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MAXIMUM MATCHINGS IN COMPLETE MULTIPARTITE GRAPHS

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Abstract. How many edges can there be in a maximum matching in a complete multipartite graph? Several cases where the answer is known are discussed, and then a new formula is given which answers this question.

1. Introduction

Suppose we have a company that has several departments, and for some project, we want the employees from different departments working in pairs. If no employee works in more than one department, then how many simultaneous pairs of employees can we have with no employee in more than one pair? That is, what is the maximum number of pairs? When can all employees be paired?

We will treat this as a problem in graph theory. We will see that employees correspond to vertices, and pairs of employees correspond to edges. Similarly, departments relate to vertex partition sets in a multipartite graph that represents the company. Furthermore, we will consider the complete multipartite graph because we allow any pair of employees in different departments to be paired together. Finally, we will see that vertex disjoint edges correspond to disjoint pairs of employees. Thus our goal is to find a matching, that is, a set of vertex disjoint edges, of largest size. We will use standard notation and definitions from graph theory [1] to look at this problem.

In essence, in this paper, we look at how many edges there can be in a maximum matching in a complete multipartite graph. We first look at several cases in which the answer is known. Except for results which are cited, all of the theorems are original. Finally, in our conclusion, we provide a new theorem which solves this problem with a simple formula.

2. Basic Definitions

First, we define a graph to be a finite set of objects, called vertices, together with a collection of unordered pairs of vertices, called edges. Pictorially, edges can be viewed as line segments connecting the vertices. (See Figure 1.) A graph is multipartite if the set of vertices in the graph can be divided into non-empty subsets, called parts, such that no two vertices in the same part have an edge connecting them. (See Figure 2.) Furthermore, a complete multipartite graph is a multipartite graph such that any two vertices that are not in the same part have an edge connecting them. We will denote a complete multipartite graph with n parts by $K_{m_1,m_2,\ldots,m_n}$ where m_i is the number of vertices in the i^{th} part of the graph. Because it will be helpful later, we will assume without loss of generality

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that $1 \leq m_1 \leq m_2 \leq \cdots \leq m_n$. Furthermore, we will call the n^{th} part the maximum part. An example of a complete multipartite graph would be $K_{2,2,3}$. (See Figure 3.)

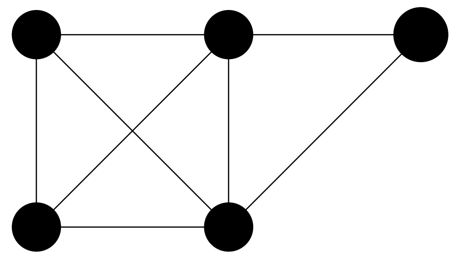

Figure 1 – A Graph.

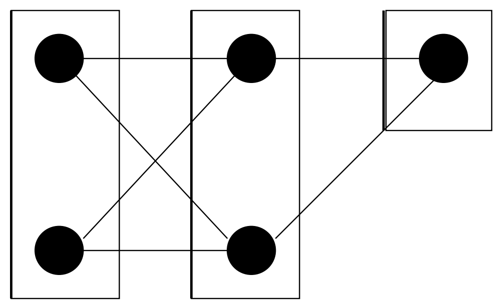

Figure 2 – A 3-Partite Graph.

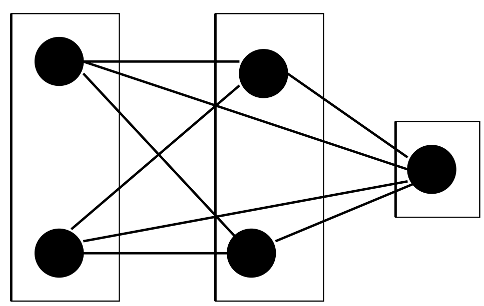

Figure 3 – A Complete 3-Partite Graph.

Two edges are said to be vertex disjoint if they do not have a vertex in common. We call a set of pairwise vertex disjoint edges a matching. Clearly, every edge connects two vertices, and every vertex in a matching lies on exactly one edge by the definition of vertex disjoint edges. (See Figure 4.) So if E equals the total number of edges in a matching, and if V equals the total number of vertices in the same matching, then $V = 2E$. A matching is called maximum if there is no other matching containing more edges. In Figure 5, the thick lines represent a maximum matching in the graph with vertices A, B, C, and D. (See Figure 5.)

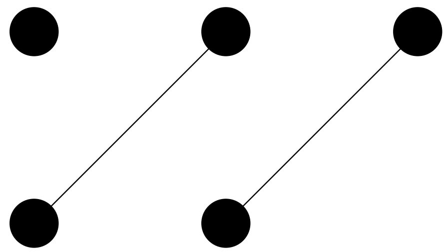

Figure 4 – Vertex Disjoint Edges.

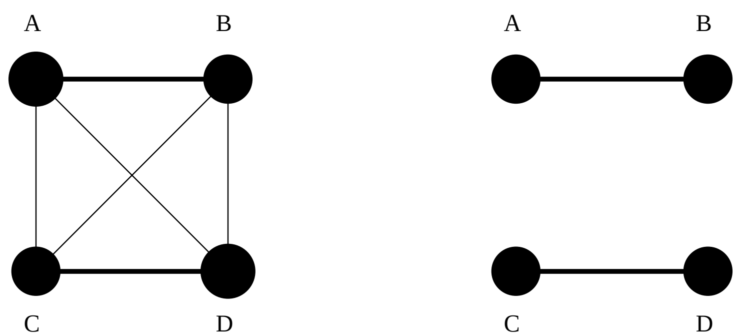

Figure 5 – A Matching.

3. A Useful Lemma

We will let M be the number of edges in, i.e. the size of, a maximum matching. Notice then, that $2M$ is the largest number of vertices used by any matching of a given graph. Since no vertex can be used more than once in a matching, the number of vertices used in a maximum matching must be less than or equal to the total number of vertices in the graph. [1]

Lemma 1. *Let T be the total number of vertices in a graph and let M be the number of edges in a maximum matching of the graph. Then $M \leq \lfloor T/2 \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. That is, the number of edges in a maximum matching of the graph is less than or equal to one half the total number of vertices in the graph.*

Observe that if a matching uses all vertices, or all but one vertex, then no larger matching can be produced, so the matching must be maximal. So when looking for maximum matchings, we will try to find matchings that use all but possibly one vertex.

4. Matchings in Complete Graphs

A complete graph is a graph for which every two vertices in the vertex set have an edge connecting them. Thus a complete graph has all possible edges. Note that a complete graph is just a complete multipartite graph for which every part consists of a single vertex. What is M in a complete graph?

To find a maximum matching, take two vertices of the graph and connect them, then repeat this process for the remaining vertices until either one or no vertices remain. Clearly this produces a maximum matching, because a larger matching would require more vertices than are in the graph. Hence, if there is an even number of vertices in the graph, $M = T/2$ (no unused vertices), and if there is an odd number of vertices in the graph, then only $T - 1$ vertices are used, so $M = (T-1)/2$. Thus $M = \lfloor T/2 \rfloor$.

5. Matchings for Complete Multipartite Graphs

Now consider a maximum matching in a complete multipartite graph with T total vertices and an arbitrary number of parts. (See Figure 6.) The size of the maximum matching has the same upper bound of $\lfloor T/2 \rfloor$. But, can a maximum matching always use all but at most one vertex? To answer this, consider the case of $K_{2,3,7}$. Since no two vertices in the part with 7 vertices can be connected by an edge, all vertices in a maximum matching from the part with 7 vertices must be connected to vertices from the other two parts. However, there are only 5 such vertices, so at least 2 of the vertices from the part with 7 vertices can not be used in any maximum matching. Clearly, $M \leq (T-2)/2 = T/2 - 1 < \lfloor T/2 \rfloor$, and hence some method of calculating M other than searching for a matching using all but possibly one of the vertices is necessary. (See Figure 7.)

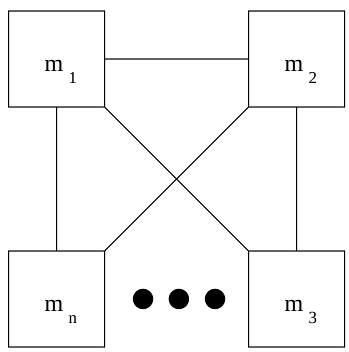

Figure 6 – A Complete Multipartite Graph $K_{m_1,m_2,\ldots,m_n}$.

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Figure 7 – A Maximum Matching for the Trivial Case $K_{2,3,7}$.

As we saw in the previous section, the difficulty with finding M for the complete multipartite graph $K_{2,3,7}$ was that the maximum part had more vertices than all of the other parts combined. Clearly, whenever the maximum part in a complete multipartite graph has more vertices than all the other parts combined, a maximum matching that uses all vertices cannot be found. Note from the example, $K_{2,3,7}$, that the least number of vertices are left unused when all edges in the matching use vertices in the maximum part. Thus, for any complete multipartite graph with more vertices in one part than in all the other parts combined, a maximum matching is obtained by connecting all vertices not in the maximum part to vertices in the maximum part. This matching uses all vertices not in the maximum part, and each of its edges is connected to a vertex in the largest part. Thus, any maximum matching in this graph has as many edges as the total number of vertices not in the maximum part. This case will be called the trivial case.

Theorem 1. Let $K_{m_1,m_2,\ldots,m_n}$ be a complete multipartite graph with m_i vertices in the i^{th} part, labeled so that $m_1 \leq m_2 \leq \cdots \leq m_n$. If $m_n \geq m_1+m_2+\cdots+m_{n-1}$, then:

(i) the number of edges in any maximum matching is $M = m_1+m_2+\cdots+m_{n-1}$;
(ii) a maximum matching is obtained by connecting all vertices in the parts with $m_1, m_2, \cdots, m_{n-1}$ vertices to vertices in the part with m_n vertices.

Now consider the 2-partite (bipartite) case, the complete bipartite graph $K_{m,n}$, where $m \leq n$. This satisfies the trivial case with $M = m$. (See Figure 8.) [1]

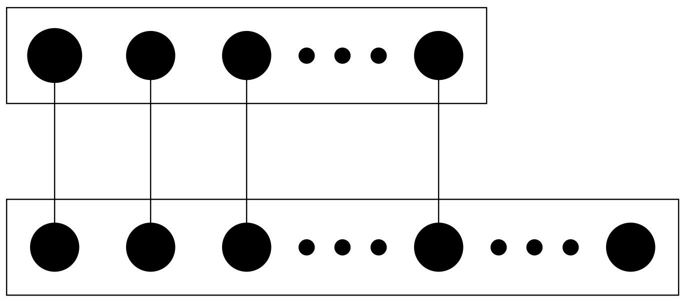

Figure 8 – A Maximum Matching for the Bipartite Case $K_{m,n}$.

6. The Nontrivial Complete Multipartite Graph:

6.1. The 3-Partite Case.

Consider the nontrivial case, the case in which the maximum part has fewer vertices than the smaller parts combined. Note that a matching using all vertices, except possibly one, must be maximal. So we will try to find an algorithm to produce such a matching if possible and if it exists, we know that $M = \lfloor T/2 \rfloor$.

First, consider the nontrivial, complete multipartite graph with 3 parts, $K_{l,m,n}$, with $l \leq m \leq n$ and $n < l + m$. We assume first that there are an even number of vertices in this graph. For convenience, let parts I, II, and III be the parts with l, m, and n vertices, respectively. Now, to produce a matching in the graph we choose $m - l$ vertex disjoint edges between parts II and III. (See Figure 9.)

Now parts I and II both have l vertices available and part III has $n-(m-l)$ vertices available. The subgraph containing only the unmatched vertices, $K_{l,l,n-(m-l)}$, is nontrivial because $n - (m - l) < l + m - (m - l) = l + l = 2l$. Note that since $n - (m - l) = n + m + l - 2m$, we see that part III has an even number of vertices. So we choose $a = \frac{1}{2}(n - (m - l))$ edges between parts I and III; and we choose the same number of edges between parts II and III. (See Figure 10.)

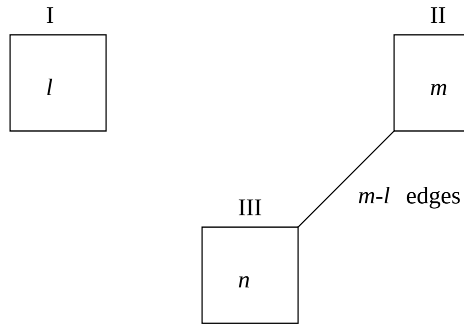

Figure 9 – Construction of a Maximum Matching for a 3-Partite Graph.

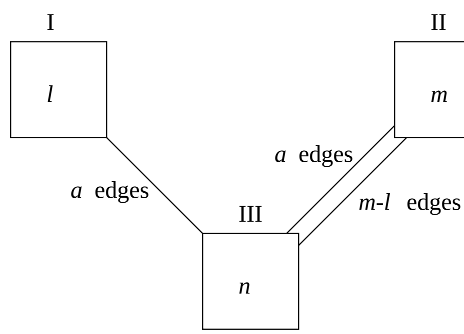

Figure 10 – Constructing a Maximum Matching Step Two: $a = [n - (m - l)]/2$.

Now $m - l$ edges were put in the matching by the first selection procedure, and $n - (m - l)$ edges put in the matching by the second selection procedure. Furthermore, we now have a complete bipartite graph consisting of all available vertices from the original graph that are not yet used in the matching. Furthermore, both parts I and II in the bipartite graph of available vertices and edges have the same number of vertices, $k = l - \frac{1}{2}(n - (m - l))$ vertices. So we choose k edges between parts I and II as we did in the trivial case. Now, all vertices in the graph have been used in the constructed matching, so $M = T/2 = (l + m + n)/2$. (See Figure 11)

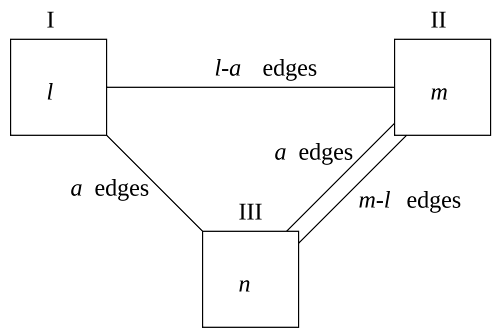

Figure 11 – Constructing a Maximum Matching: Step Three.

Now consider the case when $l+m+n$ is odd. We know $n < m+l$ so $n \le m+(l-1)$. So we choose one vertex from part I to be excluded from the matching. We have a nontrivial complete 3-partite graph with an even number of vertices if $l > 1$, and a complete bipartite graph if $l = 1$. Both of these cases are discussed above. If $n = m + (l-1)$, then we have the trivial case. Otherwise, a maximum matching in a complete 3-partite graph uses all but at most one vertex. Therefore, $M = \lfloor T/2 \rfloor$.

Theorem 2. If $K_{l,m,n}$ is a nontrivial complete 3-partite graph with $l \le m \le n$ (and $n < l+m$), then
(i) $M = \lfloor T/2 \rfloor$; and
(ii) a maximum matching is obtained by the following algorithm:

Step 1: If T is odd, then mark off one vertex in the part with l vertices to be excluded. Let $l' = l$ if T is even and $l' = l-1$ if T is odd.

Step 2: Connect $\frac{1}{2}(l'+m-n)$ vertices from the part with m vertices to the part with l' vertices.

Step 3: Connect all remaining vertices in the parts with m and l' vertices to the n vertices in the remaining part.

We now have produced a maximum matching for any nontrivial complete 3-partite graph. We note that for any maximum matching in a nontrivial, 3-partite graph with an odd number of vertices, the excluded vertex can come from any of the three parts. Furthermore, if there are an even number of vertices, the same number of edges must be used between any two parts in the graph for any maximum matching chosen, because by connecting vertices between the smaller parts, we get a trivial 3–partite subgraph with the maximum part having exactly the same number of vertices as the other two parts combined, expressed as $K_{m,m}$.

6.2. **Four or More Parts.** Now we would like to use the theorem in the previous section to find maximum matchings for all nontrivial, multipartite graphs. We will use the principle of mathematical induction to find such a matching.

Observe that $K_{m,m}$ always has a maximum matching of size m using all vertices. Furthermore, as we saw above, the graph $K_{l,m,n}$ with an even number of vertices has a maximum matching of size $T/2$ using all vertices.

Suppose for a given $n \ge 3$, we know that for any nontrivial, complete multipartite graph with n or $n-1$ parts, $K_{m_1,m_2,\ldots,m_{n-1}}$ or $K_{m_1,m_2,\ldots,m_n}$, the size of a maximum matching is $M = \lfloor T/2 \rfloor$. Now consider an arbitrary nontrivial multipartite graph with $n+1$ parts, $K_{m_1,m_2,\ldots,m_n,m_{n+1}}$ with $m_1 \le m_2 \le \cdots \le m_n \le m_{n+1}$. (See Figure 12.)

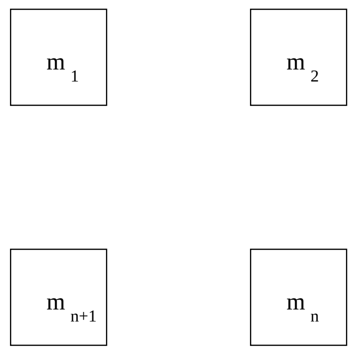

Figure 12 – A Complete Multipartite Graph with $n+1$ Parts.

Since this graph is nontrivial, $m_{n+1} < m_1 + m_2 + \cdots + m_{n-1} + m_n$. Suppose there are even number of vertices in this graph.

Choose m_1 vertex disjoint edges between the parts of size m_1 and m_{n+1}. (See Figure 13.)

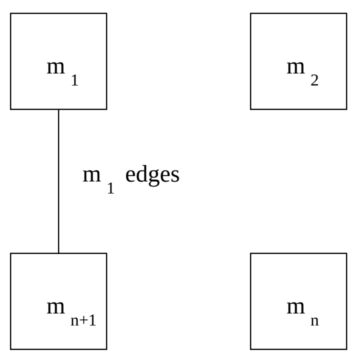

Figure 13 – Constructing a Maximum Matching: Step One.

The remaining vertices correspond to the section of the complete multipartite graph with n parts: $K' = K_{m_2,m_3,\ldots,m_n,m_{n+1}-m_1}$. This graph has either n or $n-1$ parts. Since $n+1 \ge 4$, there must be parts with m_2 and m_3 vertices, and with $m_{n+1} - m_1$ vertices if $m_1 \ne m_{n+1}$, in this graph. It is then sufficient to show that all remaining vertices can be used in the matching. Furthermore, to do this it is only necessary to show that the K' is nontrivial since it has n or $n-1$ parts. But since $\{m_i\}_{i=1}^{n+1}$ is monotonic nondecreasing, the maximum part has either $m_{n+1} - m_1$ or m_n vertices. And, we know that $m_{n+1} < m_1 + m_2 + m_{3+} \cdots + m_n$ subtracting m_1 from both sides yields $m_{n+1} - m_1 < m_2 + m_3 + \cdots + m_n$. So if the maximum part has $m_{n+1} - m_1$ vertices, then K' is nontrivial. Suppose m_n is the largest component. Then there are two cases: (i) $m_{n+1} = m_1$, in which case we are left with a multipartite graph with $n-1$ parts; and (ii) $m_{n+1} \ne m_1$. Consider the first case, $m_{n+1} = m_1$. Then K' is has $n-1$ parts, and $m_{n+1} = m_1$, so $m_1 = m_2 = \cdots = m_n = m_{n+1}$. Hence, the maximum part has m_n

vertices and $m_n = m_2 < m_2 + m_3 + \cdots + m_n$ so K' is nontrivial, since K' must have at least 2 parts, m_2 and m_n. Now consider the second case, $m_{n+1} \neq m_1$ and m_n is the size of the part with the most vertices. Since $m_1 \leq m_2$, clearly $(m_{n+1} - m_1) + m_2 + \cdots + m_{n-1} = m_{n+1} + (m_2 - m_1) + m_3 + \cdots + m_{n-1} \geq m_{n+1} + m_3 + \cdots + m_{n-1} > m_{n+1} \geq m_n$. So if the maximum part has m_n vertices, then K' is a nontrivial complete multipartite graph with either n or $n-1$ parts. By induction, all vertices will be used in any maximum matching of this graph. Furthermore, by making such a matching and adding the m_1 edges between parts of size m_1 and m_{n+1} mentioned above, a maximum matching using all vertices in the original graph is obtained. Hence, any nontrivial complete multipartite graph with an even number of vertices has a maximum matching using all vertices in the graph. Now suppose T is odd. Then choose the vertex to be excluded from the maximum matching to be from the part with m_1 vertices. Since K is nontrivial, and since there are an odd number of vertices, we know that $m_{n+1} < m_1 + m_2 + \cdots + m_n$. Therefore $m_{n+1} \leq (m_1 - 1) + m_2 + m_3 + \cdots + m_n$, so $K' = K_{m_1 - 1, m_2, m_3, \ldots, m_{n+1}}$ is a nontrivial graph, now with an even number of vertices. And since $n \geq 3$, it has either $n+1$ parts, or n parts if $m_1 = 1$. So we can find a maximum matching for K' as we did above. Consequently, we have obtained a maximum matching for K using all but one vertex. Therefore, for any nontrivial complete multipartite graph, we can find a maximum matching using all but at most one vertex. Hence for any nontrivial complete multipartite graph, $M = \lfloor T/2 \rfloor$.

Theorem 3. Given any complete multipartite graph, $K_{m_1, m_2, \ldots, m_n}$, where $m_1 \leq m_2 \leq \cdots \leq m_n$ and $m_n < m_1 + m_2 + \cdots + m_{n-1}$, the following hold:
(i) $M = \lfloor T/2 \rfloor$; and
(ii) the following algorithm produces a matching of size M :

Step 1: If there are an odd number of vertices, remove one vertex from the smallest part and then relabel the parts that still have unused vertices in order of increasing size.

Step 2: If there are exactly two nonempty parts, connect all vertices in one part to vertices in the other and stop.

Step 3: If there are exactly three nonempty parts, follow the procedure outlined in the 3-part theorem, and stop.

Step 4: Connect all vertices from a part with m_1 vertices to the maximum part, and then relabel remaining parts in order of increasing size. Return to Step 2.

6.3. **Alternative proof:** In the proof of this theorem, we have put edges into a matching and looked at graphs with fewer vertices, and fewer parts, until no vertices remained. We will now look at another proof of this theorem, one which just reduces the number of parts in the graph and inserts the edges at the end.

Suppose that given any nontrivial complete multipartite graph with n parts, $M = \lfloor T/2 \rfloor$. Then, suppose we have a graph $K_{m_1, m_2, \ldots, m_n, m_{n+1}}$ with $m_1 \leq m_2 \leq \cdots \leq m_n \leq m_{n+1}$, $m_{n+1} < m_1 + m_2 + \cdots + m_n$, and $n > 3$. Consider the complete multipartite graph with n parts, $K' = K_{m_1 + m_2, m_3, \ldots, m_n, m_{n+1}}$. (See Figure 14.)

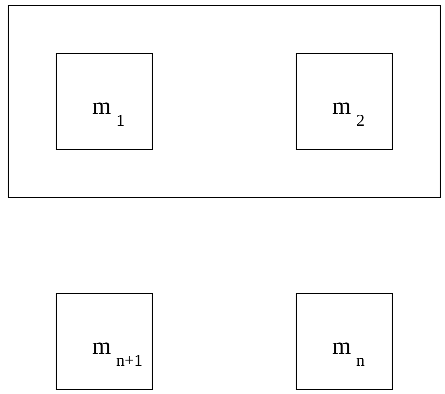

Figure 14 – Constructing a Maximum Matching: Alternative Step One

This clearly is the original $n+1$-partite graph with all edges between m_1 and m_2 removed. Clearly in K' the maximum part is either m_{n+1} or $m_1 + m_2$. Note that $m_{n+1} < m_1 + m_2 + m_3 + \cdots + m_n$. Furthermore, $m_1 \leq m_n$ and $m_2 \leq m_{n+1}$ so since $n > 3$, we know that $m_1 + m_2 \leq m_n + m_{n+1} < m_3 + m_4 + \cdots + m_n + m_{n+1}$. Therefore, the maximum part in the graph $K_{m_1+m_2,m_3,m_4,\ldots,m_n,m_{n+1}}$ has fewer vertices than the sum of the vertices in all of the other parts. Thus we have a nontrivial multipartite graph with n parts. Hence, all vertices, (except one possible odd vertex), in the graph will be used in any maximum matching of K'. Consequently, the same is true for the complete multipartite graph with $n+1$ parts, $K_{m_1,m_2,\ldots,m_n,m_{n+1}}$. Thus we know $M = \lfloor T/2 \rfloor$ for the complete multipartite graph with $n+1$ parts. So, by induction, for any nontrivial complete multipartite graph, all but at most one vertex will be unused in any maximum matching, or $M = \lfloor T/2 \rfloor$.

The importance of this proof is that we get a much more general statement about where edges go in the maximum matching. The first proof says that if all edges in the part with the least vertices are connected to vertices in the maximum part, and this process is repeated, then eventually a nontrivial complete multipartite graph with either 2 or 3 parts is obtained for which we know the arrangement of the edges in a maximum matching. So in essence we know the exact arrangement of the edges for a particular maximum matching of any complete multipartite graph. The second proof shows that the parts can be combined into fewer parts with more vertices per part until we have a nontrivial complete multipartite graph with 3 parts, and then a maximum matching can be produced. So using this second proof we can find many different edge arrangements for a maximum matching in any complete multipartite graph.

7. Combination of Results:

We now consider the multipartite graph $K_{m_1,m_2,\ldots,m_n}$ where $m_1 \leq m_2 \leq \cdots \leq m_n$. If the maximum part is trivially large, $m_n > \sum_{i=1}^{n-1} m_i$, then we see that the size of any maximum matching is $M = \sum_{i=1}^{n-1} m_i = \frac{2}{2}\sum_{i=1}^{n-1} m_i < (m_n + \sum_{i=1}^{n-1} m_i)/2 = (\sum_{i=1}^{n} m_i)/2 = T/2$. Hence in this case, the size of any maximum matching must be $M = \min\{\sum_{i=1}^{n-1} m_i, \lfloor \frac{1}{2}\sum_{i=1}^{n} m_i \rfloor\}$. Suppose instead, that the maximum part is not trivially large, $m_n < \sum_{i=1}^{n-1} m_i$. Then the size of any maximum matching is $M = \lfloor \frac{1}{2}\sum_{i=1}^{n} m_i \rfloor \leq \frac{1}{2}[(\sum_{i=1}^{n-1} m_i) + m_n] < \frac{1}{2}(2\sum_{i=1}^{n-1} m_i) = \sum_{i=1}^{n-1} m_i$. Hence in this case, the size of any maximum matching must be $M = \min\{\sum_{i=1}^{n-1} m_i, \lfloor \frac{1}{2}\sum_{i=1}^{n} m_i \rfloor\}$. Finally, if when the maximum part has exactly the same number of vertices as the total number of vertices in the smaller parts, $m_n = \sum_{i=1}^{n-1} m_i$, then the size of any maximum matching is $M = \frac{1}{2}\sum_{i=1}^{n} m_i = \frac{1}{2}(\sum_{i=1}^{n-1} m_i + m_n) = \frac{1}{2}(2\sum_{i=1}^{n-1} m_{i)} = \sum_{i=1}^{n-1} m_i$. Thus, the size of any maximum matching is $M = \min\{\sum_{i=1}^{n-1} m_i, \lfloor \frac{1}{2}\sum_{i=1}^{n} m_i \rfloor\}$. Since there are no other possibilities for the size of the maximum part in a complete multipartite graph, we get the following:

Theorem 4. Given any complete multipartite graph $K_{m_1,m_2,\ldots,m_n}$, with m_n vertices in the maximum part, the size of a maximum matching is

$$M = \min\{\sum_{i=1}^{n-1} m_i, \lfloor \frac{1}{2}\sum_{i=1}^{n} m_i \rfloor\}.$$

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