VIZING'S CONJECTURE: A SURVEY AND RECENT RESULTS

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ABSTRACT. Vizing's conjecture from 1968 asserts that the domination number of the Cartesian product of two graphs is at least as large as the product of their domination numbers. In this paper we survey the approaches to this central conjecture from domination theory and give some new results along the way. For instance, several new properties of a minimal counterexample to the conjecture are obtained and a lower bound for the domination number is proved for products of claw-free graphs with arbitrary graphs. Open problems, questions and related conjectures are discussed throughout the paper.

Keywords: *Cartesian product; domination; Vizing's conjecture*

1. Introduction

Vertex connectivity, matching number, chromatic number, crossing number, genus, and independence number are but a few examples of graph invariants. An important problem to be solved in understanding a graph invariant is "how it behaves" on graph products. Because of how the product relates to the two factors, it seems reasonable to think that the value of the invariant on the product of two graphs G and H will, in some consistent way, relate to its value—and perhaps that of other invariants—on G and H. In 1996 Nowakowski and Rall [38] explored this relationship for twelve independence, coloring and domination invariants on ten associative graph products whose edge structure depends on that of both factors.

For some invariants and products, this relationship is known and easy to verify. An example of this situation is that the chromatic number of the Cartesian product of two graphs is the maximum of their chromatic numbers. In some cases, for example the independence number of the direct product, there are proven bounds, but in general no exact formula is known in terms of the independence numbers of the two factor graphs. For still other invariants the invariant has a conjectured behavior, but the issue is far from being settled. This is the situation for the domination number on a Cartesian product. The following conjecture was made by V. G. Vizing in 1968, after being posed by him as a problem in [42]:

Conjecture 1.1. ([43]) *For every pair of finite graphs G and H,*

$$\gamma(G \,\Box\, H) \geq \gamma(G)\,\gamma(H). \qquad (1)$$

As usual, γ stands for the domination number, and $G \Box H$ stands for the Cartesian product of graphs G and H.

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Vizing's conjecture is arguably the main open problem in the area of domination theory. In this paper we present a survey of what is known about attacks on Vizing's conjecture and give some new results. We say that a graph G *satisfies Vizing's conjecture* if inequality (1) holds for every graph H.

The most successful attack in proving that the conjecture holds in special cases has been the idea of partitioning a graph into subgraphs of a particular type, as initiated by Barcalkin and German [4]. Their approach gives a large class of graphs, we call them BG-graphs, which satisfy Vizing's conjecture. Their fundamental contribution, its consequences, and independent rediscoveries are presented in Section 2.

The class of BG-graphs has been expanded in two different ways. The first class is the one of Type $\mathcal{X}$ graphs as introduced in [25]; the second was recently proposed in [10]. These two classes are presented in Sections 3 and 4, respectively. The second approach implies, among others, that chordal graphs satisfy Vizing's conjecture, a result first proved in [2].

In 2000, Clark and Suen [15] made a breakthrough by proving, using what we call the double projection approach, that $\gamma(G \Box H) \geq \frac{1}{2}\gamma(G)\,\gamma(H)$ for all graphs G and H. Aharoni and Szabó [2] applied the approach to settle the conjecture for chordal graphs. In this paper we use the double projection approach to prove that for a claw-free graph G and any graph H without isolated vertices, $\gamma(G \Box H) \geq \frac{1}{2}\alpha(G)(\gamma(H)+1)$, where α stands for the independence number. (Note that $\alpha(G) \geq \gamma(G)$ holds for any graph G.) Moreover, if H has large enough order, this bound can be improved. The double projection approach is treated in Section 5.

We follow with a section on possible minimal counterexamples to the conjecture. Among other properties we prove that a minimal counterexample G is edge-critical, that every vertex of G belongs to a minimum dominating set, and that the domination number decreases in any graph formed by identifying arbitrary vertices u and v of G.

Then, in Section 7, several additional approaches to the conjecture are briefly described, while Section 8 gives Vizing-type theorems for related domination invariants. For instance, a version of Vizing's conjecture is true for the fractional domination number. We conclude the paper with several stronger and weaker conjectures than Vizing's conjecture.

In the rest of this section definitions are given. Let $G = (V(G), E(G))$ be a finite, simple graph. For subsets of vertices, A and B, we say that B *dominates* A if $A \subseteq N[B]$; that is, if each vertex of A is in B or is adjacent to some vertex of B. When A and B are disjoint and B dominates A, then we will say that B *externally dominates* A. The *domination number* of G is the smallest cardinality, denoted $\gamma(G)$, of a set that dominates $V(G)$. If D dominates $V(G)$, we will also say that D dominates the graph G and that D is a *dominating set* of G. For a survey on domination, see for example [30].

Any dominating set of G must intersect every closed neighborhood in G. Thus, the domination number of G is at least as large as the cardinality of any set $X \subseteq V(G)$ having the property that for distinct x_1 and x_2 in X, $N[x_1] \cap N[x_2] = \emptyset$. Such a set X is called a 2-*packing*, and the maximum cardinality of a 2-packing in G is denoted $\rho(G)$ and is called the 2-*packing number* of G. The smallest cardinality of a dominating set that is also independent is denoted $i(G)$, and the *vertex independence number* of G is the maximum cardinality, $\alpha(G)$, of an independent set of vertices in G. For convenience of notation we will write $|G|$ to denote the number of vertices in G, and $g \in G$ to mean that g is a vertex of G.

If G is not a complete graph, then for any pair of vertices g_1 and g_2 that are not adjacent in G, it is clear that $\gamma(G) - 1 \le \gamma(G + g_1g_2) \le \gamma(G)$. If G has the property that $\gamma(G) - 1 = \gamma(G + g_1g_2)$ for every such pair of nonadjacent vertices, then G is *edge-critical with respect to domination* (or *edge-critical* for brevity). We will see in Observation 2.1 that to prove Vizing's conjecture, it suffices to show the inequality is true whenever one of the two graphs is edge-critical.

The *Cartesian product* $G \,\Box\, H$ of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \,\Box\, H$ if either $g_1 = g_2$ and h_1h_2 is an edge in H or $h_1 = h_2$ and g_1g_2 is an edge in G. For a vertex g of G, the subgraph of $G \,\Box\, H$ induced by the set $\{\, (g, h) \mid h \in H \,\}$ is called an *H-fiber* and is denoted by ${}^gH$. Similarly, for $h \in H$, the *G-fiber*, G^h, is the subgraph induced by $\{\, (g, h) \mid g \in G \,\}$. We will have occasion to use the fiber notation G^h and ${}^gH$ to refer instead to the set of vertices in these subgraphs; the meaning will be clear from the context. It is clear that all G-fibers are isomorphic to G and all H-fibers are isomorphic to H.

We will have need of projection maps from the Cartesian product $G \,\Box\, H$ to one of the factors G or H or to a fiber. The *projection to H* is the map $p_H : V(G \,\Box\, H) \to V(H)$ defined by $p_H(g, h) = h$. For a specified vertex x of G, the projection to the H-fiber, ${}^xH$, is the function that maps a vertex (g, h) to (x, h). Projections to G or to a G-fiber have the obvious meaning.

2. Decomposable Graphs

One of the first results that shows the truth of Vizing's conjecture for a class of graphs is due to Barcalkin and German [4]. Their theorem about the so-called *decomposable graphs* is still one of the nicest partial results on the conjecture.

Let G be a graph with domination number k. If the vertex set of G can be covered by k complete subgraphs (cliques, for short), then G is called a *decomposable graph*. Barcalkin and German proved that every decomposable graph satisfies Vizing's conjecture. They also noticed that the class of graphs that satisfy Vizing's conjecture can always be extended by using the following basic fact.

Observation 2.1. *Let G be a graph that satisfies Vizing's conjecture, and let G' be a spanning subgraph of G such that $\gamma(G') = \gamma(G)$. Then G' also satisfies Vizing's conjecture.*

Indeed, if H is any graph, then

$$\gamma(G')\gamma(H) = \gamma(G)\gamma(H) \le \gamma(G \,\Box\, H) \le \gamma(G' \,\Box\, H)\,.$$

The last inequality holds since $G' \,\Box\, H$ is a spanning subgraph of $G \,\Box\, H$.

Hence the theorem of Barcalkin and German states:

Theorem 2.2. ([4]) *If a graph G is a spanning subgraph of a decomposable graph G' such that $\gamma(G) = \gamma(G')$, then for every graph H, $\gamma(G \,\Box\, H) \ge \gamma(G)\gamma(H)$.*

Theorem 2.2 is not difficult to prove if one observes a nice feature of the partition of decomposable graphs related to the external domination of cliques.

Let G be a decomposable graph with $\gamma(G) = k$, and let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be a partition of $V(G)$ into cliques. Suppose that one wants to externally dominate a union of $\ell < k$ cliques from $\mathcal{C}$; that is, given $C_{i_1}, \ldots, C_{i_\ell} \in \mathcal{C}$ one looks for a (smallest) set of vertices in $G - (C_{i_1} \cup \cdots \cup C_{i_\ell})$ that dominates $C_{i_1} \cup \cdots \cup C_{i_\ell}$. Select and fix such a set D, and let $C_{j_1}, \ldots, C_{j_t}$ be the cliques from $\mathcal{C}$ that have a nonempty intersection with D. Then we claim that

$$\sum_{m=1}^{t} (|C_{j_m} \cap D| - 1) \geq \ell. \tag{2}$$

In words, the sum of the number of "additional" vertices (that is, the surplus to 1 of the number of vertices from D) in cliques C_{j_m} is at least the number of cliques that are externally dominated. Indeed, if the left-hand side of inequality (2) were less than ℓ, then the $\ell + t$ cliques from $\mathcal{C}$ would be dominated by fewer than $\ell + t$ vertices. Since the remaining cliques can be dominated by $k - (\ell + t)$ vertices (each clique by one of its vertices), this implies the contradiction $\gamma(G) < k$.

Proof of Theorem 2.2. By Observation 2.1 we may assume that G is a decomposable graph. Consider the Cartesian product $G \,\square\, H$, where G is a decomposable graph with $\gamma(G) = k$. Let D be a minimum dominating set of $G \,\square\, H$. The main idea of the proof is that each vertex from D will get a label from 1 to k, and for each label i, the projection to H of the vertices from D that are labeled i, is a dominating set of H. This clearly implies $|D| \geq k\,\gamma(H)$, and G satisfies the conjecture.

Let $C_1, \ldots, C_k$ be a partition of $V(G)$ into cliques. For each $h \in V(H)$ and i, $1 \leq i \leq k$, we call $C_i^h = V(C_i) \times \{h\}$ a *G-cell*. (See Figure 1 where the corresponding partition of $G \,\square\, H$ is shown, and the cell C_i^h is shaded.)

The following *simple labeling rule (SLR)* is used: if a G-cell C_i^h contains a vertex from D, then one of the vertices from $D \cap C_i^h$ gets the label i. Hence in the projection, h will also get the label i. Note that we have not yet determined the remaining labels if there is more than one vertex in $D \cap C_i^h$.

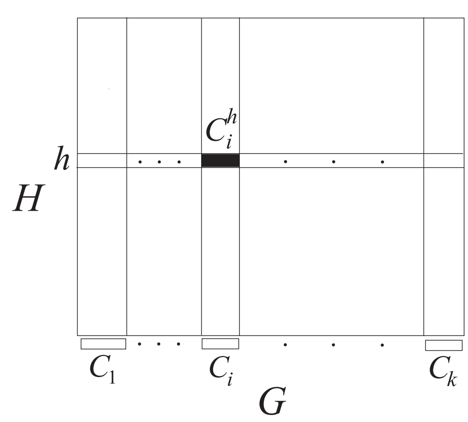

FIGURE 1. The partition of $G \,\square\, H$ into G-cells

Fix an arbitrary vertex $h \in V(H)$. We need to prove that for an arbitrary i, $1 \le i \le k$, there exists a vertex from D, labeled by i, that is projected to the neighborhood of h. There are two cases. First, if there exists a vertex of D in $V(C_i) \times N[h]$, then by SLR, there will be a vertex in the neighborhood of h to which the label i is projected, and so this case is settled.

The second case is that there is no vertex from D in $V(C_i) \times N[h]$, and we call such C_i^h a *missing G-cell* for h. Let $C_{i_1}^h, \cdots, C_{i_\ell}^h$ be the missing G-cells for h. Note that, by definition, the missing G-cells for h are dominated entirely within the G-fiber, G^h. Hence there are vertices in $D \cap G^h$ that dominate $C_{i_1}^h \cup \cdots \cup C_{i_\ell}^h$. Let $C_{j_1}^h, \ldots, C_{j_t}^h$ be the G-cells in G^h that intersect D. Since G^h is isomorphic to G, we infer by inequality (2) that

$$\sum_{m=1}^{t} (|C_{j_m}^h \cap D| - 1) \ge \ell.$$

Thus there are enough additional vertices in $D \cap G^h$ (that have not already been labeled by SLR), so that for each missing G-cell C_i^h, the label i can be given to one of the vertices in $C_{j_m}^h \cap D$, where $|C_{j_m}^h \cap D| \ge 2$. Hence in this case the label i will be projected to h. This concludes the proof of Theorem 2.2. $\square$

A graph G satisfying the hypothesis of Theorem 2.2 was said by Barcalkin and German to belong to the $\mathcal{A}$-class; we call such a graph a *BG-graph*. Several common graphs are BG-graphs, including trees, any graph with domination number 2, cycles, and any graph having a 2-packing of cardinality equal to its domination number.

The truth of the conjecture for a graph having a 2-packing of cardinality equal to its domination number was independently obtained in [33] while in [18] it was done for cycles. Also, Vizing's conjecture was proved for graphs with a certain vertex-partition property by Faudree, Schelp and Shreve [19] and by Chen, Piotrowski and Shreve [12]. It was shown in [23] that the first class is a proper subclass of the BG-graphs while the second is the same as the class of BG-graphs.

On the other hand, any bipartite graph B with $\gamma(B) < |B|/2$, that is edge-critical, is not a BG-graph. An example of such a bipartite graph is B_2 of Figure 2, formed by removing the edges of 3 vertex-disjoint 4-cycles from $K_{6,6}$. The nonbipartite graph B_1 is not edge-critical, but a short analysis shows it also is not a BG-graph.

3. The One-Half Argument and Type $\mathcal{X}$ Graphs

In 1995, Hartnell and Rall [25] found a method of partitioning $V(G)$ that is somewhat different from that of Barcalkin and German. They termed the class of graphs having such a partition, Type $\mathcal{X}$. Any decomposable graph is of Type $\mathcal{X}$. The proof of the main theorem regarding these graphs uses the simple fact that a connected graph of order at least 2 has a domination number that is at most one-half its order. The proof technique is best illustrated by using a specific example, the graph G in Figure 3.

Note first that $\gamma(G) = 3$. We shall consider $V(G)$ as being partitioned into three parts, two of which induce cliques and the other induces a star. Let $C_1 = \{r, s\}$, $C_2 = \{a, b, c\}$ and $S = \{u, v, w\}$. Note that both the star S and the clique C_2 contain a vertex whose

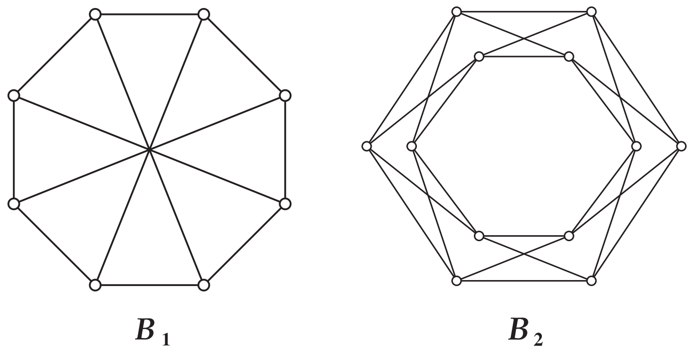

FIGURE 2. Two graphs that are not BG-graphs

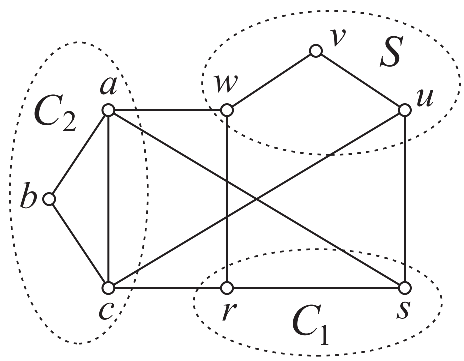

FIGURE 3. Graph G for one-half argument example

neighborhood is entirely contained in its own part, whereas both vertices of C_1 have a neighbor outside of C_1.

Let H be any graph and consider any dominating set D of $G \,\Box\, H$. We will show that $|D| \geq 3\gamma(H)$; more precisely, we will find three disjoint sets in D each with cardinality at least $\gamma(H)$. We start by designating a vertex x from each of the three parts of $V(G)$, and constructing a so-called missing fiber list $\mathcal{L}_x$.

Suppose $D \cap {}^vH$ does not dominate ${}^vH$. Let $\mathcal{L}_v$ be the set of all the vertices h in H such that (v, h) is not dominated by $D \cap {}^vH$. We call $\mathcal{L}_v$ the *missing fiber list* for the H-fiber ${}^vH$. In a similar manner make a missing fiber list $\mathcal{L}_b$ for the H-fiber ${}^bH$.

Fix a vertex, say r, of the clique C_1, and project all the elements of $D \cap {}^sH$ onto ${}^rH$. Let X represent the set of vertices in ${}^rH$ which were already in D, together with those that are images of this projection. The set of all vertices h in H such that (r, h) is not dominated by X is the missing fiber list, $\mathcal{L}_r$, for C_1. Of course, some of these missing fiber lists may be empty. The key idea is to find replacements for the vertices on these lists in the ${}^uH$-, ${}^wH$-, ${}^aH$-, and ${}^cH$-fibers.

By the nature of S noted above, if $h \in \mathcal{L}_v$ then at least one of $(u, h), (w, h)$ is in D. Similarly, if $h \in \mathcal{L}_b$, then at least one of $(a, h), (c, h)$ is in D. However, these vertices also dominate (s, h) or (r, h) (within the G^h-fiber). In showing that D contains at least $3\gamma(H)$ vertices we must be careful not to count any members of D twice.

Now project the vertices in $D \cap {}^uH$ onto ${}^wH$. Let F denote the subgraph of ${}^wH$ induced by the resulting set of vertices from $D \cap {}^wH$ together with the image of this projection of $D \cap {}^uH$.

Consider any $k \in \mathcal{L}_r$.

Assume first that $(w, k) \notin F$. This implies that $k \notin \mathcal{L}_v$ and that both of (a, k) and (c, k) are in D to dominate (s, k) and (r, k). Thus, one of these can be counted for the missing fiber list in C_1 and the other for the missing fiber list $\mathcal{L}_b$, if necessary.

Now assume that (w, k) is an isolated vertex in F and that both of (u, k) and (w, k) belong to D. If $k \in \mathcal{L}_b$, then the nature of C_2 noted above implies there are at least three members of D in the G^k-fiber. In particular, at least one of (a, k) or (c, k) is in D. Hence, (u, k) can be counted for the missing fiber list $\mathcal{L}_r$ and (w, k) for the missing fiber list $\mathcal{L}_v$ if necessary (i.e., if $k \in \mathcal{L}_v$).

Next we assume that (w, k) is an isolated vertex in F and that only one of (u, k) or (w, k) is in D. Suppose that $(u, k) \in D$. Since D dominates (r, k) it follows that $(c, k) \in D$. In addition, since (w, k) must be dominated by D from within the G^k-fiber, we may conclude that either $(v, k) \in D$ or $(a, k) \in D$. In this case we have at least three vertices from D in G^k. Hence we will be able to count one of these for missing fiber k in C_1 and still have two members of D in case k is in $\mathcal{L}_v$ or in $\mathcal{L}_b$. The case when $(w, k) \in D$ is handled in a similar manner.

Finally, we assume that (w, k) is a vertex in a component K of order two or more in F. Since the complement of a minimal dominating set is also a dominating set, K has two disjoint dominating sets A_1 and A_2. The vertices in A_1 can be counted towards dominating ${}^vH$ and those in A_2 can be counted towards dominating ${}^rH$.

This counting gives $|D| \geq 3\gamma(H) = \gamma(G)\gamma(H)$. Thus G satisfies Vizing's conjecture.

In the proof of [25, Theorem 3.1] that the graphs of Type $\mathcal{X}$ satisfy Vizing's conjecture, a similar method of partitioning and the one-half argument were used as for G in the above example. Here is the formal definition of Type $\mathcal{X}$ graphs.

Consider a graph G with $\gamma(G) = n = k + t + m + 1$ and such that $V(G)$ can be partitioned into $S \cup SC \cup BC \cup C$, where $S = S_1 \cup \cdots \cup S_k$, $BC = B_1 \cup \cdots \cup B_t$, and $C = C_1 \cup \cdots \cup C_m$. Each of $SC, B_1, \ldots, B_t, C_1, \ldots, C_m$ induces a clique. Every vertex of SC (special clique) has at least one neighbor outside SC, whereas each of $B_1, \ldots, B_t$ (the buffer cliques), say B_i, has at least one vertex, say b_i, which has no neighbors outside B_i. Each $S_i \in \{S_1, \ldots, S_k\}$ is "star-like" in that it contains a star centered at a vertex v_i which is adjacent to each vertex in $T_i = S_i - \{v_i\}$. The vertex v_i has no neighbors besides those in T_i. Although other pairs of vertices in T_i may be adjacent (and hence S_i does not necessarily induce a star), S_i does not induce a clique nor can more edges be added in the subgraph induced by S_i without lowering the domination number of G. Furthermore, there are no edges between vertices in S and vertices in C.

It should be noted that a graph of Type $\mathcal{X}$ need not have a clique having the properties of SC, and any of t, m or k is allowed to be 0. However, if such an SC is not in G, then $\gamma(G) = n = k + t + m$. Also, if SC is not present and BC is empty, but S as well as C

are not empty, then the graph is disconnected. SC cannot be the only one of these which is nonempty, since by definition its vertices must have neighbors outside SC.

Theorem 3.1. ([25]) *Let G' be a spanning subgraph of a graph G of Type $\mathcal{X}$ such that $\gamma(G') = \gamma(G)$. Then G' satisfies Vizing's conjecture.*

The same authors showed that any graph whose domination number is one more than its 2-packing number is of Type $\mathcal{X}$, thus establishing the following corollary to Theorem 3.1.

Corollary 3.2. ([25]) *If G is a graph and $\gamma(G) = \rho(G) + 1$, then G satisfies Vizing's conjecture.*

3.1. **Extending the One-Half Argument.** Our purpose in the remainder of this section is to generalize the above reasoning. Whereas the one-half argument essentially "shared" vertices from one copy of a graph between two copies, one can consider the situation where these vertices may be needed in three different copies. We do this with an example.

Consider the graph G in Figure 4. We observe that $\gamma(G) = 5$, and the 2-packing number of G is 3 (implying that G is not of Type $\mathcal{X}$). We show that G satisfies Vizing's conjecture.

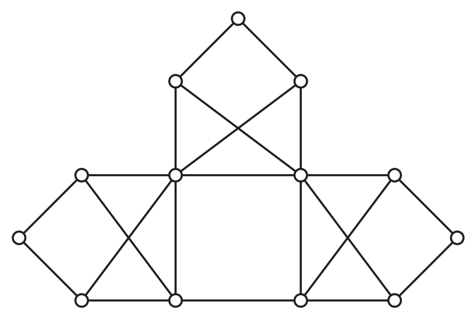

FIGURE 4. The graph G

We first establish a lemma that will be needed in the argument.

Lemma 3.3. *Let G be a connected graph of order $n \geq 2$ and assume that $V(G) = V_1 \cup V_2 \cup W$ where no vertex in V_1 is adjacent to any vertex in V_2. For $i = 1, 2$ let $\gamma_i(G)$ denote the minimum cardinality of a set of vertices of G that dominates V_i, and let $\gamma(G)$ denote the domination number of G. Then,*

$$\gamma_1(G) + \gamma_2(G) + \gamma(G) \leq n\,.$$

Proof. We prove the statement for trees. The lemma will follow in general since these three numbers $\gamma_1(G), \gamma_2(G), \gamma(G)$ are no larger than the corresponding numbers for any spanning tree of G.

The statement clearly holds for the only tree of order 2. Assume the result holds for all nontrivial trees of order at most k and let T be any tree of order $k+1$. Let $V(T) = V_1 \cup V_2 \cup W$ and assume no vertex in V_1 is adjacent to any vertex in V_2. The result clearly holds if T is a star. Otherwise, let v be a vertex of degree one at the end of a longest path in T and let w be its unique neighbor. Let R denote the other degree one neighbors (if any) of w. Let T' be the tree $T - (\{v, w\} \cup R)$.

By induction $\gamma_1(T') + \gamma_2(T') + \gamma(T') \le |T'|$, where we are assuming $V(T')$ inherits the partition from that of $V(T)$. It is clear that $\gamma(T) \le \gamma(T') + 1$. If $R = \emptyset$, then at most one of V_1, V_2 intersects $\{v, w\}$. On the other hand, if $R \neq \emptyset$, then $|T| \ge |T'| + 3$ and since w dominates $R \cup \{v\}$, it follows immediately that

$$\gamma_1(T) + \gamma_2(T) + \gamma(T) \le \gamma_1(T') + \gamma_2(T') + \gamma(T') + 3\,.$$

$\square$

We will now show that the graph G from Figure 4 satisfies Vizing's conjecture. To that end, let H be an arbitrary graph and let D be a minimum dominating set of the Cartesian product $G \,\square\, H$. For an arbitrary vertex g of G we denote the intersection of D with the fiber ${}^gH$ by D_g. Label the vertices of G and select pairs of its vertices as shown in Figure 5.

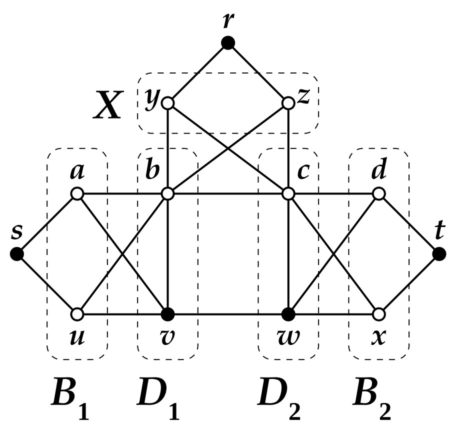

FIGURE 5. Labeled graph G

While D dominates all of $G \,\square\, H$, to show that $\gamma(G \,\square\, H) \ge \gamma(G)\gamma(H)$ we will show that $|D| \ge 5\gamma(H)$ to dominate ${}^rH \cup {}^sH \cup {}^tH \cup {}^vH \cup {}^wH$.

First we note that if each of D_r, D_s, D_t, D_v and D_w contains at least $\gamma(H)$ vertices, then $|D| \ge 5\gamma(H)$.

Thus, we assume this is not the case. We associate the color 1 (respectively, 2, 3, 4, 5) with the H-fiber ${}^rH$ (respectively, ${}^sH$, ${}^tH$, ${}^vH$, ${}^wH$). We will color each vertex of H with a subset of $\{1, 2, 3, 4, 5\}$ in such a way that the subset of $V(H)$ colored i dominates H for each $1 \le i \le 5$. The subset assigned to each vertex will accumulate during the course of the argument below. As these assignments are made, the total number of assignments will not exceed $|D|$. We begin by assigning color 1 (respectively, 2, 3, 4 or 5) to each vertex h for which (r, h) belongs to D_r (respectively, $(s, h) \in D_s$, $(t, h) \in D_t$, $(v, h) \in D_v$, $(w, h) \in D_w$).

We say that a vertex h of H is on the *missing fiber list* at r if the vertex (r, h) is not dominated by D_r. Denote this missing fiber list by $\mathcal{L}_r$. Note that if $h \in \mathcal{L}_r$, then at least one of (y, h) and (z, h) is in D. Similarly, let $\mathcal{L}_s$ and $\mathcal{L}_t$ be the missing fiber lists at s and t, respectively.

Let f_X denote the map from D_y to ${}^zH$ given by $f_X(y,h) = (z,h)$ and let D'_z denote $D_z \cup f_X(D_y)$. In an analogous way we form D'_u, D'_v, D'_w and D'_x by doing similar projections of the subset of D in the H-fibers located at clear vertices within $B_1 \times V(H), D_1 \times V(H), D_2 \times V(H)$, and $B_2 \times V(H)$ onto the H-fibers specified by the solid vertices. Let us say that a vertex h from H has *weight 1 in* B_1 if exactly one of (a,h) or (u,h) is in D. If both of these vertices are in D we say h has weight 2 in B_1. The weight of h in each of D_1, D_2, B_2 and X is defined in a similar way.

As we did above for r, s and t, we define missing fiber lists $\mathcal{L}_v$ and $\mathcal{L}_w$. However, here we say for example, that $h \in \mathcal{L}_v$ if (v,h) is not dominated by D'_v (instead of, if (v,h) is not dominated by D_v). We are not concerned with missing fiber lists at u, x and z. Rather, the focus is on the connected components of the subgraphs induced by D'_u, D'_x and D'_z.

Each such component is an isolated vertex or has order at least two. In the case where one of these components has order at least two the complement of a minimal dominating set is also a dominating set. We will make use of the fact that such a component has two disjoint dominating sets.

Consider a component C in the subgraph induced by D'_u. If C has order at least two, then choose a minimal dominating set A of C. Assign color 2 to any vertex k for which $(u,k) \in A$ and assign color 4 to k if $(u,k) \in C - A$. If C has order 1, say $C = \{(u,k)\}$, then there are two possibilities. If k has weight two in B_1, then assign colors 2 and 4 to k. Finally, assume k has weight one in B_1. We observe that k cannot be on the missing fiber lists at both s and v because D dominates both of (a,k) and (u,k) (in fact, at least one of $(s,k), (b,k), (v,k)$ is in D). If $k \in \mathcal{L}_s$, assign color 2 to k; if $k \in \mathcal{L}_v$, assign color 4 to k. The components of the subgraph induced by D'_x are handled in a similar manner.

We now make the important observation that if a vertex h is on the missing fiber list at s, then color 2 has been assigned to h or to a neighbor of h. Thus, the vertices having color 2 in their list dominate H. The same is true for the vertices having color 3 in their list. Also, if a vertex k is on the missing fiber list $\mathcal{L}_v$ and $(u,k) \in D'_u$, then either k or one of its neighbors has been assigned the color 4. Furthermore, if vertex k is in $\mathcal{L}_v$ and $(u,k) \notin D'_u$, then $(w,k) \in D$, and hence neither k nor any of its neighbors in H belongs to $\mathcal{L}_w$. Similar statements hold in regard to such a vertex of H relative to $\mathcal{L}_w$.

Suppose that $k \in \mathcal{L}_v$ and color 4 has not yet been assigned to k or one of its neighbors. Then $(u,k) \notin D'_u$ and $(w,k) \in D$. Assume that $(z,k) \notin D'_z$. Then $k \notin \mathcal{L}_r$ for otherwise the vertex (r,k) would not be dominated. Now we ask which vertex dominates (b,k). It can be neither (y,k) nor (z,k) because $(z,k) \notin D'_z$. Similarly, since $(u,k) \notin D'_u$ it is also not dominated by (u,k) or (a,k). Hence (b,k) must be dominated by (c,k). Then both (c,k) and (w,k) are in D and we may assign color 4 to k. The same can be done for color 5. If some vertex k is not dominated by a vertex assigned color 4 or 5 then $(z,k) \in D'_z$. Note also that $(z,k) \in D'_z$ when $k \in \mathcal{L}_r$. We therefore only need to deal with the vertices in D'_z.

Consider a component $C = \{(z,k)\}$ of order one in D'_z. There are three cases to handle: k is in both, neither or exactly one of $\mathcal{L}_v, \mathcal{L}_w$. Suppose that $k \in \mathcal{L}_v \cap \mathcal{L}_w$. Since D dominates both (v,k) and (w,k) this implies that k has weight at least one in both of B_1 and B_2. As noted above, either k or one of its neighbors in H has been assigned color 4. The same is true about color 5. Thus we can assign color 1 to k.

Assume next that $k \notin \mathcal{L}_v \cup \mathcal{L}_w$. In this case we assign color 1 to k.

Finally, assume that k belongs to exactly one of the two missing fiber lists $\mathcal{L}_v$ and $\mathcal{L}_w$. Without lost of generality we suppose that $k \in \mathcal{L}_v$ and k is not in $\mathcal{L}_w$. From the previous observation we may assume that $(u,k) \notin D'_u$. As noted above this implies that (w,k) belongs to D and neither k nor any of its neighbors in H belongs to $\mathcal{L}_w$. If $k \notin \mathcal{L}_r$, assign color 4 to k. If, on the other hand, k belongs to $\mathcal{L}_r$, then because D dominates both of (y,k) and (z,k) it follows that (c,k) is also in D or k has weight two in X. In both instances we assign colors 1 and 4 to k.

We are thus led to the final case in which we consider a component C in D'_z of order at least two. Let $(z,h) \in C$. As with a component of order one, we have already handled the situation in which $h \in \mathcal{L}_v \cap \mathcal{L}_w$ (from B_1 and B_2). Place (z,h) in V_1 if $h \in \mathcal{L}_v - \mathcal{L}_w$ and place (z,h) in V_2 if $h \in \mathcal{L}_w - \mathcal{L}_v$. Let $W = C - (V_1 \cup V_2)$. By our observation above no vertex of V_1 is adjacent to any vertex of V_2. We apply Lemma 3.3 and infer that there exist subsets A_1, A_2 and A of C such that A_i dominates V_i, for $1 \le i \le 2$, A dominates C and $|A_1|+|A_2|+|A| \le |C|$. These subsets of C need not be disjoint. If $(z,h) \in A_1$, assign color 4 to h. If $(z,h) \in A_2$, assign color 5 to h. Finally assign color 1 to any h for which $(z,h) \in A$. We have assigned $|A_1|+|A_2|+|A|$ colors which is at most $|C|$.

It now follows that the set of vertices having color 1 (respectively, 4 or 5) dominates H. Combining this with our earlier conclusion about colors 2 and 3 we have shown

$$\gamma(G \,\Box\, H) = |D| \ge 5\gamma(H) = \gamma(G)\gamma(H)\,.$$

It is unclear how far the above idea can be pushed.

4. Fair Reception

Recently Brešar and Rall [10] introduced the notion of a fair reception in a graph. It uses an approach similar to that of Barcalkin and German, by partitioning a graph into subgraphs that enjoy a condition like that in inequality (2). Yet the partition is more general, since arbitrary subgraphs (not only cliques) are allowed in the partition, and also there can be a special, external part Z.

Let $\{S_1,\dots,S_k\}$ be a collection of pairwise disjoint sets of vertices from a graph G with $\mathcal{S} = S_1 \cup \cdots \cup S_k$, and let $Z = V(G) - \mathcal{S}$. We say that $S_1,\dots,S_k$ form *a fair reception of size* k if for any integer $\ell, 1 \le \ell \le k$, and any choice of ℓ sets $S_{i_1},\dots,S_{i_\ell}$ from the collection, the following holds: if D externally dominates $S_{i_1} \cup \cdots \cup S_{i_\ell}$ then

$$|D \cap Z| + \sum_{j, S_j \cap D \neq \emptyset} (|S_j \cap D| - 1) \ge \ell. \tag{3}$$

That is, on the left-hand side we count all the vertices of D that are not in $\mathcal{S}$, and for vertices of D that are in some S_j, we count all but one from $D \cap S_j$.

In any graph, any nonempty subset of the vertex set forms a fair reception of size 1. Another example is obtained by taking a 2-packing and letting each set S_i consist of exactly one vertex of the 2-packing. Hence in any graph G there is a fair reception of size $\rho(G)$.

Given a graph G, the largest k such that there exists a fair reception of size k in G is denoted by $\gamma_F(G)$, and is called the *fair domination number of* G. For instance in C_5 we can let S_1 be a single vertex and the vertices in its antipodal edge be S_2, and obtain a fair reception of size 2. Thus $\gamma_F(C_5) = 2 = \gamma(C_5)$. We have the following basic observation about the fair domination number.

Proposition 4.1. ([10]) *For any graph G, $\rho(G) \le \gamma_F(G) \le \gamma(G)$.*

Proof. The first inequality has been established above. Suppose there is a graph G such that $r = \gamma(G) < \gamma_F(G) = k$. Let A be a minimum dominating set, and assume that the sets $S_1, \dots, S_k$ with $Z = V(G) - \mathcal{S}$ form a fair reception of size k in G. Since $r < k$, the set A must be disjoint from at least one of these sets. Assume that $A \cap S_i = \emptyset$ for $1 \le i \le t$ and that $A \cap S_j \ne \emptyset$ for $t+1 \le j \le k$.

The set A externally dominates $S_1 \cup \cdots \cup S_t$ and so it follows from the definition of fair reception that

$$\begin{aligned} t &\le |A \cap Z| + \sum_{j, S_j \cap A \ne \emptyset} (|S_j \cap A| - 1) \\ &= |A \cap Z| + \sum_{j=t+1}^{k} |S_j \cap A| - (k - t) \\ &= |A| - k + t\,. \end{aligned}$$

This immediately implies that $k \le |A|$, which is a contradiction. $\square$

If the partition of $V(G)$ into $\gamma_F(G)$ sets (and an eventual set Z) obeys condition (3), then similar arguments as in Theorem 2.2 can be used to deduce the following result.

Theorem 4.2. ([10]) *For all graphs G and H,*

$$\gamma(G \,\Box\, H) \ge \max\{\gamma(G)\gamma_F(H), \gamma_F(G)\gamma(H)\}.$$

An application to Vizing's conjecture is seen in the following obvious corollary.

Corollary 4.3. ([10]) *If G is a graph with $\gamma_F(G) = \gamma(G)$, then G satisfies Vizing's conjecture.*

It is also easy to see the following fact.

Proposition 4.4. ([10]) *Let G be a decomposable graph. Then a partition of the vertex set into $\gamma(G)$ cliques yields a fair reception of G of size $\gamma(G)$ (in which $\mathcal{S}$ equals $V(G)$).*

Note that a family of sets that forms a fair reception in a graph G also forms a fair reception in any spanning subgraph of G. Hence by the above proposition, the class of graphs G with $\gamma(G) = \gamma_F(G)$ contains the class of BG-graphs as well. Thus, Theorem 4.2 is a generalization of the result by Barcalkin and German. The same authors [10] also constructed an infinite family of graphs whose fair domination number and domination number are equal but which are not covered by the Type $\mathcal{X}$ results.

The fair domination number of a graph is related to the invariant γ^i introduced by Aharoni and Szabó [2]. Let $\gamma^i(G)$ denote the maximum, over all independent sets M in G, of the smallest cardinality of a set D that dominates M (i.e., such that $M \subseteq N[D]$).

Proposition 4.5. ([10]) *For any graph G, $\gamma_F(G) \ge \gamma^i(G)$.*

Proof. It is easy to see that we may assume G has no isolated vertices. Let I be an independent set of vertices in G that requires $k = \gamma^i(G)$ vertices to dominate, and suppose that $A = \{x_1, \ldots, x_k\}$ dominates I. Because G has no isolated vertices we may assume that A externally dominates I. Let the sets $S_1, \ldots, S_k$ be a partition of I such that $S_i \subseteq N(x_i)$. We claim that $S_1, \ldots, S_k$ form a fair reception in G. To (externally) dominate any subfamily of ℓ of these sets, one needs at least ℓ vertices (all of which are in Z, hence (3) will be satisfied). Indeed, otherwise we easily infer that I can be dominated by fewer than k vertices, which is a contradiction. Thus $\gamma_F(G) \geq k = \gamma^i(G)$. $\square$

As a corollary, we obtain the result of Aharoni and Szabó [2]:

Theorem 4.6. ([2]) *For any G and H, $\gamma(G \,\Box\, H) \geq \gamma^i(G)\gamma(H)$.*

Aharoni, Berger and Ziv [1] showed that γ^i and γ agree on chordal graphs. Thus:

Corollary 4.7. ([2]) *Chordal graphs satisfy Vizing's conjecture.*

Unfortunately, $\gamma^i(G)$ can be arbitrarily smaller than $\gamma(G)$, though it is at least $\rho(G)$.

Question 4.8. *What other classes of graphs satisfy $\gamma = \gamma^i$?*

The relationship between $\gamma_F(G)$ and $\gamma(G)$ is murkier. We verified recently, using a computer check of all appropriate partitions, that for the graph G from Figure 3, $\gamma_F(G) = \gamma(G) - 1$. Note that G satisfies Vizing's conjecture since it is of type $\mathcal{X}$.

The following natural questions regarding the fair domination number are unresolved:

Question 4.9. *Is there a general lower bound for $\gamma_F(G)$ in terms of $\gamma(G)$? For example, is $\gamma_F(G) \geq \gamma(G)-1$ for every connected graph G? Does there exist a constant $c > 1/2$ such that $\gamma_F(G) \geq c\,\gamma(G)$ for every graph G? Such a constant would imply (using Theorem 4.2) that*

$$\gamma(G \,\Box\, H) \geq c\,\gamma(G)\,\gamma(H)\,,$$

for every G and H, an improvement over Theorem 5.1.

5. The Double-Projection Argument

In attacking Vizing's conjecture, the following question is quite natural. Is there a constant $c > 0$ such that

$$\gamma(G \,\Box\, H) \geq c\gamma(G)\gamma(H)\,?$$

And, of course, hoping that the inequality with $c = 1$ could eventually be proved, the question was stated explicitly in [23]. It was answered in the affirmative by Clark and Suen in [15]. We next describe their idea that nicely incorporates the product structure of $G \,\Box\, H$.

Let H be a graph with $\gamma(H) = k$ and let $\{h_1, \ldots, h_k\}$ be a minimum dominating set of H. Consider a partition $\{\pi_1, \ldots, \pi_k\}$ of $V(H)$ chosen so that $h_i \in \pi_i$ and $\pi_i \subseteq N[h_i]$ for each i. Let $G_i = V(G) \times \pi_i$. For a vertex g of G the set of vertices $\{g\} \times \pi_i$ is called an *H-cell*, see Figure 6.

Let D be a minimum dominating set of $G \,\Box\, H$. For $i = 1, \ldots, k$, let n_i be the number of H-cells in G_i such that all vertices from the H-cell are dominated by D from within

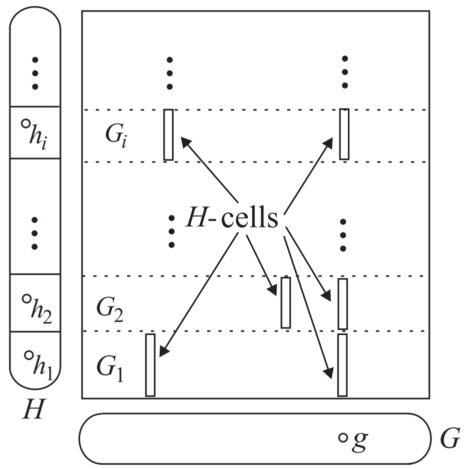

FIGURE 6. Clark-Suen partition

the corresponding H-fiber. Then, considering the projection $p_G(D \cap G_i)$, it follows that $|D \cap G_i| + n_i \geq \gamma(G)$ and thus

$$|D| + \sum_{i=1}^{k} n_i \geq \gamma(G)\gamma(H). \tag{4}$$

On the other hand, the projection $p_H(D \cap {}^gH)$ gives $\gamma(H) \leq |D \cap {}^gH| + (k - m_g)$, where m_g denotes the number of cells in ${}^gH$ that are dominated by D from within ${}^gH$. Consequently, $m_g \leq |D \cap {}^gH|$. Hence,

$$|D| \geq \sum_{g \in G} m_g\,. \tag{5}$$

Since the H-cells were counted in two ways, that is, $\sum_{i=1}^{k} n_i = \sum_{g \in G} m_g$, the inequalities (4) and (5) give:

Theorem 5.1. ([15]) *For all graphs G and H,*

$$\gamma(G \,\square\, H) \geq \frac{1}{2}\gamma(G)\,\gamma(H).$$

The factor 1/2 of Theorem 5.1 comes from the double counting of the vertices of the minimum dominating set D. Aharoni and Szabó [2] modified this approach to establish their result about chordal graphs (Corollary 4.7). Recently Suen and Tarr [39] announced the following improvement of Theorem 5.1:

$$\gamma(G \,\square\, H) \geq \frac{1}{2}\gamma(G)\,\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}\,.$$

Yet another recent improvement of Theorem 5.1 is due to Wu [44]. A Roman domination of a graph G is a labeling of its vertices with labels from $\{0, 1, 2\}$ such that if a vertex is labeled with 0 then it has a neighbor labeled with 2. The *Roman domination number*

$\gamma_R(G)$ is the minimum weight of a Roman domination. Note that $\gamma_R(G) \le 2\gamma(G)$. Hence the following announced inequality from [44]

$$\gamma_R(G \,\square\, H) \ge \gamma(G)\gamma(H)$$

is indeed an extension of Theorem 5.1.

5.1. **Claw-free Graphs.** We next demonstrate how Clark and Suen's approach can be applied to claw-free graphs.

Theorem 5.2. *Let G be a claw-free graph. Then for any graph H without isolated vertices,*

$$\gamma(G \,\square\, H) \ge \frac{1}{2}\alpha(G)(\gamma(H)+1)\,.$$

Proof. Let $A = \{g_1, \ldots, g_{\alpha(G)}\}$ be a maximum independent set of G, and let $\{h_1, \ldots, h_{\gamma(H)}\}$ be a minimum dominating set of H. Let $\Pi = \{\pi_1, \ldots, \pi_{\gamma(H)}\}$ be a corresponding partition of $V(H)$, where $h_j \in \pi_j$ and $\pi_j \subseteq N[h_j]$, $j = 1, \ldots, \gamma(H)$.

Let D be a minimum dominating set of $G \,\square\, H$. Let $x_i = |D \cap {}^{g_i}H|$, $1 \le i \le \alpha(G)$. For $1 \le i \le \alpha(G)$ and $1 \le j \le \gamma(H)$, set $d_{i,j} = 1$ if all the vertices of $\{g_i\} \times \pi_j$ are dominated within the fiber ${}^{g_i}H$, and set $d_{i,j} = 0$ otherwise.

Note first that $x_i \ge \sum_{j=1}^{\gamma(H)} d_{i,j}$. Indeed, if this was not the case, we could form a dominating set of H with cardinality smaller than $\gamma(H)$ by adding to $p_H(D \cap {}^{g_i}H)$ all vertices h_j such that $d_{i,j} = 0$.

Let $I = \{i_1, \ldots, i_r\}$ be the set of indices i, $1 \le i \le \alpha(G)$, such that $x_i = 0$. Note that it is possible that $I = \emptyset$. Note also that

$$\sum_{i=1}^{\alpha(G)} x_i \ge \alpha(G) - r\,.$$

Let $B_j = (V(G) - A) \times \pi_j$. Consider the vertices of $\{g_i\} \times \pi_j$. If $d_{i,j} = 0$, then at least one of them is not dominated from ${}^{g_i}H$, and since A is independent, this vertex must be dominated by some vertex from B_j. Moreover, if $i \in I$, then every vertex of $\{g_i\} \times \pi_j$ is dominated from B_j. Now, a vertex from B_j can dominate at most two vertices from $A \times V(H)$ since G is claw-free and A is an independent set. Therefore,

$$\begin{aligned}
\sum_{j=1}^{\gamma(H)} |D \cap B_j| &\ge \frac{1}{2}\left(\sum_{i \notin I}\sum_{j=1}^{\gamma(H)}(1 - d_{i,j}) + \sum_{i \in I}|H|\right) \\
&= \frac{1}{2}\left((\alpha(G) - r)\gamma(H) - \sum_{i \notin I}\sum_{j=1}^{\gamma(H)} d_{i,j} + r|H|\right)\,.
\end{aligned}$$

Recall that $x_i \ge \sum_{j=1}^{\gamma(H)} d_{i,j}$ and since for all $i \in I$, $d_{i,j} = 0$, then

$$\sum_{i \notin I}\sum_{j=1}^{\gamma(H)} d_{i,j} = \sum_{i=1}^{\alpha(G)}\sum_{j=1}^{\gamma(H)} d_{i,j} \le \sum_{i=1}^{\alpha(G)} x_i\,.$$

Now, using the above three inequalities, we get

$$\begin{aligned}
|D| &= \sum_{j=1}^{\gamma(H)} |D \cap B_j| + \sum_{i=1}^{\alpha(G)} x_i \\
&\geq \frac{1}{2}\left(\big(\alpha(G) - r\big)\gamma(H) - \sum_{i=1}^{\alpha(G)} x_i + r|H| \right) + \sum_{i=1}^{\alpha(G)} x_i \\
&= \frac{1}{2}\big(\alpha(G) - r\big)\gamma(H) + \frac{r}{2}|H| + \frac{1}{2}\sum_{i=1}^{\alpha(G)} x_i \\
&\geq \frac{1}{2}\big(\alpha(G) - r\big)\gamma(H) + \frac{r}{2}|H| + \frac{1}{2}\big(\alpha(G) - r\big) \\
&= \frac{1}{2}\alpha(G)\big(\gamma(H) + 1\big) + \frac{r}{2}\big(|H| - \gamma(H) - 1\big).
\end{aligned}$$

Since H has no isolated vertices, we have $|H| \geq \gamma(H) + 1$ and the proof is complete. $\square$

Since $\gamma(G) \leq \alpha(G)$ holds for any graph G, we immediately get:

Corollary 5.3. *Let G be a claw-free graph. Then for any graph H without isolated vertices,*

$$\gamma(G \,\Box\, H) \geq \frac{1}{2}\gamma(G)(\gamma(H) + 1)\,.$$

Corollary 5.3 does not follow from Theorem 4.6. This can be seen, for instance, by letting G be the so-called cocktail party graph formed by removing a perfect matching from K_{2n} for $n \geq 2$. Then G is claw-free, $\gamma(G) = 2$ and $\gamma^i(G) = 1$. Letting $H = K_2$, we get $\gamma(G \,\Box\, K_2) = 2 = \frac{1}{2}\gamma(G)(\gamma(K_2) + 1)$, but $\gamma^i(G)\gamma(K_2) = 1$. Note too that Theorem 5.2 implies that a graph G satisfies Vizing's conjecture whenever it is claw-free and satisfies $\alpha(G) = 2\gamma(G)$. An infinite family of such graphs may be constructed following the example of Figure 7.

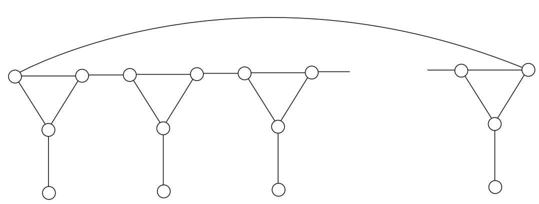

FIGURE 7. A claw-free graph with $\alpha(G) = 2\gamma(G)$

In many cases we can slightly extend Theorem 5.2 as follows.

Theorem 5.4. *Let G be a claw-free graph. Let H be a graph without isolated vertices for which $|H| \geq \Delta(H) + \gamma(H) + 2$. Then*

$$\gamma(G \,\Box\, H) \geq \frac{1}{2}\alpha(G)(\gamma(H) + 2)\,.$$

In particular, $\gamma(G \,\Box\, H) \geq \frac{1}{2}\gamma(G)(\gamma(H)+2)$.

Proof. Define $A = \{g_1, \ldots, g_{\alpha(G)}\}$, $\{h_1, \ldots, h_{\gamma(H)}\}$, D, x_i, $d_{i,j}$, and I, where $|I| = r \geq 0$, as in the proof of Theorem 5.2. In addition, let I' be the set of indices i, $1 \leq i \leq \alpha(G)$, such that $x_i = 1$. Set $|I'| = s$. With this new notation we have:

$$\sum_{i=1}^{\alpha(G)} x_i \geq 2\big(\alpha(G) - r - s\big) + s\,. \tag{6}$$

The number of vertices in $A \times V(H)$ that are not dominated from within H-fibers is at least

$$\sum_{i \notin (I \cup I')} \sum_{j=1}^{\gamma(H)} (1 - d_{i,j}) + \sum_{i \in I} |H| + \sum_{i \in I'} \big(|H| - \Delta(H) - 1\big)$$

because within an H-fiber a vertex dominates at most $\Delta(H)+1$ vertices. Therefore,

$$\begin{aligned}
\sum_{j=1}^{\gamma(H)} |D \cap B_j| \;&\geq\; \frac{1}{2}\Bigg(\sum_{i \notin (I \cup I')} \sum_{j=1}^{\gamma(H)} (1 - d_{i,j}) + \sum_{i \in I} |H| + \\
&\qquad \sum_{i \in I'} (|H| - \Delta(H) - 1) \Bigg) \\
&= \frac{1}{2}\Bigg(\big(\alpha(G) - r - s\big)\gamma(H) - \sum_{i \notin (I \cup I')} \sum_{j=1}^{\gamma(H)} d_{i,j} + \\
&\qquad r|H| + s\big(|H| - \Delta(H) - 1\big) \Bigg)\,.
\end{aligned} \tag{7}$$

Note again that $x_i \geq \sum_{j=1}^{\gamma(H)} d_{i,j}$. Since in addition $d_{i,j} = 0$ for all $i \in I$, and $d_{i,j} \geq 0$ for all $i \in I'$, we infer

$$\sum_{j=1}^{\gamma(H)} \sum_{i \notin (I \cup I')} d_{i,j} = \sum_{j=1}^{\gamma(H)} \sum_{i=1}^{\alpha(G)} d_{i,j} - \sum_{j=1}^{\gamma(H)} \sum_{i \in I'} d_{i,j} \leq \sum_{i=1}^{\alpha(G)} x_i\,.$$

Therefore, combining the above inequality with (7) and (6), we obtain

$$\begin{aligned}
|D| &= \sum_{j=1}^{\gamma(H)} |D \cap B_j| + \sum_{i=1}^{\alpha(G)} x_i \\
&\geq \frac{1}{2}\Bigg(\big(\alpha(G) - r - s\big)\gamma(H) - \sum_{i=1}^{\alpha(G)} x_i + r|H| + \\
&\quad s\big(|H| - \Delta(H) - 1\big)\Bigg) + \sum_{i=1}^{\alpha(G)} x_i \\
&\geq \frac{1}{2}\big(\alpha(G) - r - s\big)\gamma(H) + \frac{1}{2}\sum_{i=1}^{\alpha(G)} x_i + \frac{r}{2}|H| + \\
&\quad \frac{s}{2}\big(|H| - \Delta(H) - 1\big) \\
&\geq \frac{1}{2}\big(\alpha(G) - r - s\big)\gamma(H) + \big(\alpha(G) - r - s\big) + \frac{s}{2} + \frac{r}{2}|H| + \\
&\quad \frac{s}{2}\big(|H| - \Delta(H) - 1\big) \\
&= \frac{1}{2}\alpha(G)\big(\gamma(H) + 2\big) + \frac{r}{2}\big(|H| - \gamma(H) - 2\big) + \\
&\quad \frac{s}{2}\big(|H| - \Delta(H) - \gamma(H) - 2\big),
\end{aligned}$$

and the proof is complete. □

6. Properties of a Minimal Counterexample

A natural way to prove or disprove a conjecture in graph theory is to check for the existence of a minimal counterexample. Suppose Vizing's conjecture is false. Then, there exists a graph G such that for some graph H, $\gamma(G \,\square\, H) < \gamma(G)\gamma(H)$. From among all such graphs, G, choose one of smallest order. We call such a graph a *minimal counterexample.*

In this section we add to the list of properties that must be satisfied by any minimal counterexample. As a consequence it might be possible to find such a counterexample or to prove that Vizing's conjecture is actually true for any graph satisfying all such conditions. In the latter case the truth of the conjecture is established.

It is immediate that a minimal counterexample is connected. We may also assume that a minimal counterexample is edge-critical with respect to domination (see Observation 2.1). (For more information on edge-critical graphs see the survey [40].)

Suppose that u and v are distinct vertices in a graph G. Denote by G_{uv} the graph formed by identifying vertices u and v in G and then removing any parallel edges. For reference purposes let w be the vertex in the identification of u and v. (If $e = uv \in E(G)$, then G_{uv} is the usual graph obtained from G by contracting the edge e.) It is clear from the definition that $\gamma(G_{uv}) \leq \gamma(G)$. It is also easy to see that $\gamma(G_{uv} \,\square\, H) \leq \gamma(G \,\square\, H)$ for any graph H. Indeed, if D is a minimum dominating set of $G \,\square\, H$, then

$$D' = (D - ({}^{u}H \,\cup\, {}^{v}H)) \cup \{\, (w,h) \mid (u,h) \in D \text{ or } (v,h) \in D \,\}$$

dominates $G_{uv} \,\square\, H$. Using this, we can now establish another property of any minimal counterexample. Burton and Sumner [11] call graphs with this property *totally dot-critical.*

Theorem 6.1. *If G is a minimal counterexample to Vizing's conjecture, then for every pair of distinct vertices u and v of G, $\gamma(G_{uv}) < \gamma(G)$.*

Proof. Let H be a graph such that $\gamma(G \,\square\, H) < \gamma(G)\gamma(H)$, and let u and v be distinct vertices of G. Then, G_{uv} satisfies Vizing's conjecture. It follows that

$$\gamma(G_{uv})\gamma(H) \le \gamma(G_{uv} \,\square\, H) \le \gamma(G \,\square\, H) < \gamma(G)\gamma(H)\,.$$

The conclusion of the theorem follows. $\square$

Cycles of order $3n + 1$ satisfy the conclusion of Theorem 6.1 as does any graph formed from a connected graph by adding a single leaf adjacent to each vertex.

Problem 6.2. *Characterize the graphs G such that for every pair of distinct vertices u and v in G, $\gamma(G_{uv}) < \gamma(G)$.*

Among the consequences of Theorem 6.1, it follows that a minimal counterexample cannot have a vertex adjacent to two or more vertices of degree one, nor can any vertex have two neighbors, each of degree two, that are adjacent to each other. In addition, we also have the following corollary.

Corollary 6.3. *If G is a minimal counterexample to Vizing's conjecture, then for any vertex u of G, there exists a minimum dominating set D that contains u. Moreover, for any edge uv in G, there exists a minimum dominating set D such that either both u and v are in D, or u is in D and one of u or v is the only vertex not dominated by $D - \{u\}$.*

Proof. Let u and v be any two adjacent vertices of G, denote by w the vertex of G_{uv} obtained by the identification of u and v. Consider a minimum dominating set D' of G_{uv}. If the added vertex w is in D', let $D = (D' - \{w\}) \cup \{u, v\}$; otherwise let $D = D' \cup \{u\}$. In both cases, it is easy to check that D dominates G. Moreover, $|D| = |D'| + 1$, and since $\gamma(G) > \gamma(G_{uv}) = |D'|$, we deduce that D is a minimum dominating set of G that contains u.

Now, in the first case, D contains both u and v. In the second case, all the vertices in G except possibly u or v are dominated by $D - \{u\}$. Furthermore, D' contains a neighbor x of w, which is a neighbor of u or v in G. So, one of u or v is the only vertex not dominated by $D - \{u\}$. $\square$

The property given in the corollary should be considered in relation to the study of the set of vertices belonging to all, to some, or to no minimum dominating sets started by Mynhardt [37]. This study followed the initiative by Hammer et al. for stable sets [22].

In 2004 Liang Sun [41] proved that:

Theorem 6.4. ([41]) *Every graph G such that $\gamma(G) = 3$ satisfies Vizing's conjecture.*

(The paper [5] proved that the inequality in Vizing's conjecture holds if both factors have domination number 3.) This, together with the fact that any graph with domination number at most 2 is a BG-graph, implies that any G with $\gamma(G) \leq 3$ satisfies the conjecture.

There are various relationships that exist between the classes of graphs that have been shown to satisfy Vizing's conjecture. For example, every BG-graph and every graph G with $\gamma(G) \leq \rho(G) + 1$ is of Type $\mathcal{X}$. Also, $\gamma(G) = \gamma^i(G) = \gamma_F(G)$ if G is chordal.

By combining results from the first part of this survey together with those derived in this section, we see that any minimal counterexample G must satisfy all of the following conditions.

- $\gamma(G) \geq 4$;
- G is not of Type $\mathcal{X}$;
- $\gamma_F(G) < \gamma(G)$;
- G is edge-critical and $\gamma(G_{uv}) < \gamma(G)$ for all pairs of vertices u, v in G;
- Every vertex of G belongs to a minimum dominating set.

It is not hard to construct a graph that satisfies all of the above conditions, except possibly that $\gamma_F(G) < \gamma(G)$ (the invariant γ_F is often difficult to compute). Indeed, this is how we produced the graph G in Figure 4. We observe that $\gamma(G) = 5$, the 2-packing number of G is 3 (implying that G is not of Type $\mathcal{X}$), and G is edge-critical. We also think that $\gamma_F(G) < 5$, although we were not able to verify it. Hence, according to the list of conditions above, G is a possible candidate for a counterexample to Vizing's conjecture. As shown in Section 3, this is not the case.

7. Additional Approaches

Attachable Sets. Suppose that a graph G_1 has a subset S_1 of vertices such that for every graph H it is the case that $|D| \geq \gamma(G_1)\gamma(H)$ whenever D is a subset of $V(G_1 \,\square\, H)$ that dominates $(V(G_1) - S_1) \times V(H)$. Then clearly G_1 satisfies Vizing's conjecture. In this case S_1 is called an *attachable set* of G_1. For example, Hartnell and Rall showed in [24] that any independent set of vertices in C_5 is an attachable set. In fact, the following result is proved.

Theorem 7.1. ([24]) *Cycles of the form C_{3k} and C_{3k+2} have attachable sets. No cycle of the form C_{3k+1} has an attachable set.*

Another way to get a graph G' with an attachable set is to start with any graph G that satisfies the conjecture and any vertex v of G that belongs to some minimum dominating set of G. Construct G' by adding to G a new vertex v', the edge vv', and any additional subset of edges that join v' to neighbors of v in G. Then $\{v'\}$ is an attachable set of G'. If $v'w \in E(G')$ such that $w \neq v$ and w belongs to a minimum dominating set of G, then the edge vv' can be removed from G'. The set $\{v'\}$ is attachable in the resulting graph. See [24] for additional details. This provides, for example, another way to see that complete bipartite graphs satisfy Vizing's conjecture.

The use of graphs with attachable sets is illustrated by the following construction. Suppose S_i is an attachable set of G_i for $i = 1, 2$. Let G be the graph built from the disjoint union of G_1 and G_2 by adding any subset of the edges that join a vertex in S_1

with a vertex in S_2. It is easy to show that G satisfies Vizing's conjecture, and, in fact, that $S_1 \cup S_2$ is an attachable set of G.

Degree Conditions on Pairs of Graphs. Recall that one says that a graph G satisfies Vizing's conjecture if inequality (1) holds for every graph H. The majority of known results on the conjecture gives classes of graph that satisfy the conjecture. Alternatively, one could also try to prove that the inequality holds for given **pairs of graphs**, an approach followed by Clark, Ismail and Suen in [13]. We give two of their results.

Theorem 7.2. ([13]) *Let G and H be d-regular graphs where $d \le 3$ or $d \ge 27$. Then $\gamma(G \,\Box\, H) \ge \gamma(G)\gamma(H)$.*

A natural question arising from the above theorem is:

Question 7.3. *Can one prove that all cubic graphs satisfy Vizing's conjecture?*

Theorem 7.4. ([13]) *Let G and H be graphs of order at most n, and let $\delta(G), \delta(H) \ge \sqrt{n}\ln n$. Then $\gamma(G \,\Box\, H) \ge \gamma(G)\gamma(H)$.*

The approach used in [13] is the following. Clearly,

$$\gamma(G \,\Box\, H) \ge \left\lceil \frac{|G \,\Box\, H|}{\Delta(G \,\Box\, H) + 1} \right\rceil .$$

Suppose we have a general upper bound on the domination number of an arbitrary graph in terms of its number of vertices, minimum and maximum degree, and that the product of such upper bounds for G and H is bounded above by $\lceil |G \,\Box\, H|/(\Delta(G \,\Box\, H) + 1)\rceil$. Then the inequality in Vizing's conjecture holds for the pair G, H. Upper bounds applied in this approach are the following well known bound due to Arnautov [3]

$$\gamma(G) \le n\frac{1 + \ln(\delta + 1)}{\delta + 1}$$

and its extension from [14].

Pairs that Attain Equality. The reason that Vizing's conjecture is so difficult lies also in the fact that it is hard to determine or bound the domination number of a graph, especially if it is not very small. As a consequence it is very difficult to verify that a counterexample has been found. Perhaps surprisingly, many classes of pairs of graphs for which the equality is achieved in (1) have been discovered; the complete list of these classes can be found in [23]. Let us just mention the following easy example. Let G be the corona of a graph G' (that is, the graph obtained from G' by attaching a leaf to each of its vertices), and let H be the 4-cycle with a and c as nonadjacent vertices. Then $D = \{(x, a) \,|\, x \in G'\} \cup \{(y, c) \,|\, y \in G - G'\}$ is a dominating set of $G \,\Box\, H$ with $|D| = |G|$. It is easy to see that D is a minimum dominating set of $G \,\Box\, H$ and that $\gamma(G) = |G|/2$. Hence we have $\gamma(G \,\Box\, C_4) = \gamma(G)\gamma(C_4)$. Additional results regarding equality in (1) can be found in [34].

On the other hand, there are graphs for which equality is never achieved in (1) as soon as the other factor is nontrivial [26, 27, 34]. For instance, any tree that has a vertex adjacent to at least two leaves has this property.

8. Vizing-Type Theorems for Related Domination Invariants

In this section, we survey versions of Vizing's conjecture for various domination-type invariants, including fractional, total, independent, and integer domination. In particular, Theorem 5.1 can be generalized in several ways.

Fractional Domination. A function $f\colon V(G) \to [0,1]$ defined on the vertices of a graph G is called a *fractional-dominating function* if the sum of its function values over any closed neighborhood is at least 1. The weight of a fractional-dominating function is the sum of its function values over all vertices. The *fractional domination number* of G, denoted $\gamma_f(G)$, is the minimum weight of a fractional-dominating function. Note that the characteristic function of a dominating set of G is a fractional-dominating function, and so $\gamma_f(G) \le \gamma(G)$. The fractional version of Vizing's Conjecture was established by Fisher, Ryan, Domke, and Majumdar [21].

Theorem 8.1. ([21]) *For any graphs G and H, $\gamma_f(G \,\Box\, H) \ge \gamma_f(G)\,\gamma_f(H)$.*

In 2001, Brešar [5] gave a straightforward proof of the related result, originally proved by Fisher:

Theorem 8.2. ([20]) *If G and H are connected graphs, then*

$$\gamma(G \,\Box\, H) \ge \gamma_f(G)\gamma(H)\,.$$

This theorem shows that Vizing's conjecture is satisfied by graphs (e.g., trees, strongly chordal) for which the fractional domination number and domination number are equal.

The proof technique of [5] involved the following concept. Let f be a function that assigns to each vertex v of G a subset (possibly empty) of $V(H)$. For each vertex $v \in V(G)$ we require

$$\left[\bigcup_{u\in f(v)} N_H[u]\right] \cup \left[\bigcup_{z\in N_G(v)} f(z)\right] = V(H)\,. \tag{8}$$

It is clear how each such function f corresponds to a dominating set of $G \,\Box\, H$ (one forms a dominating set for $G \,\Box\, H$ by taking the union of all subsets of the form $\{v\} \times f(v)$), and conversely. The *graph domination number of G with respect to H*, denoted $\gamma_H(G)$, is defined by

$$\gamma_H(G) = \min_f \left\{ \sum_{v\in V(G)} |f(v)| \right\},$$

where the minimum is taken over all functions f as defined above and satisfying equation (8).

Observation 8.3. ([5]) *A graph G satisfies Vizing's conjecture if and only if*

$$\gamma_H(G) \ge \gamma(G)\gamma(H)$$

for all graphs H.

Total Domination. A *total dominating set* of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. It was conjectured in [31] that the product of the total domination numbers of two graphs without isolated vertices is bounded above by twice the total domination number of their Cartesian product. This conjecture was solved by Ho [**?**].

Theorem 8.4. ([**?**]) *For any graphs G and H without isolated vertices, $\gamma_t(G \,\square\, H) \geq \frac{1}{2}\gamma_t(G)\gamma_t(H)$.*

In the case when at least one of G or H is a nontrivial tree, those graphs for which $\gamma_t(G \,\square\, H) = \frac{1}{2}\gamma_t(G)\gamma_t(H)$ are characterized in [31].

Theorem 8.5. ([31]) *Let G be a nontrivial tree and H any graph without isolated vertices. Then, $\gamma_t(G \,\square\, H) = \frac{1}{2}\gamma_t(G)\gamma_t(H)$ if and only if $\gamma_t(G) = 2\gamma(G)$ and H consists of disjoint copies of K_2.*

It remains, however, an open problem to characterize the graphs G and H that achieve equality in the bound of Theorem 8.4.

Brešar et al. [8] established the following result on the total domination number of the Cartesian product $G \,\square\, H$.

Theorem 8.6. ([8]) *For any graphs G and H without isolated vertices,*

$$\gamma_t(G \,\square\, H) \geq \max\{\gamma_t(G)\rho(H), \gamma_t(H)\rho(G)\}.$$

Integer Domination. For $k \geq 1$, a function $f\colon V(G) \to \{0, 1, \ldots, k\}$ defined on the vertices of a graph G is called a $\{k\}$-dominating function, abbreviated kDF, if the sum of its function values over any closed neighborhood is at least k [16]. The weight of a kDF is the sum of its function values over all vertices. The $\{k\}$-*domination number*, denoted $\gamma_{\{k\}}(G)$, of G is the minimum weight of a kDF. Note that the characteristic function of a dominating set of G is a $\{1\}$-dominating function, and so $\gamma_{\{1\}}(G) = \gamma(G)$. This type of domination is referred to as *integer domination*. It and fractional domination are related as follows.

Theorem 8.7. ([16]) *For any graph G, $\gamma_f(G) = \min_{k\in\mathbb{N}} \gamma_{\{k\}}(G)/k$.*

The simplest version of Vizing's conjecture $\gamma_{\{k\}}(G \,\square\, H) \geq \gamma_{\{k\}}(G)\gamma_{\{k\}}(H)$ for $\{k\}$-domination is trivially false, failing even for $G = H = K_1$. Rather, the natural version is obtained from normalizing the invariant by dividing it by k. Rearranged, this conjecture is:

Conjecture 8.8. ([32]) *For any $k \geq 1$ and graphs G and H,*

$$\gamma_{\{k\}}(G \,\square\, H) \geq \frac{1}{k}\gamma_{\{k\}}(G)\gamma_{\{k\}}(H).$$

A couple of partial results are known:

Theorem 8.9. ([7]) *For any graphs G and H,*

$$\gamma_{\{k\}}(G \,\Box\, H) \geq \frac{1}{k(k+1)}\gamma_{\{k\}}(G)\gamma_{\{k\}}(H).$$

It is not difficult to see that for any graph G, $\gamma_{\{k\}}(G) \leq k\gamma(G)$. In the following theorem we use the function ψ defined by

$$\psi(G, H) = \min\bigl\{|H|(k\gamma(G) - \gamma_{\{k\}}(G)),\ |G|(k\gamma(H) - \gamma_{\{k\}}(H))\bigr\}\,.$$

Note that $\psi(G, H)$ vanishes when $\gamma_{\{k\}}(G) = k\gamma(G)$, which is true, for example, if $\gamma(G) = \rho(G)$.

Theorem 8.10. ([7]) *For any graphs G and H,*

$$2k\gamma_{\{k\}}(G \,\Box\, H) + k\psi(G, H) \geq \gamma_{\{k\}}(G)\gamma_{\{k\}}(H).$$

When $k = 1$, Theorems 8.9 and 8.10 simplify to $\gamma(G \,\Box\, H) \geq \frac{1}{2}\gamma(G)\gamma(H)$, which is Theorem 5.1. The following questions from [7] remained unresolved, even though the second one is very weak:

Question 8.11. *Is it true that for any graphs G and H, $\gamma_{\{2\}}(G \,\Box\, H) \geq \gamma(G)\gamma(H)$? Failing which, is there some k such that $\gamma_{\{k\}}(G \,\Box\, H) \geq \gamma(G)\gamma(H)$ for any pair of graphs G and H?*

Theorem 8.4 can be extended to integer total domination. For $k \geq 1$ an integer, a function $f\colon V(G) \to \{0, 1, \ldots, k\}$ is a *total $\{k\}$-dominating function*, abbreviated TkDF, if the sum of its function values over any open neighborhood is at least k. The *total $\{k\}$-domination number*, denoted $\gamma_t^{\{k\}}(G)$, of G is the minimum weight of a TkDF. Note that the characteristic function of a total dominating set of G is a total $\{1\}$-dominating function, and so $\gamma_t^{\{1\}}(G) = \gamma_t(G)$. Total $\{k\}$-domination is also referred to as *integer total domination*.

The following version of Vizing's conjecture for the total $\{k\}$-domination number holds:

Theorem 8.12. ([35]) *For $k \geq 1$ an integer, and for any graphs G and H without isolated vertices,*

$$\gamma_t^{\{k\}}(G \,\Box\, H) \geq \frac{1}{k(k+1)}\gamma_t^{\{k\}}(G)\,\gamma_t^{\{k\}}(H).$$

When $k = 1$, Theorem 8.12 gives Theorem 8.4.

Paired-Domination. A *matching* in a graph G is a set of independent edges in G. A *perfect matching* M in G is a matching in G such that every vertex of G is incident to an edge of M. A *paired-dominating set*, abbreviated PDS, of a graph G is a dominating set S of G such that the subgraph $G[S]$ induced by S contains a perfect matching M (not necessarily induced). Every graph without isolated vertices has a PDS since the end-vertices of any maximal matching form such a set. The *paired-domination number* of G, denoted by $\gamma_{\mathrm{pr}}(G)$, is the minimum cardinality of a PDS; see [28, 29].

A version of Vizing's Conjecture for the paired-domination number is studied in [8]. For this purpose, recall that for $k \geq 2$, a *k-packing* in a graph G was defined by Meir and Moon [36] as a set S of vertices of G that are pairwise at distance greater than k apart, i.e., if $u, v \in S$, then $d_G(u, v) > k$. The *k-packing number* of G, denote $\rho_k(G)$, is the maximum cardinality of a k-packing in G. We have written ρ_2 as ρ. Brešar et al. [8] observed that it is not true that for every pair of graphs G and H, $\gamma_{\rm pr}(G \,\square\, H) \geq \max\{\gamma_{\rm pr}(G)\rho(H), \gamma_{\rm pr}(H)\rho(G)\}$. For example, let G be the graph obtained from K_4 by attaching to each vertex a leaf and let $H = C_9$. Then, $\rho(G) = 4$ and $\gamma_{\rm pr}(H) = 6$, whence $\gamma_{\rm pr}(G \,\square\, H) \leq 22 < 24 = \gamma_{\rm pr}(G)\rho(H)$. However, they observed that the 3-packing number related to the paired-domination number plays a similar role as the packing number related to the usual domination number.

Theorem 8.13. ([8]) *For any graphs G and H without isolated vertices,*

$$\gamma_{\rm pr}(G \,\square\, H) \geq \max\{\gamma_{\rm pr}(G)\rho_3(H), \gamma_{\rm pr}(H)\rho_3(G)\}.$$

It was shown in [8] that every nontrivial tree T has $\gamma_{\rm pr}(T) = 2\rho_3(T)$. Hence we have the following corollary of Theorem 8.13.

Theorem 8.14. ([8]) *Let T be a nontrivial tree and H any graph without isolated vertices. Then,*

$$\gamma_{\rm pr}(T \,\square\, H) \geq \frac{1}{2}\gamma_{\rm pr}(T)\gamma_{\rm pr}(H),$$

and this bound is sharp.

We remark that it is not true in general that for any graphs G and H without isolated vertices, $\gamma_{\rm pr}(G \,\square\, H) \geq 2\rho(G)\rho(H)$. For example, letting $G = H = P_4$, we have that $\gamma_{\rm pr}(G \,\square\, H) = 6$ while $\rho(P_4) = 2$, and so $\gamma_{\rm pr}(G \,\square\, H) < 2\rho(G)\rho(H)$. On the other hand:

Theorem 8.15. ([8]) *For any graphs G and H without isolated vertices,*

$$\gamma_{\rm pr}(G \,\square\, H) \geq 2\rho_3(G)\rho_3(H).$$

Upper Domination. The maximum cardinality of a minimal dominating set in a graph G is the *upper domination number* of G, denoted by $\Gamma(G)$. In 1996, Nowakowski and Rall [38] made the natural Vizing-like conjecture for the upper domination of Cartesian products of graphs. A proof was found by Brešar.

Theorem 8.16. ([6]) *For any graphs G and H, $\Gamma(G \,\square\, H) \geq \Gamma(G)\,\Gamma(H)$.*

The maximum cardinality of a minimal total dominating set of G is the *upper total domination number* of G, denoted by $\Gamma_t(G)$. A Vizing-like bound for the upper total domination number of Cartesian products of graphs was established by Dorbec et al. [17].

Theorem 8.17. ([17]) *Let G and H be connected graphs of order at least 3 with $\Gamma_t(G) \geq \Gamma_t(H)$. Then,*

$$\Gamma_t(G \,\square\, H) \geq \frac{1}{2}\Gamma_t(G)(\Gamma_t(H) + 1),$$

and this bound is sharp.

9. Stronger and Weaker Conjectures

In this section we give several conjectures and questions. Some of them are stronger than Vizing's conjecture meaning an affirmative answer would imply the conjecture, while others would follow from the truth of Vizing's conjecture.

The bg-Conjecture. The *clique cover number* $\Theta(G)$ of a graph G is the minimum number of complete subgraphs of G that cover $V(G)$. Note that $\Theta(G) = \chi(\overline{G})$, and $\gamma(G) \leq \Theta(G)$ since a dominating set for G can be formed by choosing a single vertex from each of the $\Theta(G)$ cliques that belong to the cover. Barcalkin and German's [4] decomposable graphs are those with $\gamma(G) = \Theta(G)$. Denote by $\mathcal{E}(G)$ the collection of all edge-critical graphs, G', such that G is a spanning subgraph of G' and $\gamma(G') = \gamma(G)$. Theorem 2.2 can now be stated using these invariants.

Theorem 9.1. *If there exists a graph $G' \in \mathcal{E}(G)$ such that $\Theta(G') = \gamma(G)$, then for every graph H, $\gamma(G \,\square\, H) \geq \gamma(G)\,\gamma(H)$.*

Of course, not all graphs satisfy the hypothesis of Theorem 2.2, but it suggests another conjecture whose truth would imply Vizing's conjecture. Let $\mathrm{bg}(G)$ denote the minimum value of $\Theta(G')$, where the minimum is taken over all $G' \in \mathcal{E}(G)$. Clearly, for any graph G, $\gamma(G) \leq \mathrm{bg}(G)$, and G satisfies the hypothesis of Theorem 2.2 if and only if these two invariants have the same value.

If we could show that $\gamma(G \,\square\, H) \geq \gamma(G)\,\mathrm{bg}(H)$, then Vizing's conjecture would follow. This inequality is not true, in general. This can be seen by letting $G = B_1$ and $H = B_2$ from Figure 2; $\gamma(B_1) = 3$, $\mathrm{bg}(B_1) = 4$, $\gamma(B_2) = 4$ and $\mathrm{bg}(B_2) = 6$. The set $(\{a, b, c, d\} \times \{1, 7\}) \cup (\{e, f, g, h\} \times \{2, 8\})$ dominates $B_1 \square B_2$ and thus, $\gamma(B_1 \square B_2) \leq 16 < \gamma(B_1)\,\mathrm{bg}(B_2)$.

However, the truth of the following conjecture would also establish Vizing's conjecture.

Conjecture 9.2. *For any pair of graphs G and H,*

$$\gamma(G \,\square\, H) \geq \min\{\,\mathrm{bg}(G)\gamma(H),\ \mathrm{bg}(H)\gamma(G)\,\}\,.$$

The same graph B_2 as above could possibly produce a counterexample to Conjecture 9.2 if one can show that $\gamma(B_2 \,\square\, B_2)$ is less than 24.

Rainbow Domination. Let G be a graph and let f be a function that assigns to each vertex a set of colors chosen from the set $\{1, \ldots, k\}$; that is, $f\colon V(G) \to \mathcal{P}(\{1, \ldots, k\})$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$, we have

$$\bigcup_{u \in N(v)} f(u) = \{1, \ldots, k\},$$

then f is called a *k-rainbow dominating function* (kRDF) of G. The *weight*, $w(f)$, of a function f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. Given a graph G, the minimum weight of a kRDF is called the *k-rainbow domination number of* G, which we denote by $\gamma_{\mathrm{r}k}(G)$. (A 1RDF is just a dominating set.)

Rainbow domination in a graph G has a natural connection with the study of $\gamma(G \square K_k)$. It is easy to verify the following equality.

Observation 9.3. ([9]) *For $k \geq 1$ and for any graph G, $\gamma_{\mathrm{r}k}(G) = \gamma(G \,\square\, K_k)$.*

The introduction of rainbow domination was motivated by the study of paired-domination in Cartesian products of graphs, where certain upper bounds can be expressed in terms of rainbow domination. The following innocent question posed in [9] remains open.

Question 9.4. *Is it true that for any graphs G and H, $\gamma_{\mathrm{r}2}(G \,\Box\, H) \geq \gamma(G)\gamma(H)$?*

Since $2\gamma(G \,\Box\, H) \geq \gamma_{\mathrm{r}2}(G \,\Box\, H)$, this conjecture is stronger than the result of Clark and Suen (Theorem 5.1), and since $\gamma_{\mathrm{r}2}(G \,\Box\, H) \geq \gamma(G \,\Box\, H)$ it is a consequence of Vizing's conjecture. Even if $\gamma_{\mathrm{r}2}$ is replaced by $\gamma_{\mathrm{r}k}$ for an arbitrary k in Question 9.4, we do not know how to prove the resulting inequality.

Independent Domination. There are a number of possible inequalities similar to Vizing's conjecture for independent domination number. Several stronger ones are false:

Observation 9.5. *There exist nontrivial graphs G and H such that $i(G \Box H) < i(G)\gamma(H)$.*

Proof. Let G be the graph of order 11 that is constructed from K_3 by adding 2 leaves adjacent to one vertex x of K_3, 3 leaves adjacent to a second vertex y of K_3 and 3 leaves adjacent to the third vertex z of K_3. It is clear that $i(G) = 6$. Let $H = \overline{G}$; then $\gamma(H) = 2 = i(H)$. However, $i(G \,\Box\, H) = 11$. $\Box$

This pair of graphs also shows that there are graphs with $\gamma(G \,\Box\, H) < i(G)\gamma(H)$, and $i(G \,\Box\, H) < i(G)i(H)$. Nevertheless, here is a related conjecture.

Conjecture 9.6. *For all graphs G and H,*

$$\gamma(G \,\Box\, H) \geq \min\{i(G)\gamma(H), i(H)\gamma(G)\}\,.$$

The truth of Conjecture 9.6 would imply Vizing's conjecture. On the other hand, the following conjecture

$$i(G \,\Box\, H) \geq \gamma(G)\gamma(H)$$

is a consequence of Vizing's conjecture. Perhaps this could be proven without first proving Vizing's conjecture.

Partition Conjecture. Vizing's conjecture would follow from the following conjecture.

Conjecture 9.7. *For any graph G, there exists a partition of $V(G)$ into $\gamma(G)$ sets $A_1, \ldots, A_{\gamma(G)}$ such that for any graph H, there is a minimum dominating set, D, of $G \Box H$ such that the projection $p_H(D \cap (A_i \times V(H)))$ dominates H for all i, $1 \leq i \leq \gamma(G)$.*

10. Conclusion - work in progress

Vizing's conjecture is that the domination number of the Cartesian product of graphs G and H is at least as large as the product of their domination numbers. Starting with the paper of Barcalkin and German [4], the inequality has been proven for all H and several families of G that admit a suitable partition, such as chordal graphs. In a different direction, a relaxed inequality was proven for all pairs by Clark and Suen [15]. In contrast, several similar bounds were established for other, related graph invariants. In this paper we provided additional properties of a minimal counterexample, if it exists, and improved bounds on claw-free graphs.

A common thread running through almost all the progress is to bound the size of a dominating set of $G \square H$ by partitioning it or projecting it and thereby relating it to dominating sets of G and H. It is unclear whether this approach will be able to prove the conjecture. On the other hand, a few researchers suspect that it might not be true after all, and base their doubt on the fact that the conjectured inequality is proven sharp for several rather different families of pairs of graphs (so there "should" also be pairs of graphs which contradict the conjecture...). Indeed, forty years later, Vizing's conjecture remains unresolved. Even partial results have proven difficult. It will be interesting to see what the next decades will bring.

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References

[1] R. Aharoni, E. Berger, and R. Ziv. A tree version of Kőnig's theorem. *Combinatorica*, 22(3):335–343, 2002.

[2] R. Aharoni and T. Szabó. Vizing's conjecture for chordal graphs. *Discrete Math.*, 309(6):1766–1768, 2009.

[3] V. I. Arnautov. Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices. *Prikl. Mat. i Programmirovanie*, (11):3–8, 126, 1974.

[4] A. M. Barcalkin and L. F. German. The external stability number of the Cartesian product of graphs. *Bul. Akad. Stiinte RSS Moldoven.*, (1):5–8, 94, 1979.

[5] B. Brešar. On Vizing's conjecture. *Discuss. Math. Graph Theory*, 21(1):5–11, 2001.

[6] B. Brešar. Vizing-like conjecture for the upper domination of Cartesian products of graphs — the proof. *Electron. J. Combin.*, 12:Note 12, 6 pp. (electronic), 2005.

[7] B. Brešar, M. A. Henning, and S. Klavžar. On integer domination in graphs and Vizing-like problems. *Taiwanese J. Math.*, 10(5):1317–1328, 2006.

[8] B. Brešar, M. A. Henning, and D. F. Rall. Paired-domination of Cartesian products of graphs. *Util. Math.*, 73:255–265, 2007.

[9] B. Brešar, M. A. Henning, and D. F. Rall. Rainbow domination in graphs. *Taiwanese J. Math.*, 12(1):213–225, 2008.

[10] B. Brešar and D. F. Rall. Fair reception and Vizing's conjecture. *J. Graph Theory*, 61:45–54, 2009.

[11] T. Burton and D. P. Summer. Domination dot-critical graphs. *Discrete Math.*, 306(1):11–18, 2006.

[12] G. Chen, W. Piotrowski, and W. Shreve. A partition approach to Vizing's conjecture. *J. Graph Theory*, 21(1):103–111, 1996.

[13] W. E. Clark, M. E. H. Ismail, and S. Suen. Application of upper and lower bounds for the domination number to Vizing's conjecture. *Ars Combin.*, 69:97–108, 2003.

[14] W. E. Clark, B. Shekhtman, S. Suen, and D. C. Fisher. Upper bounds for the domination number of a graph. *Congr. Numer.*, 132:99–123, 1998.

[15] W. E. Clark and S. Suen. An inequality related to Vizing's conjecture. *Electron. J. Combin.*, 7:Note 4, 3 pp., 2000.

[16] G. S. Domke, S. T. Hedetniemi, R. C. Laskar, and G. Fricke. Relationships between integer and fractional parameters of graphs. In *Graph theory, combinatorics, and applications, Vol. 1 (Kalamazoo, MI, 1988)*, pages 371–387. Wiley, New York, 1991.

[17] P. Dorbec, M. A. Henning, and D. F. Rall. On the upper total domination number of Cartesian products of graphs. *J. Comb. Optim.*, 16(1):68–80, 2008.

[18] M. El-Zahar and C. M. Pareek. Domination number of products of graphs. *Ars Combin.*, 31:223–227, 1991.

[19] R. J. Faudree, R. H. Schelp, and W. E. Shreve. The domination number for the product of graphs. *Congr. Numer.*, 79:29–33, 1990.

[20] D. C. Fisher. Domination, fractional domination, 2-packing, and graph products. *SIAM J. Discrete Math.*, 7(3):493–498, 1994.

[21] D. C. Fisher, J. Ryan, G. Domke, and A. Majumdar. Fractional domination of strong direct products. *Discrete Appl. Math.*, 50(1):89–91, 1994.

[22] P. L. Hammer, P. Hansen, and B. Simeone. Vertices belonging to all or to no maximum stable sets of a graph. *SIAM J. Algebraic Discrete Methods*, 3(4):511–522, 1982.

[23] B. Hartnell and D. F. Rall. Domination in Cartesian products: Vizing's conjecture. In *Domination in graphs*, volume 209 of *Monogr. Textbooks Pure Appl. Math.*, pages 163–189. Dekker, New York, 1998.

[24] B. L. Hartnell and D. F. Rall. On Vizing's conjecture. *Congr. Numer.*, 82:87–96, 1991.

[25] B. L. Hartnell and D. F. Rall. Vizing's conjecture and the one-half argument. *Discuss. Math. Graph Theory*, 15(2):205–216, 1995.

[26] B. L. Hartnell and D. F. Rall. Lower bounds for dominating Cartesian products. *J. Combin. Math. Combin. Comput.*, 31:219–226, 1999.

[27] B. L. Hartnell and D. F. Rall. Improving some bounds for dominating Cartesian products. *Discuss. Math. Graph Theory*, 23(2):261–272, 2003.

[28] T. W. Haynes and P. J. Slater. Paired-domination and the paired-domatic number. *Congr. Numer.*, 109:65–72, 1995.

[29] T. W. Haynes and P. J. Slater. Paired-domination in graphs. *Networks*, 32(3):199–206, 1998.

[30] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater. *Fundamentals of Domination in Graphs*. Marcel Dekker Inc., New York, 1998.

[31] M. A. Henning and D. F. Rall. On the total domination number of Cartesian products of graphs. *Graphs Combin.*, 21(1):63–69, 2005.

[32] X. Hou and Y. Lu. On the $\{k\}$-domination number of Cartesian products of graphs. *Discrete Math.*, 309:3413–3419, 2009.

[33] M. S. Jacobson and L. F. Kinch. On the domination of the products of graphs. II. Trees. *J. Graph Theory*, 10(1):97–106, 1986.

[34] S. M. Khamis and Kh. M. Nazzal. Equality in Vizing's conjecture fixing one factor of the Cartesian product. *Ars Combin.*, 96:375–384, 2010.

[35] N. Li and X. Hou. On the total $\{k\}$-domination number of Cartesian products of graphs. *J. Comb. Optim.*, 18(2):173–178, 2009.

[36] A. Meir and J. W. Moon. Relations between packing and covering numbers of a tree. *Pacific J. Math.*, 61(1):225–233, 1975.

[37] C. M. Mynhardt. Vertices contained in every minimum dominating set of a tree. *J. Graph Theory*, 31(3):163–177, 1999.

[38] R. J. Nowakowski and D. F. Rall. Associative graph products and their independence, domination and coloring numbers. *Discuss. Math. Graph Theory*, 16(1):53–79, 1996.

[39] S. Suen and J. Tarr. An improved inequality related to Vizing's conjecture. Manuscript, 2010.

[40] D. P. Sumner and E. Wojcicka. Graphs critical with respect to the domination number. In *Domination in Graphs*, volume 209 of *Monogr. Textbooks Pure Appl. Math.*, pages 439–469. Dekker, New York, 1998.

[41] L. Sun. A result on Vizing's conjecture. *Discrete Math.*, 275(1-3):363–366, 2004.

[42] V. G. Vizing. The cartesian product of graphs. *Vyčisl. Sistemy No.*, 9:30–43, 1963.

[43] V. G. Vizing. Some unsolved problems in graph theory. *Uspehi Mat. Nauk*, 23(6 (144)):117–134, 1968.

[44] Y. Wu. An improvement on Vizing's conjecture. Manuscript, 2009.

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