On the packing chromatic number of some lattices

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Abstract

For a positive integer k, a k-packing in a graph G is a subset A of vertices such that the distance between any two distinct vertices from A is more than k. The packing chromatic number of G is the smallest integer m such that the vertex set of G can be partitioned as $V_1, V_2, \ldots, V_m$ where V_i is an i-packing for each i. It is proved that the planar triangular lattice $\mathcal{T}$ and the 3-dimensional integer lattice $\mathbb{Z}^3$ do not have finite packing chromatic numbers.

Keywords: packing chromatic number; k-packing; integer lattice; triangular lattice

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1 Introduction

Let $G = (V, E)$ be a graph, finite or infinite, and let n be a positive integer. For a vertex x in G, the *ball of radius n centered at x* is the set, $B_n(x)$, of all vertices in V whose distance in G from x is no more than n. That is, $B_n(x) = \{v \in V(G) \,|\, d_G(x, v) \le n\}$. The *sphere of radius n centered at x* is the subset of the ball of radius n centered at x defined by $\partial B_n(x) = \{v \in B_n(x) \,|\, d_G(x, v) = n\}$. We will use $B_n(x)$ and $\partial B_n(x)$ to refer either to the set of vertices or to the subgraph of G induced by this set of vertices in G. The meaning will be clear from the context. Although we are dealing with infinite graphs, for a finite subgraph H of G, the order of H will be denoted $|H|$, and $E(H)$ will denote the set of edges in H.

For a positive integer r, a subset A of V is an *r-packing* if the distance in G between each pair of distinct vertices in A is more than r. The number r is called a *width* of the packing A. Note that if A is an r-packing and $r \ge 2$, then A is also an $(r-1)$-packing. An independent set is thus a 1-packing, and a 2-packing is a collection of vertices with pairwise disjoint closed neighborhoods. We

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are interested in partitioning the vertex set of a graph into the minimum number of packings, each having a distinguishing width. Specifically, the *packing chromatic number* of a graph G, denoted $\chi_\rho(G)$, is the smallest positive integer k for which there exists a map $c : V(G) \to \{1, 2, \ldots, k\}$ such that $V_r = c^{-1}(r)$ is an r-packing in G for each $1 \le r \le k$. Such a function c is called a *packing coloring* of G. This type of coloring, which is more restrictive than the usual proper coloring, is the only type of coloring we consider in this paper. Thus, for convenience we simply refer to a packing coloring as a coloring, and we use the term to refer either to c or to the accompanying induced partition $V_1, V_2, \ldots, V_k$ of $V(G)$. If no such coloring of G exists for any positive integer k, then we say G has *infinite packing chromatic number.*

The *Cartesian product* of two graphs G and H is the graph, $G \,\square\, H$, whose vertex set is the (set) Cartesian product $V(G) \times V(H)$. Two vertices of $G \,\square\, H$ are adjacent if they are the same in one coordinate and adjacent in the other coordinate. That is, the edge set of $G \,\square\, H$ consists of

$$\{[(g,h),(g,h')] \,|\, hh' \in E(H), g \in V(G)\} \,\cup\, \{[(g,h),(g',h)] \,|\, gg' \in E(G), h \in V(H)\}\,.$$

The *two-way infinite path* with the integers as vertex set will be denoted P_∞. Let $\mathbb{Z}^2$ denote the *planar integer lattice* (i.e., $\mathbb{Z}^2 = P_\infty \,\square\, P_\infty$) while $\mathbb{Z}^3 = P_\infty \,\square\, P_\infty \,\square\, P_\infty$ is the *3-dimensional integer lattice.* If all edges of the form $[(i,j),(i+1,j-1)]$ are added to the graph $\mathbb{Z}^2$ we obtain the *planar triangular lattice* denoted by $\mathcal{T}$.

The notion of the packing chromatic number was first introduced in [3] where it was called the broadcast chromatic number. Brešar et al. [1] chose the name packing chromatic number because it is both a partitioning (coloring) and a packing concept. Most of the results that have appeared about this invariant concern finite graphs. Not surprisingly, computing the packing chromatic number of an arbitrary graph is computationally difficult. Goddard et al. proved in [3] that deciding whether $\chi_\rho(G) \le 4$ is NP-hard for a finite simple graph G.

Concerning infinite graphs, not much is known about the packing chromatic number. It is easy to see that the two-way infinite path has packing chromatic number 3. The values of $\chi_\rho(P_n \,\square\, P_\infty)$ for $n = 2, 3, 4$ and 5 are 5, 7, 8 and 9 respectively, as shown in [3]. Also in [3] it was shown that for the planar integer lattice $\mathbb{Z}^2$, $9 \le \chi_\rho(\mathbb{Z}^2) \le 23$. The lower bound of 9 follows from the result for infinite grids of width 5 listed above, while the upper bound was constructive. In [5] Sloper showed that the infinite 3-regular tree has packing chromatic number 7. It was established in [1] that the infinite hexagonal lattice, $\mathcal{H}$, has packing chromatic number bounded between six and eight, inclusive. Vesel [6] showed by computer analysis that $\chi_\rho(\mathcal{H}) \ge 7$, and then a coloring of $\mathcal{H}$ using seven colors was discovered by Fiala and Lidicky [2], thus proving that $\chi_\rho(\mathcal{H}) = 7$.

The remainder of the paper is organized as follows. In the next section we show that the planar triangular lattice $\mathcal{T}$ admits no finite packing coloring. Finally, in Section 3 we prove that $\mathbb{Z}^3$ has no packing coloring that uses a finite number of colors.

2 Triangular Lattice

In this section we prove that $\mathcal{T}$ has no packing coloring using only a finite number of colors. Our approach is to show that for an arbitrary vertex x in $\mathcal{T}$, there is a large enough radius M such that it is impossible to partition $B_M(x)$ into sets $A_1, A_2, \ldots, A_{2p+1}$ where each A_i is an i-packing. Since $\chi_\rho(H) \le \chi_\rho(G)$ whenever H is a subgraph of G, the result for $\mathcal{T}$ follows immediately.

The proof of the following lemma is clear from the definition of $\mathcal{T}$.

Lemma 1 *If x is a vertex in $\mathcal{T}$ and n is a positive integer, then the subgraph of $\mathcal{T}$ induced by $\partial B_n(x)$ is isomorphic to the cycle C_{6n}. Thus, in particular, $\partial B_n(x)$ has $6n$ vertices and $6n$ edges.*

Lemma 2 *If x is a vertex in $\mathcal{T}$ and n is a positive integer, then $|B_n(x)| = 3n^2 + 3n + 1$.*

Proof. The graph $\mathcal{T}$ is regular of degree 6 and so $|B_1(x)| = 7$. Suppose the statement is true for $n = k$. We observe that $B_{k+1}(x) \setminus B_k(x) = \partial B_{k+1}(x)$. Therefore, by Lemma 1, $|B_{k+1}(x)| = |B_k(x)| + |\partial B_{k+1}(x)| = (3k^2 + 3k + 1) + 6(k + 1) = 3(k + 1)^2 + 3(k + 1) + 1$ and the result follows by induction. ∎

Lemma 3 *If x is a vertex in $\mathcal{T}$ and n is a positive integer, then the number of triangular faces in $B_n(x)$ is $6n^2$.*

Proof. The truth of the statement for $n = 1$ follows directly from the definition of $\mathcal{T}$. Suppose that the statement is true for $n = k$. By Lemma 1 the subgraph $\partial B_k(x)$ has $6k$ edges and $\partial B_{k+1}(x)$ has $6k + 6$ edges, each of which belongs to exactly one triangular face in $B_{k+1}(x) \setminus B_{k-1}(x)$. Also, each triangular face in the subgraph $B_{k+1}(x) \setminus B_{k-1}(x)$ has either one edge in $\partial B_k(x)$ or one edge in $\partial B_{k+1}(x)$, but not both. This implies that $B_{k+1}(x) \setminus B_{k-1}(x)$ has $12k + 6$ triangular faces, and hence $B_{k+1}(x)$ has $6k^2 + (12k + 6) = 6(k + 1)^2$ triangular faces, and the proof is complete. ∎

Lemma 4 *If x is a vertex in $\mathcal{T}$ and $\epsilon > 0$ is a real number, then there exists a positive integer M_0 such that whenever M is an integer with $M \geq M_0$ and whenever A is an independent set in $B_M(x)$, then $\frac{|A|}{|B_M(x)|} < \frac{1}{3} + \epsilon$.*

Proof. Let n be a positive integer and let A be an independent set in $B_n(x)$. Then for each $a \in A$, $T_a = B_1(a)$ is a planar graph with 6 triangular faces. Since A is independent, T_a and T_b can have no triangular face in common for $a \neq b$, a,$b \in A$.
Next observe that each face of any such T_a must be a face of $B_{n+1}(x)$. It follows that the total number of triangular faces in $\bigcup_{a \in A} T_a$ must not exceed the number of triangular faces in $B_{n+1}(x)$. Thus by Lemma 3, we have $6|A| \leq 6(n+1)^2$. On the other hand, by Lemma 2, we have $|B_n(x)| = 3n^2+3n+1$. Hence $\frac{|A|}{|B_n(x)|} \leq \frac{(n+1)^2}{3n^2+3n+1}$. The conclusion of the lemma now follows from the fact that

$$\lim_{n\to\infty} \frac{(n + 1)^2}{3n^2 + 3n + 1} = \frac{1}{3}\,.$$

∎

Lemma 5 *Let x be any vertex of $\mathcal{T}$ and let r be a positive integer. There exists a positive integer M_r, such that whenever $M \geq M_r$ and $\{B_r(a) \,|\, a \in A\}$ is a collection of pairwise disjoint subsets for $A \subseteq V(B_M(x))$ it follows that $\frac{|A|}{|B_M(x)|} < \frac{1}{3r(r+1)}$.*

Proof. Let n and r be positive integers and let $A \subseteq B_n(x)$ such that $\{B_r(a) \,|\, a \in A\}$ is pairwise disjoint. Observe that $\bigcup_{a\in A} B_r(a) \subseteq B_{n+r}(x)$, and hence $|\bigcup_{a\in A} B_r(a)| \leq |B_{n+r}(x)|$. Then Lemma 2 implies that $|A|(3r^2 + 3r + 1) \leq 3(n + r)^2 + 3(n + r) + 1$, and hence it follows that

$$\frac{|A|}{|B_n(x)|} \leq \frac{3(n + r)^2 + 3(n + r) + 1}{(3r^2 + 3r + 1)(3n^2 + 3n + 1)}\,.$$

But

$$\lim_{n\to\infty} \frac{3(n+r)^2+3(n+r)+1}{(3r^2+3r+1)(3n^2+3n+1)} = \frac{1}{3r^2+3r+1} < \frac{1}{3r(r+1)}\,,$$

and therefore the conclusion of the lemma follows. ∎

Theorem 6 *The planar triangular lattice $\mathcal{T}$ has infinite packing chromatic number.*

Proof. Suppose for a contradiction that for some positive integer p, $\mathcal{T}$ has a packing coloring with a range contained in the set $I = \{1, 2, \ldots, 2p+1\}$. Choose a fixed vertex x in $\mathcal{T}$ and for each $i \in I$, let A_i be the $i-$packing in $\mathcal{T}$ consisting of all the vertices in $\mathcal{T}$ labeled i by this coloring.
Now observe that for any $i \in I$, $i > 1$, if a and b are in A_i and $a \neq b$ then we have $B_{r_i}(a) \cap B_{r_i}(b) = \emptyset$ where $r_i = \lfloor \frac{i}{2} \rfloor$. Thus by Lemma 5, there is a positive integer M_i such that if $M \geq M_i$, then $\frac{|A_i \cap B_M(x)|}{|B_M(x)|} < \frac{1}{3r_i(r_i+1)}$. Furthermore, by Lemma 3, there is an integer M_1 such that if $M \geq M_1$ then $\frac{|A_1 \cap B_M(x)|}{|B_M(x)|} < \frac{1}{3} + \frac{1}{3(p+1)}$.
Set $K = \max_{i\in I}\{M_i\}$. Since we are dealing with a packing coloring of the whole of $\mathcal{T}$, the fraction f of labeled vertices in $B_K(x)$ is equal to 1. On the other hand we can strictly bound this fraction as follows,

$$f = \sum_{i=1}^{2p+1} \frac{|A_i \cap B_K(x)|}{|B_K(x)|} < \frac{1}{3} + \frac{1}{3(p+1)} + \sum_{i=2}^{2p+1} \frac{1}{3r_i(r_i+1)}. \tag{1}$$

However, recalling the definition of r_i, we see that

$$\sum_{i=2}^{2p+1} \frac{1}{3r_i(r_i+1)} = 2\sum_{i=1}^{p} \frac{1}{3i(i+1)} = \frac{2}{3}\sum_{i=1}^{p}\left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{2}{3}\left(1 - \frac{1}{p+1}\right). \tag{2}$$

By combining inequality (1) and equation (2) we infer that $f < 1$, a contradiction. ∎

3 Three Dimensional Integer Lattice

In this section we investigate chromatic packings of $\mathbb{Z}^3$ and prove that the three dimensional integer lattice has infinite packing chromatic number. The proof of the first lemma is straightforward and is omitted.

Lemma 7 *If x is a vertex in $\mathbb{Z}^3$ and n is a positive integer, then $\partial B_n(x)$ is isomorphic to the complement of K_{4n^2+2}. In particular, $|\partial B_n(x)| = 4n^2+2$ and $|E(\partial B_n(x))| = 0$.*

Lemma 8 *If x is a vertex in $\mathbb{Z}^3$ and n is a positive integer then $|B_n(x)| = \frac{4}{3}n^3 + 2n^2 + \frac{8}{3}n + 1$.*

Proof. For $n = 1$ this is easy to check.
Suppose that the result is true for $n = k$. We observe that $B_{k+1}(x) \setminus B_k(x) = \partial B_{k+1}(x)$. Therefore,

$$\begin{aligned} |B_{k+1}(x)| &= |B_k(x)| + |\partial B_{k+1}(x)| \\ &= (\frac{4}{3}k^3 + 2k^2 + \frac{8}{3}k + 1) + (4(k+1)^2 + 2) \\ &= \frac{4}{3}(k+1)^3 + 2(k+1)^2 + \frac{8}{3}(k+1) + 1, \end{aligned}$$

and the proof is complete. ∎

In addition to using the fact that certain balls in $\mathbb{Z}^3$ are vertex-disjoint, as we did when studying $\mathcal{T}$ in Section 2, our proof technique here will also use the fact that certain odd radius balls in $\mathbb{Z}^3$ are edge-disjoint although they may share vertices. For example, consider the path P of order eleven whose vertices are $v_1, v_2, \ldots, v_{11}$ in the natural order. The set $\{v_1, v_{11}\}$ is a 9-packing in P. The balls of radius 5, $B_5(v_1)$ and $B_5(v_{11})$, share the vertex v_6 but they have no edges in common. The following result makes this precise in the case of bipartite graphs.

Lemma 9 *Let G be a bipartite graph and let $r = 2k+1$ be a positive odd integer. If A is any r-packing of G and x and y are distinct vertices in A, then the induced subgraphs $B_{k+1}(x)$ and $B_{k+1}(y)$ have no edges in common.*

Proof. Suppose the edge $e = ab$ belongs to both induced subgraphs, $B_{k+1}(x)$ and $B_{k+1}(y)$. By definition, $d_G(x,a) \le k+1$, $d_G(x,b) \le k+1$, $d_G(y,a) \le k+1$, and $d_G(y,b) \le k+1$. Since G is bipartite, $d_G(x,a) \ne d_G(x,b)$ and $d_G(y,a) \ne d_G(y,b)$. We may assume that $d_G(x,a) < d_G(x,b)$, and hence $d_G(x,a) \le k$. Therefore,

$$d_G(x,y) \le d_G(x,a) + d_G(a,y) \le 2k+1 = r\,,$$

contrary to the fact that A is an r-packing. ∎

The next result counts the number of edges in a ball of radius n in $\mathbb{Z}^3$.

Lemma 10 *If x is a vertex in $\mathbb{Z}^3$ and n is a positive integer, then the number of edges in the induced subgraph $B_n(x)$ is $4n^3 + 2n$.*

Proof. The statement is clearly true for $n = 1$. Suppose it is true for $n = k$. The subgraph $\partial B_k(x)$ has $4k^2 + 2$ vertices by Lemma 7. Considering these vertices in the induced subgraph $B_k(x)$ we see that six are incident with one edge, $12k - 12$ are incident with two edges, and $4k^2 - 12k + 8$ are incident with three edges of $B_k(x)$. But $\mathbb{Z}^3$ is regular of degree six, and each edge that belongs to the subgraph $B_{k+1}(x)$ but not to $B_k(x)$ is incident with exactly one vertex of the boundary $\partial B_k(x)$. Therefore, by the inductive hypothesis the total number of edges in $B_{k+1}(x)$ is

$$4k^3 + 2k + 6\cdot 5 + (12k-12)4 + (4k^2 - 12k + 8)3 = 4(k+1)^3 + 2(k+1)\,,$$

thus completing the induction step. ∎

Lemma 11 *Let x be a vertex in $\mathbb{Z}^3$, let $\epsilon > 0$ be a real number and let r be a positive integer. There exists a positive integer M_r such that for every integer $M \ge M_r$, if $A \subseteq V(B_M(x))$ has the property that $\{B_r(a) \mid a \in A\}$ is a collection of pairwise disjoint subsets of $V(\mathbb{Z}^3)$, then*

$$\frac{|A|}{|B_M(x)|} < \frac{3}{4r^3 + 6r^2 + 8r + 3} + \frac{\epsilon}{2^{r+1}}\,.$$

Proof. Let n be a positive integer and let $A \subseteq V(B_n(x))$ such that $\{B_r(a) \,|\, a \in A\}$ is a family of pairwise disjoint subsets of $V(\mathbb{Z}^3)$. Observe that $\bigcup_{a \in A} B_r(a) \subseteq B_{n+r}(x)$, and hence $|\bigcup_{a \in A} B_r(a)| \leq |B_{n+r}(x)|$. Then Lemma 8 implies that

$$|A|(4r^3 + 6r^2 + 8r + 3) \leq (4(n+r)^3 + 6(n+r)^2 + 8(n+r) + 3)\,,$$

and so we obtain

$$\frac{|A|}{|B_n(x)|} \leq \frac{4(n+r)^3 + 6(n+r)^2 + 8(n+r) + 3}{(4r^3 + 6r^2 + 8r + 3)\frac{1}{3}(4n^3 + 6n^2 + 8n + 3)}\,.$$

But, since r is fixed, the conclusion of the lemma follows from the fact that

$$\lim_{n \to \infty} \frac{4(n+r)^3 + 6(n+r)^2 + 8(n+r) + 3}{(4r^3 + 6r^2 + 8r + 3)\frac{1}{3}(4n^3 + 6n^2 + 8n + 3)} = \frac{3}{4r^3 + 6r^2 + 8r + 3}\,.$$

∎

Lemma 12 *Let x be a vertex in $\mathbb{Z}^3$, let $\epsilon > 0$ be a real number and let r be a positive integer. There exists a positive integer M_r such that for every integer $M \geq M_r$, if $A \subseteq V(B_M(x))$ has the property that $\{E(B_r(a)) \,|\, a \in A\}$ is a family of pairwise disjoint subsets of $E(\mathbb{Z}^3)$, then*

$$\frac{|A|}{|B_M(x)|} < \frac{3}{4r^3 + 2r} + \frac{\epsilon}{2^{r+1}}\,.$$

Proof. Let n be a positive integer and let $A \subseteq V(B_n(x))$ be such that the family of subsets $\{E(B_r(a)) | a \in A\}$ of $E(\mathbb{Z}^3)$ is pairwise disjoint. Following the same line of reasoning as in the proof of Lemma 11 and using Lemma 10, we deduce $|A|(4r^3 + 2r) \leq \big(4(n+r)^3 + 2(n+r)\big)$ and hence

$$\frac{|A|}{|B_n(x)|} \leq \frac{4(n+r)^3 + 2(n+r)}{(4r^3 + 2r)\frac{1}{3}(4n^3 + 6n^2 + 8n + 3)}\,.$$

Similar to before, the conclusion is now implied by

$$\lim_{n \to \infty} \frac{4(n+r)^3 + 2(n+r)}{(4r^3 + 2r)\frac{1}{3}(4n^3 + 6n^2 + 8n + 3)} = \frac{3}{4r^3 + 2r}\,.$$

∎

Theorem 13 *The three dimensional integer lattice $\mathbb{Z}^3$ has infinite packing chromatic number.*

Proof. Suppose for a contradiction that for some positive integer p, there is a packing coloring c of $\mathbb{Z}^3$ whose range is contained in the set $I = \{1, 2, \ldots, 2p\}$. Choose a fixed vertex x in $\mathbb{Z}^3$ and for each $i \in I$, let A_i be the $i-$packing in $\mathbb{Z}^3$ consisting of all the vertices in $\mathbb{Z}^3$ labeled i by c. Let $\epsilon = \frac{388}{8400}$. Now observe that for any even integer $i \in I$, if a and b are in A_i and $a \neq b$, then we have $B_{r_i}(a) \cap B_{r_i}(b) = \emptyset$ where $r_i = \frac{i}{2}$. Thus by Lemma 11, there is an integer M_i such that if M is an integer with $M \geq M_i$, then

$$\frac{|A_i \cap B_M(x)|}{|B_M(x)|} < \frac{3}{4r_i^3 + 6r_i^2 + 8r_i + 3} + \frac{\epsilon}{2^{r_i+1}}\,.$$

Furthermore, for any odd integer $i \in I$, if a and b are in A_i and $a \neq b$ then by Lemma 9 we see that $E(B_{r_i}(a)) \cap E(B_{r_i}(b)) = \emptyset$ where $r_i = \frac{i+1}{2}$. Thus, by Lemma 12, there is an integer M_i such that if M is an integer with $M \geq M_i$, then

$$\frac{|A_i \cap B_M(x)|}{|B_M(x)|} < \frac{3}{4r_i^3 + 2r_i} + \frac{\epsilon}{2^{r_i+1}}\,.$$

Set $K = \max_{i \in I}\{M_i\}$. Since we are dealing with a packing coloring of the whole of $\mathbb{Z}^3$, the fraction f of labeled vertices in $B_K(x)$ is equal to 1. On the other hand we can strictly bound this fraction as follows.

$$\begin{aligned}
f = \sum_{i=1}^{2p} \frac{|A_i \cap B_K(x)|}{|B_K(x)|} \quad &= \quad \sum_{i=1}^{p} \frac{|A_{2i} \cap B_K(x)|}{|B_K(x)|} + \sum_{i=1}^{p} \frac{|A_{2i-1} \cap B_K(x)|}{|B_K(x)|} \\
&< \quad \sum_{i=1}^{p} \left(\frac{3}{4i^3 + 6i^2 + 8i + 3} + \frac{\epsilon}{2^{i+1}}\right) + \sum_{i=1}^{p} \left(\frac{3}{4i^3 + 2i} + \frac{\epsilon}{2^{i+1}}\right) \\
&< \quad \sum_{i=1}^{\infty} \left(\frac{3}{4i^3 + 6i^2 + 8i + 3} + \frac{\epsilon}{2^{i+1}}\right) + \sum_{i=1}^{\infty} \left(\frac{3}{4i^3 + 2i} + \frac{\epsilon}{2^{i+1}}\right) \\
&< \quad \sum_{i=1}^{2} \frac{3}{4i^3 + 6i^2 + 8i + 3} + \sum_{i=1}^{2} \frac{3}{4i^3 + 2i} + 2\sum_{i=3}^{\infty} \frac{3}{4i^3} + 2\sum_{i=1}^{\infty} \frac{\epsilon}{2^{i+1}} \\
&< \quad \sum_{i=1}^{2} \frac{3}{4i^3 + 6i^2 + 8i + 3} + \sum_{i=1}^{2} \frac{3}{4i^3 + 2i} + \int_2^{\infty} \frac{3dt}{2t^3} + \epsilon \\
&= \quad \left(\frac{1}{7} + \frac{1}{25}\right) + \left(\frac{1}{2} + \frac{1}{12}\right) + \frac{3}{16} + \frac{388}{8400} = \frac{8399}{8400} < 1,
\end{aligned}$$

a contradiction.

∎

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