Fair reception and Vizing's conjecture

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Abstract

In this paper we introduce the concept of fair reception of a graph which is related to its domination number. We prove that all graphs G with a fair reception of size $\gamma(G)$ satisfy Vizing's conjecture on the domination number of Cartesian product graphs, by which we extend the well-known result of Barcalkin and German concerning decomposable graphs. Combining our concept with a result of Aharoni and Szabó we obtain an alternative proof of the fact that chordal graphs satisfy Vizing's conjecture. A new infinite family of graphs that satisfy Vizing's conjecture is also presented.

Keywords: Cartesian product of graphs; domination

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1 Introduction

More than forty years ago Vizing proposed the following conjecture on the domination number of the Cartesian product of two graphs.

Vizing's Conjecture *For any graphs G and H, $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$.*

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The conjecture was not given great attention in the first years of its existence, while since the 1980's several authors considered it, and showed the truth of the conjecture for some special families of graphs; cf. surveys [6] and [8, Section 8.6], and some more recent results on the topic [3, 4, 9]. One of the oldest partial results is a beautiful, far-reaching theorem of Barcalkin and German [2] which shows that the inequality in the conjecture is true if one of the graphs belongs to the so-called class $\mathcal{A}$ (that can be defined in a rather simple and natural fashion), and the other graph is arbitrary. This result was later extended by Hartnell and Rall to a more general family of graphs [5]. In this note we present a different generalization of the class $\mathcal{A}$ by using the concept of a *fair reception*. In addition, our result yields the following general lower bound on the domination number of Cartesian product of graphs: $\gamma(G\,\Box\,H) \geq \max\{\gamma(G)\gamma_F(H), \gamma_F(G)\gamma(H)\}$, where $\gamma_F(G)$ stands for the *fair domination number* of a graph G that we introduce in this paper.

In the rest of this section we present the most basic definitions. In the next section we first introduce the concept of fair reception and then prove the main theorem, the lower bound for $\gamma(G\,\Box\,H)$. Then we show that this is a generalization of the theorem of Barcalkin and German, present an infinite family of graphs F_k for which $\gamma(F_k) = \gamma_F(F_k)$ and prove that they are not in the class $\mathcal{A}$ from [2]. We follow with a connection to the concept of independence-domination number γ^i of a graph, as introduced by Aharoni and Szabó [1]. We prove that $\gamma_F(G) \geq \gamma^i(G)$ for any graph G which as a by-product yields that chordal graphs satisfy Vizing's conjecture. In the last section we present a possible approach to the conjecture by using a slightly modified fair reception method.

The *Cartesian product* $G\,\Box\,H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. The subgraph of $G\,\Box\,H$ induced by $\{g\} \times V(H)$ is isomorphic to H. It is called an *H-fiber* and is denoted ${}^gH$. Similarly one defines the *G-fiber*, G^h, for a vertex h of H.

For a graph $G = (V(G), E(G))$, a set S is a *dominating set* if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. We say that the set of vertices X from G *externally dominates* set $U \subset V(G)$ if $U \cap X = \emptyset$ and for every $u \in U$ there exists an $x \in X$ such that $ux \in E(G)$.

2 Fair reception

Let $S_1, \ldots, S_k$ be pair-wise disjoint sets of vertices from a graph $G = (V, E)$ with $\mathcal{S} = S_1 \cup S_2 \cup \cdots \cup S_k$, and let $Z = V \setminus \mathcal{S}$. We say that the sets $S_1, \ldots, S_k$ form *a fair reception of size* k if for any integer $\ell, 1 \leq \ell \leq k$, and any choice of ℓ sets $S_{i_1}, \ldots, S_{i_\ell}$ the following holds: if D externally dominates $S_{i_1} \cup \cdots \cup S_{i_\ell}$ then

$$|D \cap Z| + \sum_{j, S_j \cap D \neq \emptyset} (|S_j \cap D| - 1) \geq \ell. \qquad (1)$$

That is, on the left-hand side we count all the vertices of D that are not in $\mathcal{S}$, and for vertices of D that are in some S_j, we count all but one from $D \cap S_j$.

In any graph, any subset of the vertex set forms a fair reception of size 1. Another example is related to the concept of *2-packing* of a graph G which is a set of vertices in G that are pair-wise at distance more than 2. By letting each set S_i consist of exactly one vertex of a 2-packing T we get a fair reception of size $|T|$. (Recall that the *2-packing number* $\rho_2(G)$ of a graph G is the size of a largest 2-packing in G. Hence in any graph G there is a fair reception of size $\rho_2(G)$.)

Given a graph G, the largest k such that there exists a fair reception of size k in G is denoted by $\gamma_F(G)$, and called the *fair domination number of* G. (For instance in C_5 we can put a vertex to S_1 and the antipodal edge to S_2, and obtain a fair reception of size 2. Thus $\gamma_F(C_5) = 2 = \gamma(C_5)$.) We begin with the following basic observation about the fair domination number.

Proposition 1 *For any graph G, $\rho_2(G) \le \gamma_F(G) \le \gamma(G)$.*

Proof. The first inequality has been established above. Suppose there is a graph G such that $r = \gamma(G) < \gamma_F(G) = k$. Let A be a minimum dominating set and assume that the sets $S_1, S_2, \ldots, S_k$ form a fair reception of size k in G. Since $r < k$ the set A must be disjoint from at least one of these sets. Assume that $A \cap S_i = \emptyset$ for $1 \le i \le t$ and that $A \cap S_j \ne \emptyset$ for $t+1 \le j \le k$.

As in our definition let $\mathcal{S} = S_1 \cup S_2 \cup \cdots \cup S_k$ and $Z = V \setminus \mathcal{S}$. The set A externally dominates $S_1 \cup S_2 \cup \cdots \cup S_t$ and so it follows from the definition of fair reception that

$$t \le |A \cap Z| + \sum_{j, S_j \cap A \ne \emptyset} (|S_j \cap A| - 1) = |A \cap Z| + \sum_{j=t+1}^{k} |S_j \cap A| - (k-t) = |A| - k + t\,.$$

This immediately implies that $k \le |A|$, which is a contradiction. □

We continue with the following (our main) result.

Theorem 2 *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be arbitrary graphs. Then*

$$\gamma(G \,\square\, H) \ge \max\{\gamma(G)\gamma_F(H), \gamma_F(G)\gamma(H)\}.$$

Proof. Let D be a minimum dominating set of $G \,\square\, H$. Let $S_1, S_2, \ldots, S_k$ be pair-wise disjoint sets of vertices from H that form a fair reception of H, where $k = \gamma_F(H)$. As above, we denote $\mathcal{S} = S_1 \cup S_2 \cup \cdots S_k$, and $Z = V(H) \setminus \mathcal{S}$. Let $D_i = D \cap (V(G) \times S_i)$ and denote by D_i' the projection of D_i to G. Note that D_i' contains vertices x of G such that $(x, y) \in D$ for some $y \in S_i$, and $|D_i| \ge |D_i'|$. (Note that $|D_i| = |D_i'|$ only if there is exactly one such y for each $x \in D_i'$.) Let $d_i = |D_i| - |D_i'|$ and $d = |(V(G) \times Z) \cap D|$. Moreover, for every $x \in V(G)$ let

$$d_i^x = \begin{cases} |{}^xH \cap D_i| - 1 & , \quad \text{if } |{}^xH \cap D_i| > 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

and

$$d^x = |(\{x\} \times Z) \cap D|.$$

By letting $D^x = D \cap {}^xH$, the sum $d^x + \sum_{i=1}^k d_i^x$ represents the number of vertices in D^x that are not counted within $|D_1'| + \cdots + |D_k'|$.

For each i we denote by

$$T_i = \{x \in V(G) \,:\, x \text{ is not dominated by } D_i'\}.$$

Clearly $D_i' \cup T_i$ is a dominating set of G, hence $|D_i'| + |T_i| = |D_i' \cup T_i| \geq \gamma(G)$. For $x \in V(G)$ let T^x be the set of those integers i from 1 to k for which $x \in T_i$. By the definitions of T^x and T_i, it is clear that

$$\sum_{x \in V(G)} |T^x| = \sum_{i=1}^{k} |T_i|. \tag{2}$$

Note that for $x \in V(G)$, and any $i \in T^x$, the set of vertices $\{(x, y) \,:\, y \in S_i\}$ is not dominated by D_i. Hence all these vertices must be (externally) dominated by the vertices from D^x. Since ${}^xH$ is isomorphic to H this means that the projection $p_H(D^x)$ of D^x onto H externally dominates the sets S_i for all $i \in T^x$. Since $S_1, S_2, \ldots, S_k$ form a fair reception of H, we have

$$d^x + \sum_{i=1}^{k} d_i^x \geq |T^x|. \tag{3}$$

Now, we infer

$$\begin{aligned}
|D| = \sum_{i=1}^{k} |D_i| + d &= \sum_{i=1}^{k} (|D_i'| + d_i) + d \\
&= \sum_{i=1}^{k} |D_i'| + \sum_{i=1}^{k} \sum_{x \in V(G)} d_i^x + d \\
&= \sum_{i=1}^{k} |D_i'| + \sum_{x \in V(G)} (\sum_{i=1}^{k} d_i^x + d^x) \\
&\geq \sum_{i=1}^{k} |D_i'| + \sum_{x \in V(G)} |T^x| \qquad \text{(by (3))} \\
&= \sum_{i=1}^{k} |D_i'| + \sum_{i=1}^{k} |T_i| \qquad \text{(by (2))} \\
&\geq k\gamma(G) = \gamma_F(H)\gamma(G)\,.
\end{aligned}$$

An observation that we may reverse the roles of G and H concludes the proof. $\square$

Corollary 3 *If G is a graph with $\gamma_F(G) = \gamma(G)$, then G satisfies Vizing's conjecture.*

Barcalkin and German [2] introduced the class of so-called *decomposable graphs*, and showed that they satisfy Vizing's conjecture. Recall that a graph is *decomposable* if its vertex set can be partitioned into $\gamma(G)$ subsets, each of which induces a complete subgraph (we can also say that G partitions into $\gamma(G)$ cliques). The following fact is easy to see.

Proposition 4 *Let G be a decomposable graph. Then a partition of the vertex set into $\gamma(G)$ cliques yields a fair reception of G of size $\gamma(G)$ (in which $\mathcal{S}$ equals $V(G)$).*

Note that a family of sets that form a fair reception in a graph G also form a fair reception in any spanning subgraph of G. Hence by the above proposition, the class of graphs G with $\gamma(G) = \gamma_F(G)$ contains the class $\mathcal{A}$ (of graphs that can be realized as spanning subgraphs of decomposable graphs with the same domination number). Thus, Theorem 2 is a generalization of the result by Barcalkin and German [2]. To see that this generalization is not just an equivalent statement, we will present a class of graphs that are not in $\mathcal{A}$, yet $\gamma(G) = \gamma_F(G)$ holds for these graphs.

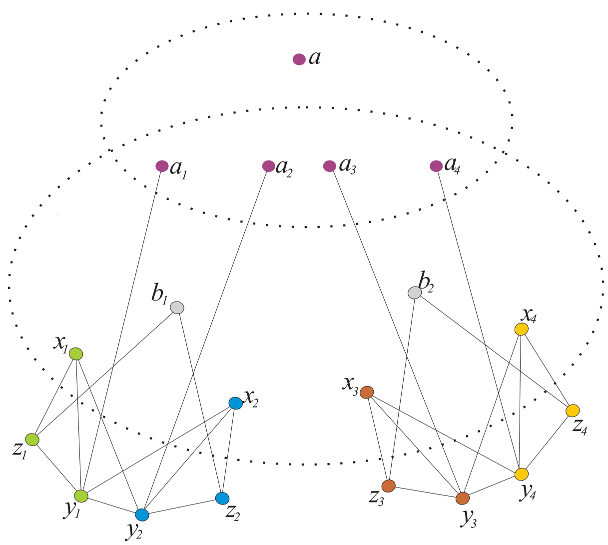

Figure 1: Graph F_2

Let F_k be the graph with

$$V(F_k) = \{a, a_1, a_2, \ldots, a_{2k}, b_1, b_2, \ldots, b_k, x_1, x_2, \ldots, x_{2k}, y_1, y_2, \ldots, y_{2k}, z_1, z_2, \ldots, z_{2k}\},$$

and the edge set defined as follows. Vertices $a, a_1, a_2, \ldots, a_{2k}$ form a clique, and vertices $a_1, a_2, \ldots, a_{2k}, b_1, b_2, \ldots, b_k, x_1, x_2, \ldots, x_{2k}$ form another clique; for $i = 1, \ldots, k$

we have: $x_{2i-1}, x_{2i}, y_{2i-1}$ and y_{2i} form a clique, and b_i is adjacent to z_{2i-1} and to z_{2i}; finally for $i = 1, \ldots, 2k$ we have that a_i is adjacent to y_i, z_i is adjacent to y_i, and x_i is adjacent to z_i. See Figure 1 where F_2 is depicted (the first two cliques are marked by dotted circles, to avoid the drawing with too many edges).

We will now prove that for any k, the graph F_k satisfies each of the following claims. For the sake of simplified notation, let

$$E' = \{a_i y_j : j \notin \{i, i+1\} \text{ if } i \text{ is odd, or } j \notin \{i-1, i\} \text{ if } i \text{ is even}\},$$

a set of edges from $\overline{F_k}$.

1. $\rho_2(F_k) = k + 1$;

2. $\gamma(F_k) = 2k + 1$;

3. $\gamma_F(F_k) = 2k + 1$;

4. For any edge $e \in E(\overline{F_k}) \setminus E'$, $\gamma(F_k + e) < \gamma(F_k)$;

5. F_k is not in class $\mathcal{A}$;

To prove these we will make use of the following notation. For each $1 \leq i \leq 2k$, let $S_i = \{x_i, y_i, z_i\}$, let $S_{2k+1} = \{a, a_1, a_2, \ldots, a_{2k}\}$, and for $1 \leq j \leq k$ we set $C_j = S_{2j-1} \cup S_{2j} \cup \{b_j\}$. Sometimes one of these names will denote the subgraph induced by this set of vertices and other times it will refer only to the set of vertices. The meaning will be clear from the context. In addition, let $B = \{b_1, b_2, \ldots, b_k\}$, $X = \{x_1, x_2, \ldots, x_{2k}\}$, $Y = \{y_1, y_2, \ldots, y_{2k}\}$ and $Z = \{z_1, z_2, \ldots, z_{2k}\}$.

To prove Claim 1 we note that each C_j has diameter two, S_{2k+1} is complete, and $V(F_k) = S_{2k+1} \cup C_1 \cup \cdots \cup C_k$. By using the pigeon hole principle it now follows that $\{a\} \cup \{z_2, z_4, \ldots, z_{2k}\}$ is a maximum 2-packing and hence $\rho_2(F_k) = k + 1$.

Let D be a minimum dominating set of F_k. For every j, $1 \leq j \leq k$, $D \cap C_j \neq \emptyset$ since $N[z_{2j}] \subseteq C_j$. Also, because $N[a] \subseteq S_{2k+1}$ it follows that $D \cap S_{2k+1} \neq \emptyset$. Suppose that $|D \cap C_j| = 1$ for some j. By considering the various possibilities for this intersection, it is clear that $D \cap C_j = \{b_j\}$. But now it follows that both of a_{2j-1} and a_{2j} belong to D in order to dominate y_{2j-1} and y_{2j}. Let $I = \{i : 1 \leq i \leq k, |D \cap C_i| = 1\}$ and let $J = \{1, 2, \ldots, k\} \setminus I$. If $|I| = 0$, then $|D| \geq 2k + 1$. Otherwise $|D| = 3|I| + 2|J| = 2k + |I|$, and this is clearly minimized when $|I| = 1$. Since $Z \cup \{a\}$ dominates F_k, it now is clear that $\gamma(F_k) = 2k + 1$. Therefore Claim 2 is proved.

We will now show that $S_1, S_2, \ldots, S_{2k+1}$ form a fair reception of size $2k + 1$. For this purpose let $\mathcal{S} = S_1 \cup S_2 \cup \cdots \cup S_{2k+1}$. Let $\mathcal{T} = \{S_{i_1}, S_{i_2}, \ldots, S_{i_\ell}\}$ be a subcollection of these sets and suppose that D is an external dominating set of $\cup_{j=1}^{\ell} S_{i_j}$. It is clear that $2k+1 \neq i_j$ for any j since the vertex a cannot be dominated from outside S_{2k+1}. For each j, $1 \leq j \leq \ell$, clearly $b_{\lceil \frac{i_j}{2} \rceil} \in D$ to dominate z_{i_j}. Let t be odd. If $S_t, S_{t+1} \in \mathcal{T}$, then $b_{\frac{t+1}{2}} \in D$ and $a_t, a_{t+1} \in D$. If only one of S_t and S_{t+1}

belongs to $\mathcal{T}$ (say S_t), then $b_{\frac{t+1}{2}} \in D$ and at least one of x_{t+1}, y_{t+1}, a_t also belongs to D. It now follows that

$$|D \cap (V(F_k) \setminus \mathcal{S})| + \sum_{j, S_j \cap D \neq \emptyset} (|S_j \cap D| - 1) \geq \ell,$$

and hence $S_1, S_2, \ldots, S_{2k+1}$ form a fair reception of size $2k+1$ in the graph F_k. Therefore, $\gamma_F(F_k) = 2k+1$.

To prove that $\gamma(F_k + e) < \gamma(F_k)$ for each edge $e \in E(\overline{F_k}) \setminus E'$ we provide a table that gives a dominating set of $F_k + e$ of order $2k$ for each of the possible types of edges e. To make the notation tractable we make some simplifying assumptions that are completely general. For example, if an edge of the form az_i is being considered, we may assume $i = 1$.

e	Dominating set of $F_k + e$	e	Dominating set of $F_k + e$
ax_1	$\{x_1, x_2, \ldots, x_{2k}\}$	ab_1	$\{b_1, x_2, \ldots, x_{2k}\}$
ay_1	$\{y_1, x_2, \ldots, x_{2k}\}$	az_1	$\{z_1, x_2, \ldots, x_{2k}\}$
a_1y_2	$\{a_1, b_1, x_3, \ldots, x_{2k}\}$	a_1z_1	$\{a_1, x_2, \ldots, x_{2k}\}$
a_1z_2	$\{a_1, x_1, x_3, \ldots, x_{2k}\}$	x_1y_3	$\{x_1, x_2, b_2, a_4, x_5, x_6, \ldots, x_{2k}\}$
x_1z_2	$\{a, x_1, x_3, \ldots, x_{2k}\}$	x_1z_3	$\{a, x_1, x_2, x_4, \ldots, x_{2k}\}$
b_1y_1	$\{b_1, a_2, x_3, \ldots, x_{2k}\}$	b_1y_3	$\{b_1, b_2, a_4, x_2, x_5, \ldots, x_{2k}\}$
b_1z_3	$\{b_1, x_2, a_3, x_4, x_5, \ldots, x_{2k}\}$	y_1y_3	$\{y_1, x_2, b_2, a_4, x_5, \ldots, x_{2k}\}$
y_1z_2	$\{y_1, a, x_3, x_4, \ldots, x_{2k}\}$	y_1z_3	$\{a, y_1, x_2, x_4, x_5, \ldots, x_{2k}\}$
z_1z_2	$\{z_1, a_2, x_3, x_4, \ldots, x_{2k}\}$	z_1z_3	$\{z_1, x_2, a_3, x_4, x_5, \ldots, x_{2k}\}$

By Claim 4, to prove that F_k does not belong to the class $\mathcal{A}$ it is sufficient to prove that for any subset $E'' \subset E'$ such that $\gamma(F_k + E'') = 2k+1 = \gamma(F_k)$, it is not possible to partition $V(F_k) = V(F_k + E'')$ into $2k+1$ complete subgraphs of $F_k + E''$. Let E'' be such a subset of edges. Note that E'' does not contain any set of edges of the form $\{a_iy_j, a_iy_{j+1}\}$ for j odd, since $\gamma(F_k + E'') = 2k+1$. This implies that if C is a complete subgraph of $F_k + E''$, then either C is already complete in F_k or C contains at most one vertex from any set of the form $\{z_j, y_j, z_{j+1}, y_{j+1}\}$ for j odd. By definition it is easy to see that at least $2k$ complete subgraphs are needed to cover Z, and none of these $2k$ complete subgraphs contains the vertex a. Also, if one of these cliques intersects B (at, say b_i), then two additional cliques will be required to cover C_i. Therefore, because $N[a] = S_{2k+1}$ it now follows that at least $2k+2$ complete subgraphs of $F_k + E''$ will be needed in any clique partition of $V(F_k + E'')$, and Claim 5 is established.

We now show that the graphs F_k for $k > 1$ do not belong to the class of graphs introduced by Hartnell and Rall in [5]. Hence the graphs G with $\gamma_F(G) = \gamma(G)$ present a generalization of the class $\mathcal{A}$ that is distinct from class $\mathcal{X}$. Since the description of $\mathcal{X}$ is rather lengthy, we refer the reader to its definition on page 211 in [5].

Proposition 5 *For any integer k, $k > 1$, the graph F_k is not a spanning subgraph of a graph G from class $\mathcal{X}$ such that $\gamma(F_k) = \gamma(G)$.*

Proof. We prove the statement for $k = 2$; see Figure 1. The general proof is similar but is notationally complex. Assume for the sake of contradiction that there is such a graph $G \in \mathcal{X}$. Because of Claim 4, we may assume, as in our proof of Claim 5 above, that $G = F_2 + E''$ for some subset E'' of $\{a_1y_3, a_1y_4, a_2y_3, a_2y_4, a_3y_1, a_3y_2, a_4y_1, a_4y_2\}$, subject to the requirement that $\gamma(G) = 5 = \gamma(F_2)$.

The graph G has only one simplicial vertex, namely a, and hence G has at most one buffer clique. Since G is not decomposable, there must be at least one "star-like" subgraph in the partition of $V(G)$. We denote these by $T_1, \ldots, T_r$ to avoid confusion with earlier notation. Each T_i has a center vertex, t_i, all of whose neighbors belong to T_i. By considering each of the vertices of G, one can show that the set of centers of these star-like subgraphs is a subset of one of $\{z_1, z_3\}, \{z_1, y_3\}, \{y_1, z_3\}$ or of an obvious symmetric counterpart. (For example, $\{z_2, y_3\}$ could be a set of such centers.) If u is a vertex of a star-like subgraph T_i, $u \neq t_i$, then by definition each neighbor of u that is not in T_i belongs either to a buffer clique or to a special clique. It is now straightforward to check that any choice of these centers forces G to have more than one special clique. This contradicts the structure of a graph in $\mathcal{X}$ and thus completes the proof. □

It is still not clear whether every graph G from the class $\mathcal{X}$ admits a fair reception of size $\gamma(G)$, but we think that this is not the case.

The fair domination number of a graph is related to another graph invariant that was proposed by Aharoni and Szabó [1] as follows. Given a graph G and an independent set I of vertices in G, the least size of a set of vertices in G that dominates I is denoted by $\gamma_I(G)$, and by $\gamma^i(G)$ we denote the largest $\gamma_I(G)$ over all independent sets I in G. The invariant γ^i was used in the proof that chordal graphs satisfy Vizing's conjecture. In particular, it was shown that for any chordal graph $\gamma^i(G) = \gamma(G)$ [1]. We establish the following relation with our concept.

Proposition 6 *For an arbitrary graph G, $\gamma_F(G) \geq \gamma^i(G)$.*

Proof. Note that for any independent set I there exists a set S of size $\gamma_I(G)$ that externally dominates I (that is $S \cap I = \emptyset$). Let I be an independent set of vertices in a graph G with $\gamma_I(G) = \gamma^i(G)$, and let $S = \{x_1, \ldots, x_k\}$ be a set of vertices of size $k = \gamma^i(G)$ that externally dominates I. Let the sets $S_1, \ldots, S_k$ be a partition of I such that $S_i \subset N(x_i)$ (where $N(x_i)$ is the neighborhood of x_i). We claim that $S_1, \ldots, S_k$ form a fair reception in G. Indeed, to (externally) dominate any subfamily of ℓ sets S_i, one needs at least ℓ vertices, otherwise we easily infer that $\gamma_I(G) < k$ which is a contradiction. Hence $\gamma_F(G) \geq k = \gamma^i(G)$. □

Combining Propositions 1 and 6 we obtain the following chain of inequalities for an arbitrary chordal graph G:

$$\gamma^i(G) \leq \gamma_F(G) \leq \gamma(G) = \gamma^i(G).$$

Hence $\gamma_i(G) = \gamma_F(G) = \gamma(G)$ and by Theorem 2 we infer

Corollary 7 [1] *If G is a chordal graph and H an arbitrary graph then $\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H)$.*

We conclude this section with the following natural question.

Problem 1 *Is there a general lower bound for $\gamma_F(G)$ in terms of a function of $\gamma(G)$? Say $\gamma_F(G) \geq \gamma(G) - 1$?*

3 Concluding remarks

Although we did not find an example of a graph for which it would be easy to prove that $\gamma_F(G) < \gamma(G)$, we believe there are such graphs. For instance, graph G_1 in Figure 2 seems to be a good candidate. This graph was used in [5] as an example from the class $\mathcal{X}$, hence Vizing's conjecture is known to hold for this graph. We shall now present another proof of this fact which is based on a partition similar to the fair reception.

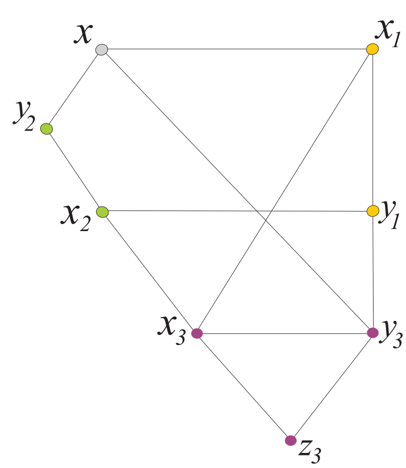

Figure 2: Almost fair-reception of graph G_1

Consider the partition of the vertex set as seen in Figure 2. It is formed by the sets $S_1 = \{x_1, y_1\}, S_2 = \{x_2, y_2\}$, and $S_3 = \{x_3, y_3, z_3\}$. The sets do not form a fair reception because S_1 can be externally dominated by $x_2 \in S_2$ and $x_3 \in S_3$, and so the left-hand side of inequality (3) is equal to 0 in this case. One can readily check (by looking at each combination of subsets from $\{S_1, S_2, S_3\}$) that this is the only such case in which the condition from (3) is not fulfilled. Without formally defining it, it seems natural to call this kind of partition an *almost fair reception*.

The proof of Theorem 2 is based in part on the fact that the union of the set of vertices D_i' that project to G from D_i and the set $T_i = V(G) \setminus (\cup_{v \in D_i'} N[v])$

(corresponding to the set of cells $T_i \times S_i$ that are missed by D_i) form a dominating set. This is clearly a very strong requirement which may be needed only in the case when T_i is a 2-packing. One might weaken this condition by taking instead of T_i just some subset T_i' of G that dominates T_i, and still we have $|D_i'| + |T_i'| \geq \gamma(G)$.

Consider now the Cartesian product $G \Box G_1$ of an arbitrary graph G with the graph G_1 (see also Figure 3). From the structure of G_1 in which $z_3 \in S_3$ has no neighbors outside S_3 we get $T_3 = \emptyset$. Next, for any vertex $a \in T_2$ the vertex (a, x) must be in D to ensure that the cell $\{a\} \times S_2$, missed by D_2, is dominated. Also, if $a \in T_1 \cap T_2$ then $(a, x) \in D$, and moreover $|(\{a\} \times S_3) \cap D| \geq 2$. However, for $b \in T_1 \setminus T_2$ the cell $\{b\} \times S_1$ that is missed by D_1 could be dominated by (b, x_2) and (b, x_3), while (b, x) need not be in D. For all such cells the vertex (b, x) is dominated by (b, y_2) or (b, y_3) or it is dominated from the G^x-fiber. The former two cases imply $|(\{b\} \times S_2) \cap D| = 2$, or $|(\{b\} \times S_3) \cap D| \geq 2$, respectively, and one of the "additional" vertices can be used in the recount for the missing cells of T_1 (let us denote the set of such vertices b by A_1). The latter case, that (b, x) is dominated from the G^x-fiber, needs some more investigation. It is clear that b cannot be adjacent to an $a \in T_2$, since then $(b, x_2) \in D$ would dominate a vertex from the cell $\{a\} \times S_2$, a contradiction. Thus, those $(a, x) \in D$ that are used in the recount for the missing cells of T_2 do not dominate (b, x), where $b \in T_1 \setminus T_2$. Let T_1' be a set of vertices $c \in V(G) \setminus (T_2 \setminus T_1)$ such that vertices $(c, x) \in D$ dominate all vertices (b, x) where $b \in T_1 \setminus A_1$. It is clear that $|D_1'| + |T_1'| + |A_1| \geq \gamma(G)$, and vertices from T_1' and A_1 have not been counted within $|D_2'|$ or $|D_3'|$ nor have been used in the recount for the missing cells of T_2. This shows that Vizing's conjecture holds for graph G_1.

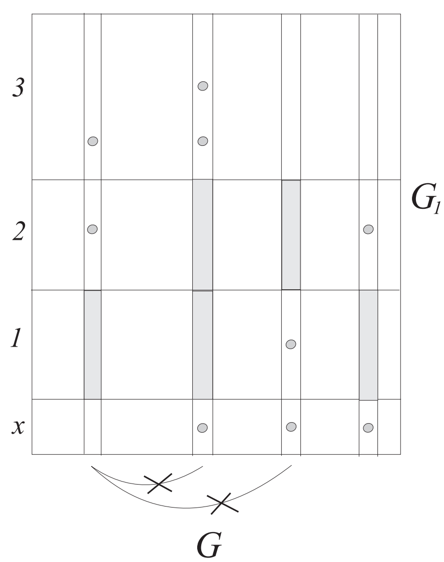

Figure 3: Cartesian product with graph G_1.

We think that by a similar but more tedious arguing one can prove that all graphs from the class $\mathcal{X}$ satisfy Vizing's conjecture. However, the desire is to use a similar method for other classes of graphs, for which the conjecture is still open.

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