \frac{\pi}{p}\left[\frac{\left[\frac{n+1}{2}\right]}{2}\right]\left[\frac{\left[\frac{n+1}{2}\right]+1}{2}\right]\right),$$

Received by the editors August 12, 1995.
1991 *Mathematics Subject Classification.* 53C20.
Key words and phrases. Lens space; n-extent; 4-dimensional, compact, simply connected, positively curved manifold; $\mathbb{Z}_p$-action.
Research supported by NSF Grant number DMS9423945 under the REU program. The authors would like to thank professors M. Kalka and D. G. Yang for their invaluable help with this project.

where $[x]$ *is the largest integer that is less than or equal to* x*. In particular,*

$$xt\left(S^3(p,m)\right) \geqslant \frac{\pi}{4}\left(1+\frac{1}{p}\right). \tag{2}$$

The main purpose of this paper is to conjecture, given strong numerical evidence, that the above inequalities should be equalities.

What is remarkable is that $xt_n\left(S^3(p,m)\right)$ appears to be independent of m. In particular, the above inequality implies that $xt_5\left(S^3(p,m)\right) > \frac{\pi}{3}$ for $p=1$ and the primes $p=2,3,5,7$, while the numerical evidence suggests that $xt_5\left(S^3(p,m)\right) < \frac{\pi}{3}$ for all prime $p \geqslant 11$, which complements the result that $xt_5\left(S^3(p,m)\right) < \frac{\pi}{3}$ for primes $p \geqslant 41$ by D. G. Yang in [5].

Our approach used a computer program to approximate $xt_5\left(S^3(p,m)\right)$. We found that no existing software packages could be used for this purpose, because they require smooth functions, and our distance function is not smooth. The situation is further complicated by the $\mathbb{Z}_p$-action since we have to use the distance function on S^3. Thus, a specialized program had to be written; an explanation of the code can be found in section (4).

2. Numerical Evidence of the N-Extent of $S^3(p,m)$

Our research has primarily dealt with finding the 5-extent of $S^3(p,m)$; however, we have extended this to studying the general n-extent of $S^3(p,m)$. We shall use the following parametrization of S^3. Consider S^3 as the unit sphere in $\mathbb{C}^2$, then each point on S^3 is uniquely represented by a pair of complex numbers $(z_1,z_2)\in\mathbb{C}^2$ with $|z_1|^2+|z_2|^2=1$. The parametrization (r,θ,ϕ) is then given by

$$z_1 = e^{i\theta}\sin r$$
$$z_2 = e^{i\phi}\cos r,$$

where $0\leqslant r\leqslant\frac{\pi}{2}$ and $\theta,\phi\in[0,2\pi]$.

In this parametrization, the metric of S^3 is given by $dr^2+\sin^2 r d\theta^2+\cos^2 r d\phi^2$; however, to find the distance between two points, we must know the geodesic containing them. One therefore notes that, considering points on S^3 as vectors, their dot product is equal to the cosine of the angle between them, and this angle times the radius of the sphere is the distance between the points along the great circle containing them. Since we are considering a unit sphere, we can just take the arc cosine of the dot product to get the distance. Hence, for two points $\left(e^{i\theta_1}\sin r_1, e^{i\phi_1}\cos r_1\right)$ and $\left(e^{i\theta_2}\sin r_2, e^{i\phi_2}\cos r_2\right)$, the distance is equal to

$$\arccos\left(\sin r_1\sin r_2\cos\left(\theta_1-\theta_2\right)+\cos r_1\cos r_2\cos\left(\phi_1-\phi_2\right)\right). \tag{3}$$

The orbit of a point is the set of images of the group elements applied to that point. For example, the orbit of (z_1,z_2) is the set of points

$$\left(e^{ig\theta_p}z_1, e^{igm\theta_p}z_2\right),$$

where $1\leqslant g\leqslant p$ and $\theta_p=\frac{2\pi}{p}$. For $S^3(p,m)$ we need to compute the minimal distance between orbits of points:

$$\inf_{g\in\mathbb{Z}_p}\quad\arccos\Big(\sin r_1\sin r_2\cos\left(\theta_1-\left(\theta_2+g\theta_p\right)\right) \tag{4}$$
$$+\cos r_1\cos r_2\cos\left(\phi_1-\left(\phi_2+gm\theta_p\right)\right)\Big)$$

where $0 < m < p$ (each running of the program tests only one value of m, see section (4)). Equipped with formula (4), one can estimate $xt_5\left(S^3(p,m)\right)$ by approximating the manifold S^3 with a finite grid of points. We attempted to partition the r, θ, and ϕ values finely enough so that our error would be less than the difference between $\frac{\pi}{3}$ and our lower bound. Our partitioning allowed for interesting results, but was not fine enough to support a proof.

Definition 4. An *antipodal configuration* on $S^3(p,m)$ is a set of n points such that, given the above parametrization, $\left[\frac{n}{2}\right]$ and $\left[\frac{n+1}{2}\right]$ points lie on the circles $\left(e^{i\theta},0\right)$ and $\left(0,e^{i\phi}\right)$, not necessarily respectively, forming $\left[\frac{n}{2}\right]$- and $\left[\frac{n+1}{2}\right]$-extenders. (Note that this is not a unique configuration.)

There is much numerical evidence that the n-extent of $S^3(p,m)$ can be achieved with an antipodal configuration. Evidence for the 5-, 4- and 3-extents is strong, whereas it is obvious for the 2-extent (which is just the diameter). Of course, there may be many possible point configurations which achieve the n-extent of a lens space, but if an antipodal configuration always *can* achieve the n-extent, then we *may* choose it.

Definition 5. An *n-extender* is a set of n points on a compact metric space where the n-extent is achieved.

Conjecture . *In the lens space $S^3(p,m)$, with the above-mentioned parametrization, the n-extent can always be achieved with an antipodal configuration. This makes the inequality in the Proposition into a strict equality.*

Evidence. We ran the program given in section (5) to find the 5-extent, as well as variations to find the 4- and 3-extents, with intervals as small as $\frac{\pi}{35}$. Every time the n-extender was an antipodal configuration. □

Lemma 1. *The n-extent of a compact metric space X is a non-increasing function of n:*

$$xt_n(X) \geqslant xt_{n+1}(X).$$

Proof. Let $X_* \subset X$ be the set of points of an $n+1$-extender, and let $x_i, x_j \in X_*$. Then

$$\begin{aligned} xt_{n+1}(X_*) &= xt_{n+1}(X) \\ &= \frac{1}{2}\binom{n+1}{2}^{-1}\sum_{i=1}^{n+1}\sum_{\substack{j=1\\ j\neq i}}^{n+1} d(x_i,x_j) \\ &= \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{n}\sum_{\substack{j=1\\ j\neq i}}^{n+1} d(x_i,x_j). \end{aligned}$$

where we have divided by two because each distance has been counted twice. Now take

$$\min_i\left\{\frac{1}{n}\sum_{\substack{j=1\\ j\neq i}}^{n+1} d(x_i,x_j)\right\}$$

and let x_q be the x_i where this minimum is achieved. Noting that $xt_{n+1}(X_*)$ is the average of

$$\frac{1}{n}\sum_{\substack{j=1\\ j\neq i}}^{n+1} d(x_i, x_j)$$

over all i, it is clear that

$$\begin{aligned} xt_n(X_* - x_q) &= \frac{1}{n}\sum_{\substack{i=1\\ i\neq q}}^{n+1}\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i,q}}^{n+1} d(x_i, x_j) \\ &\geqslant \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{n}\sum_{\substack{j=1\\ j\neq i}}^{n+1} d(x_i, x_j) = xt_{n+1}(X_*). \end{aligned}$$

Because $X_* - x_q \subset X$, it is clear that $xt_n(X_* - x_q) \leqslant xt_n(X)$ and

$$xt_{n+1}(X) = xt_{n+1}(X_*) \leqslant xt_n(X_* - x_q) \leqslant xt_n(X).$$

□

Proof of Proposition. Let $k = \left[\frac{n}{2}\right]$ and $l = \left[\frac{n+1}{2}\right]$ be the number of points on $\left(e^{i\theta}, 0\right)$ and $\left(0, e^{i\phi}\right)$ respectively. The distance between any two points, one on each circle, is $\frac{\pi}{2}$, and there are kl of them. The maximal total distance for k points on $S^1/\mathbb{Z}_p$ is achieved by the same configuration that achieves the maximal average distance, i.e. the k-extender. It is a fact (see [1]) that

$$xt_n(S^m) = \frac{\pi}{2 - \left[\frac{n+1}{2}\right]^{-1}}$$

for any m. On S^1 the only effect of the $\mathbb{Z}_p$-action is to divide the circumference by p, so

$$xt_n\left(S^1/\mathbb{Z}_p\right) = \frac{\pi}{p\left(2 - \left[\frac{n+1}{2}\right]^{-1}\right)}.$$

Thus, the maximal total distance for k points on $\left(e^{i\theta}, 0\right)$ is

$$xt_k\left(S^1/\mathbb{Z}_p\right)\binom{k}{2} = \frac{\pi}{p}\left[\frac{k}{2}\right]\left[\frac{k+1}{2}\right]$$

and a similar result follows for l. Thus the sum is

$$\frac{\pi}{2}kl + \frac{\pi}{p}\left[\frac{k}{2}\right]\left[\frac{k+1}{2}\right] + \frac{\pi}{p}\left[\frac{l}{2}\right]\left[\frac{l+1}{2}\right].$$

Divide this by $\binom{n}{2}$, and we get formula (1). For $n = 4k$,

$$xt_{4k}\left(S^3(p,m)\right) \geqslant \frac{k\pi}{4k-1}\left(1 + \frac{1}{p}\right).$$

Since $xt_n(M)$ is non-increasing in n, we obtain

$$\begin{aligned} xt\left(S^3(p,m)\right) &= \lim_{k\to\infty} xt_{4k}\left(S^3(p,m)\right) \\ &\geqslant \frac{\pi}{4}\left(1 + \frac{1}{p}\right). \end{aligned}$$

□

Remark 1. Some simple algebra will show that

$$xt_{4n+3}\left(S^3(p,m)\right) = xt_{4n+4}\left(S^3(p,m)\right).$$

Remark 2. For $n = 5$, the conjecture yields

$$xt_5\left(S^3(p,m)\right) = \frac{3\pi}{10}\left(1+\frac{1}{p}\right). \tag{5}$$

This delineates all the primes for which D. G. Yang's proof will work. The conjectured 5-extent is less than $\frac{\pi}{3}$ for prime $p \geqslant 11$, but for $p = 2$, 3, 5, and 7, it is greater than $\frac{\pi}{3}$.

Remark 3. Consideration of higher extents shows that the number of isolated fixed points for a compact, positively curved, 4-manifold M with a nontrivial isometric $\mathbb{Z}_p$-action is at most 7 for a prime $p = 7$ and at most 9 for a prime $p = 5$. In fact,

$$xt_{4k}\left(S^3(3,m)\right) \geqslant \frac{4k\pi}{3(4k-1)} > \frac{\pi}{3},$$

and since the n-extent is an decreasing function of p and a non-increasing function of n, $xt_n\left(S^3(2,m)\right) \geqslant xt_n\left(S^3(3,m)\right) > \frac{\pi}{3}$ for all integer $n \geqslant 2$. Thus, there is no maximal number of isolated fixed points for $p = 2, 3$.

An interesting result is the behavior of the 5-extenders as r_2 (see section (4)) is moved from $\frac{\pi}{2}$ to 0. All the points are ordered in the r-coordinate, such that given $r_2 \in \left[0, \frac{\pi}{2}\right]$, $r_2 \leqslant r_3 \leqslant r_4 \leqslant r_5 = \frac{\pi}{2}$. As r_2 is moved from $\frac{\pi}{2}$ to 0 and more points are allowed, we observe that the 5-extent is monotonically increasing. Also, the four free points of the extender have r_2 as their r-coordinate until we pass r = $\frac{\pi}{4}$, at which time two points take r-value r_2, and the other two have r-value $\frac{\pi}{2}$ - r_2; in other words, the two pairs of points are equal r-distance away from $\frac{\pi}{4}$. When $r_2 = 0$, we see that the points are in an antipodal configuration.

Note that to find the 5-extent, we must take the supremum over all 5-point configurations for a given r_2. However, all data implies that the 5-extent is achieved when $r_2 = 0$, i.e. there are two fixed points, one at either end. Our data further implies that the 5-extent *can* be achieved with the antipodal configuration.

3. Applications to Group Actions

We now show how our results apply to 4-manifolds. For a more complete and thorough explanation, see D. G. Yang [5]. We start by recalling Lemmas 9 and 10 in [5]:

Lemma 2 (Lemma 9 of [5]). *Let $F(\mathbb{Z}_p, M)$ be the fixed point set of the isometric $\mathbb{Z}_p$-action on M. Then*

1. $\chi(M) \equiv \chi(F(\mathbb{Z}_p, M))\,(mod\ p)$
2. *Each connected component of $F(\mathbb{Z}_p, M)$ is a totally geodesic submanifold of even codimension.*
3. *$F(\mathbb{Z}_p, M)$ is nonempty and*

$$F(\mathbb{Z}_p, M) = \begin{cases} \chi(F(\mathbb{Z}_p, M)) \ \textit{isolated points} \\ \textit{or} \quad S^2 \cup (\chi(F(\mathbb{Z}_p, M)) - 2) \ \textit{isolated points,} \end{cases}$$

where M is a 4-dimensional, compact, positively curved, orientable manifold with a nontrivial, isometric $\mathbb{Z}_p$-action.

Lemma 3 (From Lemma 10 of [5]). *Let M be a simply connected 4-manifold with positive sectional curvature and a $\mathbb{Z}_p$ symmetry for some prime p. Assume that for every free isometric $\mathbb{Z}_p$-action on S^3, $xt_n\left(S^3/Z_p\right) \leqslant \frac{\pi}{3}$. Then $F(\mathbb{Z}_p, M)$ contains at most n isolated fixed points.*

Our numerical data's application lies in Lemma 10, where the assumption that $xt_n\left(S^3/\mathbb{Z}_p\right) \leqslant \frac{\pi}{3}$ is made. To get our desired result, we first use Toponogov's comparison theorem [4] to show that for each triple (i, j, k) of distinct integers in $[1, n+1]$ we have $\alpha_{ijk} + \alpha_{jki} + \alpha_{kij} > \pi$, where α_{ijk} is the union of arc_{ij} and arc_{jk}, and thus

$$\sum_{i=1}^{n+1} \sum_{i<j<k\leqslant n+1} \alpha_{ijk} > \binom{n+1}{3}\pi.$$

On the other hand, since $xt_n\left(S^3(p,m)\right) < \frac{\pi}{3}$,

$$\sum_{i=1}^{n+1} \sum_{i<j<k\leqslant n+1} \alpha_{ijk} < \binom{n+1}{3}\pi.$$

We do this by seeing that a $\mathbb{Z}_p$-action on $y \in M$ (an isolated fixed point) induces a free isometric $\mathbb{Z}_p$-action on $S^3_y \subset T_y M$, which is isometrically $S^3(p, m)$ for some integer m with $1 \leqslant m \leqslant p-1$. The contradiction becomes clear, and so we have our desired result of $\mathrm{F}(\mathbb{Z}_p, M)$ containing at most n isolated fixed points.

To see a more useful application of our research, we outline D. G. Yang's proof. By way of Lemma 2, it follows that $\mathrm{F}(\mathbb{Z}_p, M)$ contains at most n isolated fixed points. Using part 3 of Lemma 2, $\chi\left(F\left(\mathbb{Z}_p, M\right)\right) \leqslant n+2$ since the largest number of isolated points in the fixed point set is $\chi\left(F\left(\mathbb{Z}_p, M\right)\right) - 2$. Noting that $\chi\left(F\left(S^1, M\right)\right) = \chi\left(F\left(\mathbb{Z}_p, M\right)\right)$, we obtain the expression

$$\chi(M) = \chi\left(F\left(\mathbb{Z}_p, M\right)\right) \leqslant n+2.$$

We will now introduce the concept of a Betti number $\mathrm{B}(M) = \sum_{i=0}^{4} dimH_i(M)$ for 4-manifolds. Since $\dim H_1(M) = \dim H_3(M) = 0$,

$$B(M) = \sum_{i=0}^{4} dimH_i(M) = \chi(M) = 2 + dimH_2(M) \geqslant 2.$$

Combining this with our earlier expression, we end up with

$$2 \leqslant B(M) = \chi(M) = \chi\left(F\left(\mathbb{Z}_p, M\right)\right) \leqslant n+2.$$

This is the desired estimate for the Euler characteristic $\chi(M)$.

4. Appendix A

The program that we used is named 5extent.c and is written in the C programming language. It approximates $xt_5\left(S^3(p,m)\right)$; a few simple modifications allow computation of the n-extent, detailed at the end of this section. To run it, compile 5extent.c on your system and run the .out file with these parameters: N_1 N_2 N_3 P M R. The meaning and permissible values of these numbers are discussed below.

To compute the 5-extent, S^3 is covered with a grid of points determined by N_1, N_2, N_3. The program will consider all points in the form: $\frac{\pi i}{2(N_1-1)}, \frac{2\pi j}{\mathrm{P}N_2}, \frac{2\pi k}{N_3}$, where $0 \leqslant i < N_1$, $0 \leqslant j < N_2$, $0 \leqslant k < N_3$ and $i, j, k, N_1, N_2, N_3 \in \mathbb{Z}$. Note that θ's interval is not $[0, 2\pi]$; the $\mathbb{Z}_p$ action effectively partitions either θ or ϕ, and we have chosen θ. Note also that r = 0 and r = $\frac{\pi}{2}$ are both included, whereas

$\theta = \frac{2\pi}{P}$ and $\phi = 2\pi$ are not distinct from $\theta = 0, \phi = 0$, respectively, and so are excluded. This partitioning applies to all n-extents.

To obtain the error for a particular partitioning, we find half the maximal distance between two neighboring points on the grid and use the triangle inequality. The distance between two neighboring points on the grid is given by

$$\arccos\left(\sin(r_0)\sin(r_0+\Delta r)\cos(\Delta\theta)+\cos(r_0)\cos(r_0+\Delta r)\cos(\Delta\phi)\right).$$

Taking the partial derivative with respect to r_0, one sees that the r_0 which gives the maximal error is dependent upon $\Delta\theta$ and $\Delta\phi$. When $\Delta\phi$ is larger than $\Delta\theta$, the maximal error occurs at $r_0 = 0$ so that $\delta = \arccos(\cos(\Delta r)\cos(\Delta\phi))$ is half the length of the diagonal between two neighboring points on the grid. When $\Delta\theta$ is larger than $\Delta\phi$ the maximal error occurs at $r_0 = \frac{\pi}{2} - \Delta r$, where $\delta = \arccos\left(\sin\left(\frac{\pi}{2}-\Delta r\right)\cos(\Delta\theta)\right) = \arccos(\cos(\Delta r)\cos(\Delta\theta))$ is half the length of the diagonal. Therefore, we use the expression $\delta = \arccos\left(\cos\left(\frac{\pi}{4(N_1-1)}\right)\cos\left(\frac{\pi}{N}\right)\right)$, where $N = \min\{\mathrm{P}N_2, N_3\}$. By the triangle inequality, the distance between two points on $S^3(p,m)$ is less than or equal to the sum of the distance approximated by our grid and 2δ. Since there are ten distances in the 5-extent, we obtain $\frac{20}{10}\arccos\left(\cos\left(\frac{\pi}{4(N_1-1)}\right)\cos\left(\frac{\pi}{N}\right)\right)$; however, the homogeneity of $S^3(p,m)$ allows us to fix one point, so the error is

$$\varepsilon = \frac{16}{10}\arccos\left(\cos\left(\frac{\pi}{4(N_1-1)}\right)\cos\left(\frac{\pi}{N}\right)\right) \tag{6}$$

The error for the n-extent is easily calculated in the same manner.

The next parameter, P, is simply a prime number denoting which $\mathbb{Z}_p$ subgroup of the S^1 group the user wishes to test. M, the next parameter, corresponds to m in the $\Psi_{1,m}$ map of (2). In the program, M is sent into the function "dist," where it is called Mgen. In "dist," orbits of points are considered by the $\Psi_{1,m}$ map: each application of the map rotates the θ-coordinate by $\frac{2\pi}{\mathrm{P}}$, while the ϕ-coordinate is rotated by $\frac{2\mathrm{M}\pi}{\mathrm{P}}$. g is the number of these applications. Although $1 \leqslant \mathrm{M} \leqslant \mathrm{P}-1$, one need only test for $\mathrm{M} \leqslant \frac{\mathrm{P}+1}{2}$ due to the symmetry of the rotations. The supremum of these results is $xt_5\left(S^3/\mathbb{Z}_p\right)$.

The final parameter, R, correlates to the r-coordinate of our second point (r_2), such that $R \in [0, N_1-1] \subset \mathbb{Z}$ and $r_2 = \frac{R\pi}{2(N_1-1)}$. The homogeneity of $S^3(p,m)$ allows us to fix one of our five points. Because the $\mathbb{Z}_p$-action is isometric, we may allow the θ- (or ϕ-) coordinate of our fixed point to vary, and instead fix the θ-coordinate of a different point. Two choices are optimal: $(0,\theta_1,0)$ and $(r_2,0,\phi_2)$; $\left(\frac{\pi}{2},0,\phi_1\right)$ and $(r_2,\theta_2,0)$. Since at $r_2 = 0$ our point is $\left(0,e^{i\phi}\right)$ and at $r_2 = \frac{\pi}{2}$ our point is $\left(e^{i\theta},0\right)$, we can let $\theta_1 = 0$ in the first case and let $\phi_1 = 0$ in the second case. Our program uses the second option because the smaller partitioning of θ allows the program to run faster. The program is written so that $r_2 \leqslant r_3 \leqslant r_4 \leqslant r_5 \leqslant \frac{\pi}{2}$. Making R a parameter saves computing time and allows observation of some interesting behavior (c.f. section (2)). To find the 5-extent for $S^3(p,m)$ for a certain P and M, the user must test all possible values of R for each lens space.

A naive approach would be to compute the distance for every configuration of 5 points. This includes an excessive amount of repetition in calculation. Our program computes the distance between every point on the grid generated by N_1, N_2, N_3, and stores the results in an array. The distance is calculated in the function "dist" by taking the minimal distance between the orbits of two points. When finding the distance between two orbits, one can fix a point in one of the orbits, because the $\mathbb{Z}_p$-action is isometric. The program proceeds to consider every configuration of five points on the grid. For each one it takes the ten appropriate distances in the array and averages them, then checks whether this is greater than the current maximum. When the program is done, it will print out the 5-extender and the 5-extent.

Looking at the program, note that the coordinates of the third point correlate to j_1, j_2, j_3, the fourth point k_1, k_2, k_3, and the fifth point l_1, l_2, l_3. To find the 4-extent, remove one of these points entirely - the three loops for that point and the four distances associated to it. To find the 3- or 2-extent, remove two or three points respectively. For 6-extents and higher, simply add in another point, with three loops and five additional distances. When altering the code, remember that there are $\binom{n}{2}$ distances; you must divide by this number after adding up the distances.

A word of caution is necessary for those intending to run this program; the enormity of the number of calculations may cause problems. A "segmentation fault" may mean your computer does not have enough available memory for holding the array of distances. You must lessen the number of partitions or get a bigger computer.

5. Appendix B

5-Extent.c

\# include <stdio.h>
\# include <math.h>
\# define Pi 3.14159

float dist(int f_1, int f_2, int f_3, int f_4, int f_5, int f_6, int $part_1$,
int $part_2$, int $part_3$, int $prime$, int $Mgen$);
/* This calculates the distance between any two points */
/* on our grid in the lens space $S^3(p, m)$. */

int main(int argc, char *argv[ ])
{
int i_1, int i_2, int i_3, int i_4, int i_5, int i_6; / * These are index variables. */
int h_2, int j_1, int j_2, int j_3, int k_1, int k_2, int k_3, int l_1, int l_2, int l_3;
/* These are the indices of our points. */
int e_1, int e_2, int e_3; /* More index variables*/
int N_1 = atoi(argv[1]); /* Number of r partitions. */
int N_2 = atoi(argv[2]); /* Number of θ partitions. *
int N_3 = atoi(argv[3]); /* Number of ϕ partitions. */
int P = atoi(argv[4]);
int M = atoi(argv[5]); /* Member of the $\mathbb{Z}_p$ group. */
int R = atoi(argv[6]);
float D[N_1][N_2][N_3][N_1][N_2][N_3]; /* This array stores all the distances. */
float ext_5; /* Holds the current supremum. */
float $extco$[12]; /* Holds coordinates of the 5-extender. */
float d_1, d_2, d_3, d_4; /* Holds sums of distances. */
int n_2; /* Most of the time this is identical to N_2. */

/* The following code fills the array D with all possible distances */
/* on our mesh of $S^3(p,m)$. The second r value, */
/* denoted by i_4, starts at i_1, eliminating redundant */
/* calculations through ordering of points. */
for $(i_1 = (N_1 - 1)\,; i_1 > (R-1)\,; i_1 - -)$
for $(i_2 = 0; i_2 < N_2; i_2 + +)$
for $(i_3 = 0; i_3 < N_3; i_3 + +)$
for $(i_4 = i_1; i_4 > (R-1)\,; i_4 - -)$
for $(i_5 = 0; i_5 < N_2; i_5 + +)$
for $(i_6 = 0; i_6 < N_3; i_6 + +)$
D$[i_1][i_2][i_3][i_4][i_5][i_6]$
= dist($i_1, i_2, i_3, i_4, i_5, i_6, N_1, N_2, N_3, P, M$);

/* This section varies the j, k, and l points around the mesh, */
/* as well as the θ coordinate of the R point, summing up */
/* the ten distances. */
$ext_5 = 0.0$; /* The 5-extent will be greater than zero. */
$n_2 = N_2$;
if $(R == 0)$ $n_2 = 1$; /* If R is 0, there is no θ coordinate. */
for $(h_2 = 0; h_2 < n_2; h_2 + +)$
{
$d_1 =$ D$[N_1 - 1][0][0][R][h_2][0]$;
for $(j_1 = (N_1 - 1)\,; j_1 > (R-1)\,; j_1 - -)$
for $(j_2 = 0; j_2 < N_2; j_2 + +)$
for $(j_3 = 0; j_3 < N_3; j_3 + +)$
{
$d_2 = d_1 +$ D$[N_1 - 1][0][0][j_1][j_2][j_3]$ +
D$[j_1][j_2][j_3][R][h_2][0]$;
for $(k_1 = (N_1 - 1)\,; k_1 > (j_1 - 1)\,; k_1 - -)$
for $(k_2 = 0; k_2 < N_2; k_2 + +)$
for $(k_3 = 0; k_3 < N_3; k_3 + +)$
{
$d_3 = d_2 +$ D$[N_1 - 1][0][0][k_1][k_2][k_3]$ +
D$[k_1][k_2][k_3][R][h_2][0]$ +
D$[k_1][k_2][k_3][j_1][j_2][j_3]$;
for $(l_1 = (N_1 - 1)\,; l_1 > (k_1 - 1)\,; l_1 - -)$
for $(l_2 = 0; l_2 < N_2; l_2 + +)$
for $(l_3 = 0; l_3 < N_3; l_3 + +)$
{
$d_4 = d_3 +$ D$[N_1 - 1][0][0][l_1][l_2][l_3]$ +
D$[l_1][l_2][l_3][R][h_2][0]$ +
D$[l_1][l_2][l_3][j_1][j_2][j_3]$ +
D$[l_1][l_2][l_3][k_1][k_2][k_3]$;
$d_4 = d_4$ / 10; /* Average the distance sum. */
if $(d_4 > ext_5)$
{
$ext_5 = d_4$;
$extco[0] = R$;

$extco[1] = h_2;$
$extco[2] = 0;$
$extco[3] = j_1;$
$extco[4] = j_2;$
$extco[5] = j_3;$
$extco[6] = k_1;$
$extco[7] = k_2;$
$extco[8] = k_3;$
$extco[9] = l_1;$
$extco[10] = l_2;$
$extco[11] = l_3;$
}
}
}
}
}

/* By this stage, we have our 5-extent for the given mesh. */
/* Now we compute the coordinates of the 5-extenders. */
for $(e_1 = 0; e_1 < 10; e_1 += 3)$
$extco[e_1] = extco[e_1] * \mathrm{Pi} \;/\; (2 * (N_1 - 1));$
for $(e_2 = 1; e_2 < 11; e_2 += 3)$
$extco[e_2] = extco[e_2] * 2 * \mathrm{Pi} \;/\; (N_2 * P);$
for $(e_3 = 2; e_3 < 12; e_3 += 3)$
$extco[e_3] = extco[e_3] * 2 * \mathrm{Pi} \;/\; N_3;$

/* The following section prints our results. */
printf("The 5-extent for %d partitions of r, \n", N_1);
printf("%d partitions of theta, and %d partitions of phi, \n", N_2, N_3);
printf("with $\mathbb{Z}_{(\%d,\%d)}$ action, is %f. \n", P, M, ext_5);
printf("The coordinates are: \n");
printf("%f 0.000000 0.000000 \n");
printf("%f %f %f \n", $extco[0]$, $extco[1]$, $extco[2]$);
printf("%f %f %f \n", $extco[3]$, $extco[4]$, $extco[5]$);
printf("%f %f %f \n", $extco[6]$, $extco[7]$, $extco[8]$);
printf("%f %f %f \n", $extco[9]$, $extco[10]$, $extco[11]$);
return 0;
}

/* This function will calculate the distance between points */
/* on $S^3\,(p, m)$ by finding the minimal distance between orbits */
/* of points in S^3 under the $\mathbb{Z}_p$ action. Since we are */
/* dealing with the dot product and not the arc cosine of the */
/* dot product, we must find the maximum over the orbits. */
float dist(int f_1, int f_2, int f_3, int f_4, int f_5,
int f_6, int $part_1$, int $part_2$, int $part_3$, int $prime$, int $Mgen$);
{
int g; /* Member of the $\mathbb{Z}_p$ group. */

float r_1, t_1, p_1, r_2, t_2, p_2; /* The coordinates of the two points. */
float $cosAB$; /* This represents the dot product. */
float $dorb$; /* The current maximal, orbital distance. */
$r_1 = f_1 * \mathrm{P}i/(2*(part_1 - 1))$;
$t_1 = f_2 * 2 * \mathrm{P}i/(part_2 * prime)$;
$p_1 = f_3 * 2 * \mathrm{P}i/part_3$;
$r_2 = f_4 * \mathrm{P}i/(2*(part_1 - 1))$;
$t_2 = f_5 * 2 * \mathrm{P}i/(part_2 * prime)$;
$p_2 = f_6 * 2 * \mathrm{P}i/part_3$;
$dorb = -1.0$;

for $(g = 0; g < prime; g++)$
{
$$cosAB = (sin(r_1)) * (sin(r_2)) * (cos(t_1 - t_2 - g * 2 * \mathrm{P}i/prime)) + (cos(r_1)) * (cos(r_2)) * (cos(p_1 - p_2 - g * Mgen * 2 * \mathrm{P}i/prime));$$
if $(cosAB > dorb)$ $dorb = cosAB$;
}
return acos($dorb$);
}

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