Packing chromatic vertex-critical graphs

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Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that the vertex set of G can be partitioned into sets V_i, $i \in [k]$, where vertices in V_i are pairwise at distance at least $i+1$. Packing chromatic vertex-critical graphs, χ_ρ-critical for short, are introduced as the graphs G for which $\chi_\rho(G-x) < \chi_\rho(G)$ holds for every vertex x of G. If $\chi_\rho(G) = k$, then G is k-χ_ρ-critical. It is shown that if G is χ_ρ-critical, then the set $\{\chi_\rho(G)-\chi_\rho(G-x) :\ x \in V(G)\}$ can be almost arbitrary. The 3-χ_ρ-critical graphs are characterized, and 4-χ_ρ-critical graphs are characterized in the case when they contain a cycle of length at least 5 which is not congruent to 0 modulo 4. It is shown that for every integer $k \geq 2$ there exists a k-χ_ρ-critical tree and that a k-χ_ρ-critical caterpillar exists if and only if $k \leq 7$. Cartesian products are also considered and in particular it is proved that if G and H are vertex-transitive graphs and $\mathrm{diam}(G) + \mathrm{diam}(H) \leq \chi_\rho(G)$, then $G \,\square\, H$ is χ_ρ-critical.

Key words: packing chromatic number; packing chromatic vertex-critical graph; tree; caterpillar; Cartesian product of graphs, vertex-transitive graph

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1 Introduction

Given a graph G and a positive integer i, an *i-packing* in G is a subset W of the vertex set of G such that the distance between any two distinct vertices from W is greater than i. Note that in this terminology independent sets are precisely 1-packings. The *packing chromatic number* of G is the smallest integer k such that the vertex set of G can be partitioned into sets $V_1, \ldots, V_k$, where V_i is an i-packing for each $i \in [k] = \{1, \ldots, k\}$. This invariant is well defined on any graph G and is denoted $\chi_\rho(G)$.

More generally, for a nondecreasing sequence $S = (s_1, \dots, s_k)$ of positive integers, the mapping $c : V(G) \longrightarrow [k]$ is an *S-packing coloring* if for any i in $[k]$ the set $c^{-1}(i)$ is an s_i-packing [12].

The packing chromatic number was introduced in [11] under the name broadcast chromatic number; the present name is used since [5]. Among the many developments on the packing chromatic number we point out the very exciting development on the class of subcubic graphs. The question whether in subcubic graphs the packing chromatic number is bounded by an absolute constant was asked in the seminal paper [11]. Gastineau and Togni were the first to find an explicit cubic graph G with large packing chromatic number, more precisely $\chi_\rho(G) = 13$, and asked whether 13 is an actual upper bound for χ_ρ in the class of cubic graphs [9]. That this is not the case was first demonstrated in [6]. Then, in a breakthrough result, Balogh, Kostochka and Liu [1] proved that almost every n-vertex cubic graph of girth at least g has the packing chromatic number greater than k. Brešar and Ferme complemented the result with an explicit infinite family of subcubic graphs with unbounded packing chromatic number [3]. The packing chromatic number of subcubic outerplanar graphs has been thoroughly investigated in [4, 10], while for the packing chromatic number of some additional classes of cubic graphs see [17]. From the other investigations on the packing chromatic number we emphasize that the decision version of the packing chromatic number is NP-complete on trees [8] and that the packing chromatic number of the infinite square lattice is from the set $\{13, 14, 15\}$ [2].

An important consideration when investigating a graph invariant is to understand critical graphs with respect to the invariant. The most prominent example is probably the concept of vertex-critical graphs with respect to the chromatic number; that is, graphs G such that $\chi(G - x) < \chi(G)$ for every vertex x of G, cf. [16]. In this paper we study vertex-critical graphs with respect to the packing chromatic number. The importance of this idea arose during the study of the relationship between the packing chromatic number, chromatic number, and clique number in [7]. We say that a graph G is *packing chromatic vertex-critical*, *χ_ρ-critical* for short, if $\chi_\rho(G - x) < \chi_\rho(G)$ holds for every vertex x of G. If $\chi_\rho(G) = k$, then we also say that G is *k-χ_ρ-critical* (or more simply k-critical). To further simplify the terminology we will typically say a *critical graph* when considering a packing chromatic vertex-critical graph.

We proceed as follows. In the next section some definitions are given, known results recalled, and new preliminary results proved. In Section 3 we demonstrate that if G is a χ_ρ-critical graph, then the set $\{\chi_\rho(G) - \chi_\rho(G - x) :\ x \in V(G)\}$ can be almost arbitrary. In the subsequent section we characterize 3-critical graphs and do the same for 4-critical graphs that contain a cycle of length at least 5 and not congruent to 0 modulo 4. In the case that graphs contain cycles congruent to 0 modulo 4, a partial characterization of 4-critical graphs is given along with some infinite families of such graphs. In Section 5 we concentrate on critical trees and first show that for every integer $k \geq 2$ there exists a k-χ_ρ-critical tree. In the main result of the section we prove that a k-χ_ρ-critical caterpillar exists if and only if $k \leq 7$. We follow with a section on Cartesian products which turned out to yield many critical graphs. For instance, $K_{1,3} \,\square\, P_3$ is 6-χ_ρ-critical even though neither $K_{1,3}$ nor P_3 is critical. In the main result of the section we prove that if G and H are vertex-transitive graphs and $\mathrm{diam}(G) + \mathrm{diam}(H) \le \chi_\rho(G)$, then $G \,\square\, H$ is χ_ρ-critical. We conclude the paper with several open problems and directions for future investigation.

2 Preliminaries

A *support vertex* of a graph is a vertex adjacent to a vertex of degree 1, that is, to a *leaf*. By the *distance* between two vertices of a graph we mean the usual shortest-path distance. The diameter, $\mathrm{diam}(G)$, of a graph G is the maximum distance between all pairs of vertices of G. If H is a subgraph of a graph G, then for distinct vertices u and v in H, the distance between u and v in H is at least the distance between u and v in G. Hence, if W is an i-packing in G, it follows that $W \cap V(H)$ is an i-packing in H. This gives the following obvious but useful lemma.

Lemma 2.1 *If H is a subgraph of a graph G, then $\chi_\rho(G) \ge \chi_\rho(H)$.*

In any packing coloring of a graph with diameter d, each color d or larger can be assigned to at most one vertex. The following is a direct consequence of this fact.

Observation 2.2 *Suppose G is a graph with $\mathrm{diam}(G) = d \le k = \chi_\rho(G)$. If $c : V(G) \longrightarrow [k]$ is a packing coloring and x is a vertex of G such that $d \le c(x) \le k$, then $\chi_\rho(G - x) < \chi_\rho(G)$.*

The following result from the seminal paper will be used several times.

Proposition 2.3 ([11, Proposition 2.1]) *If G is a graph with order $n(G)$, then $\chi_\rho(G) \le n(G) - \alpha(G) + 1$, with equality if $\mathrm{diam}(G) = 2$.*

If x is a vertex of a graph G, then the difference between $\chi_\rho(G) - \chi_\rho(G - x)$ can be made arbitrary large, see [6]. The situation is different for leafs as the next observation asserts.

Lemma 2.4 *If x is a leaf of a graph G, then $\chi_\rho(G) - 1 \le \chi_\rho(G - x) \le \chi_\rho(G)$.*

Proof. Clearly, $\chi_\rho(G - x) \le \chi_\rho(G)$. Suppose now that $\chi_\rho(G - x) = k$. Then using an optimal packing coloring of $G - x$ and using color $k + 1$ for the vertex x in G we infer that $\chi_\rho(G) \le k + 1$. Hence $\chi_\rho(G) \le k + 1 = \chi_\rho(G - x) + 1$. $\square$

We next list some general properties of critical graphs.

Lemma 2.5 *If G is a χ_ρ-critical graph, then G is connected.*

Proof. Suppose G has components $A_1, \ldots, A_r$. Since $\chi_\rho(G) = \max\{\chi_\rho(A_i)\}$, there exists a component, say A_j, that has packing chromatic number $\chi_\rho(G)$. If $r \geq 2$ and x is a vertex in $G - A_j$, then it is clear that $\chi_\rho(G - x) = \chi_\rho(A_j) = \chi_\rho(G)$, which contradicts the assumption that G is χ_ρ-critical. Therefore, $r = 1$ and hence G is connected. □

If a graph is k-critical and some of its edges can be removed without decreasing the packing chromatic number, then this edge-reduced graph is also k-critical as the next result shows.

Proposition 2.6 *Let k be a positive integer and let G be a k-critical graph. If F is any subset of the edges of G such that $\chi_\rho(G - F) = k$, then $G - F$ is also k-critical.*

Proof. Let x be any vertex in a k-critical graph G and let F be any set of edges of G such that $\chi_\rho(G - F) = k$. If F' is that subset of edges in F that are not incident with x, then $(G - F) - x = (G - x) - F'$, and $(G - x) - F'$ is a subgraph of $G - x$. Since G is k-critical, it now follows from Lemma 2.1 that

$$\chi_\rho((G - F) - x) = \chi_\rho((G - x) - F') \leq \chi_\rho(G - x) < k\,,$$

and therefore $G - F$ is k-critical. □

The following corollary is a special case of Proposition 2.6, but we state it anyway for its later use.

Corollary 2.7 *Let H be a graph with $\chi_\rho(H) = k$ and let u and v be nonadjacent vertices in H. If H is not k-critical, then $H + uv$ is not k-critical.*

Finally, knowing the packing chromatic number of paths and cycles, we get that C_n is critical if and only if $n \in \{3, 4\}$, or $n \geq 5$ and $n \not\equiv 0 \pmod 4$. It is also easy to verify that if $n \geq 2$, then $K_{n,n}$ is an $(n+1)$-χ_ρ-critical graph.

3 Vertex-deleted subgraphs of χ_ρ-critical graphs

In this section we show that the condition $\chi_\rho(G - x) < \chi_\rho(G)$ for G to be χ_ρ-critical cannot be replaced with $\chi_\rho(G - x) = \chi_\rho(G) - 1$ (as it is the case with the chromatic critical graphs). Moreover, we show that if G is a χ_ρ-critical graph, then the set of differences

$$\Delta_{\chi_\rho}(G) = \{\chi_\rho(G) - \chi_\rho(G - x) :\ x \in V(G)\}$$

can be almost arbitrary. Before stating the main result, we consider three particular examples.

Clearly, if G is vertex-transitive and critical, then $|\Delta_{\chi_\rho}(G)| = 1$. For instance, $\chi_\rho(K_{r,r} - w) = \chi_\rho(K_{r,r}) - 1$ for every w and every $r \geq 1$, hence $\Delta_{\chi_\rho}(K_{r,r}) = \{1\}$.

Consider next the Petersen graph P. Since $\mathrm{diam}(P) = 2$, Proposition 2.3 implies that $\chi_\rho(P) = 7$ and in $P - x$ there are three vertices pairwise at distance 3 that can be colored with 2. From here a 5-χ_ρ-coloring of $P - x$ is easy to obtain. In conclusion, $\chi_\rho(P - x) = \chi_\rho(P) - 2$ for every vertex x and thus $\Delta_{\chi_\rho}(P) = \{2\}$.

Next, let $2 \le r \le s$, and let G be the graph constructed from identifying a vertex from K_r with a vertex from K_s. The packing chromatic number of G is $r + s - 2$. If w is the vertex shared by the two complete graphs, then $\chi_\rho(G - w) = s - 1$. For any other vertex x in G, we have $\chi_\rho(G - x) = r + s - 3$. Hence $\Delta_{\chi_\rho}(G) = \{1, r - 1\}$.

Theorem 3.1 *Let $S = \{1, s_1, \ldots, s_r\}$, be a set of positive integers. If for every $i \in [r]$ we have $\displaystyle\sum_{j=1, j\neq i}^{r} s_j \ge s_i - 1$, then there exists a χ_ρ-critical graph G such that $\Delta_{\chi_\rho}(G) = S$.*

Proof. Suppose first that $r \ge 2$. Let $V(K_r) = \{x_1, \ldots, x_r\}$. Let $G(s_1, \ldots, s_r)$ be the graph obtained from K_r such that for every $i \in [r]$, a vertex of a complete graph X_i of order $s_i + 2$ is identified with x_i. See Fig. 1 for an example of this construction.

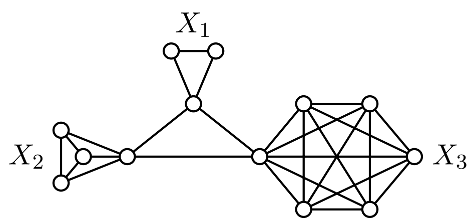

Figure 1: The graph $G(1, 2, 4)$.

To simplify the notation, set $G = G(s_1, \ldots, s_r)$ in the sequel. Note first that

$$|V(G)| = \sum_{i=1}^{r} |V(X_i)| = \sum_{i=1}^{r} (s_i + 2) = \sum_{i=1}^{r} s_i + 2r\,.$$

If c is a χ_ρ-coloring of G, then the vertices of X_i, $i \in [r]$, receive different colors. In particular, $|c^{-1}(1)| \le r$ and $|c^{-1}(2)| \le r$. Moreover, since $\mathrm{diam}(G) = 3$, $|c^{-1}(\ell)| \le 1$ for any $\ell \ge 3$. Since $s_i \ge 2$ and so $s_i + 2 \ge 4$, $i \in [r]$, in each X_i colors 1 and 2 can be used. It follows that

$$\chi_\rho(G) = 2 + (|V(G)| - 2r) = 2 + \sum_{i=1}^{r} s_i\,. \tag{1}$$

Since $r \ge 2$, for at least one s_i we have $s_i \ge 3$. Assume without loss of generality that $s_1 \ge 3$. Let $u \in X_1$ be an arbitrary vertex different from x_1. Then $G - u$ is isomorphic to $G(s_1-1, s_2, \ldots, s_r)$ (where it is possible that $s_1 - 1 = s_i$ for some $i \geq 2$). Then by (1) we get $\chi_\rho(G-u) = 2 + (s_1-1) + \sum_{i=2}^r s_i = \sum_{i=1}^r s_i + 1 = \chi_\rho(G) - 1$. This shows that $1 \in \Delta_{\chi_\rho}(G)$.

Consider next the vertex deleted subgraph $G - x_i$, $i \in [r]$. Since x_i is a cut vertex, we infer, having in mind (1), that

$$\begin{aligned}
\chi_\rho(G - x_i) &= \max\{\chi_\rho(K_{s_i+1}), \chi_\rho(G(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_r))\} \\
&= \max\{s_i + 1, 2 + \sum_{j=1, j\neq i}^{r} s_j\} \\
&= 2 + \sum_{j=1, j\neq i}^{r} s_j \,,
\end{aligned}$$

where the last equality follows since we have assumed that $\displaystyle\sum_{j=1,j\neq i}^{r} s_j \geq s_i - 1$. From here and (1) we get that $\chi_\rho(G) - \chi_\rho(G - x_i) = s_i$, that is, $s_i \in \Delta_{\chi_\rho}(G)$ for $i \in [r]$. This proves the theorem for $r \geq 2$.

Let now $S = \{1, s\}$, where $s \geq 2$. Let G be the graph obtained from two disjoint copies of K_{s+1} by identifying a vertex from one copy with a vertex from the other copy. Let x be the identified vertex. Then $|V(G)| = 2s+1$ and $\chi_\rho(G) = 1 + (2s+1-2) = 2s$. If u is a vertex of G different from x, then $\chi_\rho(G-u) = 1 + (2s-2) = 2s-1$, so that $\chi_\rho(G) - \chi_\rho(G-u) = 1$. Finally $\chi_\rho(G-x) = \chi_\rho(K_s) = s$, so that $\chi_\rho(G) - \chi_\rho(G-x) = s$ and we are done. □

4 On 3-χ_ρ-critical and 4-χ_ρ-critical graphs

Clearly, K_2 is the unique 2-χ_ρ-critical graph. To characterize the 3-χ_ρ-critical graphs we recall from [11, Proposition 3.1] that if G is a connected graph, then $\chi_\rho(G) = 2$ if and only if G is a nontrivial star.

Proposition 4.1 *A graph G is 3-χ_ρ-critical if and only if $G \in \{C_3, P_4, C_4\}$.*

Proof. Let G be a 3-χ_ρ-critical graph. Then G is connected by Lemma 2.5. Clearly, $|V(G)| \geq 3$. If $|V(G)| = 3$, then $G \in \{C_3, P_3\}$, giving us the 3-χ_ρ-critical graph C_3. Assume in the rest that $|V(G)| \geq 4$ and let u be an arbitrary vertex of G. Then $\chi_\rho(G-u) = 2$ and hence by the above remark, $G - u$ is a disjoint union of stars. If at least two of these stars contain an edge, then G contains a P_5 and hence cannot be 3-χ_ρ-critical. On the other hand, at least one of the stars, say S_1, must contain an edge, for otherwise G, being connected, would itself be a star. Then $G - u$ contains at most one isolated vertex, for otherwise removing one such vertex from G would yield a graph with $\chi_\rho = 3$. To summarize thus far, $G - u$ contains the star S_1 with at least one edge and contains at most one isolated vertex.

Suppose first that $G - u = S_1$. As $|V(G)| \geq 4$, S_1 has at least two leaves. If u is adjacent to the center of S_1, then since G is a not a star, u must be adjacent to at least one leaf of S_1. Removing in G another leaf of S_1 leaves a graph with $\chi_\rho = 3$ (it contains a C_3). So u is not adjacent to the center of S_1 and is hence adjacent to at least one leaf in S_1. If S_1 has at least three leaves, then G again cannot be 3-χ_ρ-critical. In conclusion, S_1 has exactly two leaves. If u is adjacent to exactly one of them, we get P_4, and if it is adjacent to both, we get C_4.

Suppose next that $G - u$ is the disjoint union of S_1 and an isolated vertex, say w. If S_1 has at least two leaves, then similarly as in the first case we infer that G is not 3-χ_ρ-critical. And if $S_1 = K_2$, then we either find the 3-χ_ρ-critical P_4 or the triangle with a pendant edge which is not 3-χ_ρ-critical.

Since it is clear that each of C_3, P_4, and C_4 is 3-χ_ρ-critical, the proof is complete. □

We next turn our attention to 4-critical graphs.

Lemma 4.2 *If a graph G contains a cycle C whose order is larger than 3 and is not divisible by 4 such that $V(G) - V(C) \neq \emptyset$, then G is not 4-critical.*

Proof. Suppose that $\chi_\rho(G) = 4$, for otherwise there is nothing to be proved. Let C be a cycle in G of order $n > 3$ such that $n \not\equiv 0 \pmod 4$ and $V(G) \neq V(C)$. For any $x \in V(G - C)$, the cycle C is a subgraph of $G - x$ and thus by Lemma 2.1, $4 = \chi_\rho(G) \geq \chi_\rho(G - x) \geq \chi_\rho(C) = 4$. Therefore, G is not 4-critical. □

For the sake of simplicity we call a cycle in a graph G that does not contain all the vertices of G a *proper cycle*. We are now able to characterize the 4-χ_ρ-critical graphs that contain a cycle of order at least 5 that is not divisible by 4.

Theorem 4.3 *If G is a graph that contains a cycle of length $n \geq 5$, where $n \not\equiv 0 \pmod 4$, then G is 4-critical if and only if one of the following holds.*

- *$n = 5$ and G is one of the graphs in Fig. 2,*
- *$n = 6$ and G is one of four bipartite graphs obtained by adding some subset of chords to the 6-cycle,*
- *$n \geq 7$ and G is isomorphic to C_n.*

Proof. Let n be an integer, $n \geq 5$, such that $n \not\equiv 0 \pmod 4$. Note first that $\chi_\rho(C_n) = 4$ and each vertex-deleted subgraph of C_n is the path P_{n-1}, which has packing chromatic number 3. Thus C_n is 4-χ_ρ-critical. Let G be a 4-χ_ρ-critical graph that contains a cycle C of order n. By Lemma 4.2 it follows that C is not a proper cycle; that is,

C is a Hamiltonian cycle. We let C have vertices $v_0, v_1, \ldots, v_{n-1}$ and edges v_iv_{i+1} for $0 \le i \le n-1$, where the subscripts are computed modulo n. A chord v_iv_j of C is called an r-*chord* if the shortest distance between v_i and v_j on C is r. Note that C has r-chords for $2 \le r \le \lfloor \frac{n}{2} \rfloor$.

First we let $n = 5$. All the chords of the 5-cycle are 2-chords. One can easily check that the five graphs of Fig. 2 are 4-critical and that if chords are added in any other way, the resulting graph is not 4-critical. Second, let $n = 6$. By Proposition 2.3, $\chi_\rho(K_{3,3}) = 4$ and since each vertex-deleted subgraph of $K_{3,3}$ is isomorphic to $K_{2,3}$, Proposition 2.3 implies that $\chi_\rho(K_{2,3}) = 3$. Therefore, $K_{3,3}$ is 4-χ_ρ-critical. Since C_6 is also 4-critical, it follows from Proposition 2.6 that the two subgraphs of $K_{3,3}$ obtained by deleting one or two independent edges from $K_{3,3}$ are all 4-critical. On the other hand, if any 2-chord is added to a 6-cycle, the resulting graph contains a proper cycle of order 5 and is therefore not 4-critical.

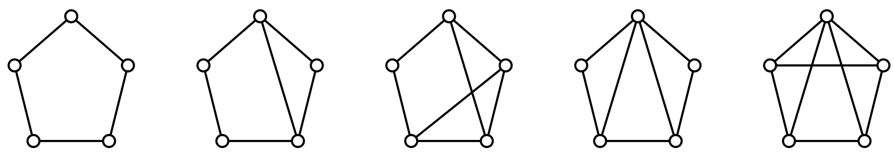

Figure 2: 4-critical graphs obtained from C_5 by adding chords.

Now, let $n \ge 7$. Let G_r be the graph formed by adding the r-chord v_0v_r to C. We claim that for each $2 \le r \le \lfloor \frac{n}{2} \rfloor$, the graph G_r is not 4-critical. It will then follow by Corollary 2.7 that no supergraph of G_r is 4-critical. This will show that G is isomorphic to the n-cycle.

We consider first the case $r = 2$. The graph G_2 contains proper cycles of order 3 and $n-1$. If $n - 1 \not\equiv 0 \pmod 4$, then by Lemma 4.2 we conclude that G_2 is not 4-critical. Therefore, we assume that $n \equiv 1 \pmod 4$. In this case $n \ge 9$ and we infer that G_r contains a vertex x such that $G_r - x$ has a subgraph isomorphic to the graph H in Fig. 3. We claim that $\chi_\rho(H) \ge 4$. Let $c : V(H) \to [k]$ be a k-packing coloring. If $c(v_0) = 1$, then $c(v_1)$, $c(v_2)$ and $c(v_{n-1})$ are three distinct colors, which implies $k \ge 4$. Similarly, $c(v_2) = 1$ implies $k \ge 4$. Hence, $k \ge 4$ unless $\{c(v_0), c(v_2)\} = \{2, 3\}$. However, if $\{c(v_0), c(v_2)\} = \{2, 3\}$, then it is easy to see that one of $c(v_1)$, $c(v_3)$, $c(v_4)$, $c(v_{n-1})$ or $c(v_{n-2})$ is at least 4. Therefore, $\chi_\rho(H) \ge 4$ and by Lemma 2.1 it follows that G_2 is not 4-critical.

Now assume that $r \ge 3$. Recall that $n \ge 7$. The graph G_r has proper cycles of orders $r + 1$ and $n - r + 1$. Note that $r + 1 > 3$ and $n - r + 1 > 3$. A straightforward computation shows that either $r+1 \not\equiv 0 \pmod 4$ or $n-r+1 \not\equiv 0 \pmod 4$ except when $n \equiv 2 \pmod 4$ and $r \equiv 3 \pmod 4$. Therefore, if $n \not\equiv 2 \pmod 4$ or $r \not\equiv 3 \pmod 4$, then it follows by Lemma 4.2 that G_r is not 4-critical. Thus we assume that $n \equiv 2 \pmod 4$ and $r \equiv 3 \pmod 4$. Suppose that G_r is 4-critical and let $c : V(G_r - v_{r+2}) \to [3]$ be a 3-packing coloring of $G_r - v_{r+2}$. Let L be the subgraph of $G_r - v_{r+2}$ shown in Fig. 3. If

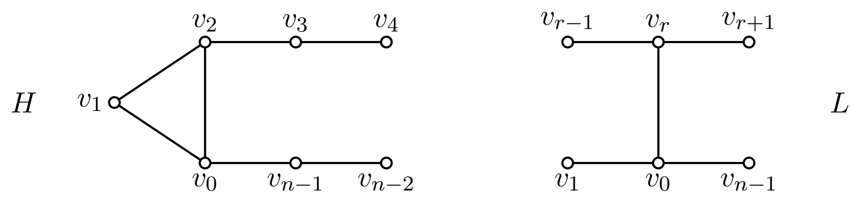

Figure 3: Subgraphs H and L.

$c(v_0) = 1$, then $c(v_1)$, $c(v_{n-1})$ and $c(v_r)$ must all be different, which is a contradiction. Therefore, $c(v_0) \neq 1$, and similarly $c(v_r) \neq 1$. Without loss of generality we assume that $c(v_0) = 3$ and $c(v_r) = 2$. This implies that $c(v_1) = 1$ and $c(v_{r-1}) = 1$. If $r = 3$, we arrive at a contradiction since then v_1 and v_{r-1} are adjacent. For $r > 3$ the vertex v_{r-2} is distance 2 from v_r and distance 3 from v_0, which is again a contradiction. This implies that $\chi_\rho(G_r - v_{r+2}) \geq 4$ and hence G_r is not 4-critical.

Therefore, for $n \geq 7$ such that $n \not\equiv 0 \pmod 4$ we have shown that if a 4-χ_ρ-critical graph G contains a cycle of order n, then $G = C_n$. This completes the proof of the theorem. □

There remains the case when the longest cycle of a 4-packing chromatic graph has order n congruent to 0 modulo 4. If such a graph is 4-critical and $n \geq 8$, then none of its cycles of order eight or more can have chords since any such chord creates a proper cycle whose order is at least 5 and is not congruent to 0 modulo 4. For a partial characterization of 4-critical graphs all of whose cycles have order congruent to 0 modulo 4 we first prove the following.

Lemma 4.4 *Let G_{2k}, $k \geq 3$, be the graph obtained from the path P_{2k} by attaching a leaf to each of the vertices at distance 2 from an endvertex. Then $\chi_\rho(G_{2k}) = 4$.*

Proof. Let the vertices of P_{2k} be $v_1, \ldots, v_{2k}$, and let v_3' and v_{2k-2}' be the leaves attached to v_3 and v_{2k-2}, respectively. Coloring the vertices of P_{2k} with the pattern $1, 2, 1, 3, 1, 2, 1, 3, \ldots$, and the vertices v_3' and v_{2k-2}' with 4 and 1, respectively, we get $\chi_\rho(G_{2k}) \leq 4$. For the rest of the proof we first state the following observation.

Fact. In any 3-χ_ρ-coloring of a path, if x and y are adjacent vertices both of which have a distance at least 2 from the ends of the path, then at least one of x and y must be colored 1.

Suppose now that G_{2k} admits a 3-packing coloring c. Then $c(v_3) \neq 1$ and $c(v_{2k-2}) \neq 1$. Suppose that $c(v_3) = 2$. By the above Fact $c(v_4) \neq 3$, that is, $c(v_4) = 1$. If $k = 3$, this is a contradiction, so let $k > 3$. Then $c(v_5) = 3$. Repeatedly using the Fact we deduce that $c(v_{2k-2}) = 1$ must hold, a contradiction again. Therefore $c(v_3) = 3$. Then we first infer that either $c(v_2) = 2$ and $c(v_1) = 1$, or $c(v_2) = 1$ and $c(v_1) = 2$. In the first case $c(v_4) = 1$ clearly holds. In the second case, assuming that $c(v_4) = 2$, we get that v_6 cannot be properly colored. Hence in either case, $c(v_4) = 1$. But then, using the Fact as many times as required it follows again that $c(v_{2k-2}) = 1$. Hence c does not exist and so $\chi_\rho(G_{2k}) \geq 4$. $\square$

Let $\mathcal{C}$ be the class of graphs that contain exactly one cycle and have an arbitrary number of attached leaves to each of the vertices of the cycle. Recall that the *net graph* is the graph formed by attaching a single leaf to each vertex of a K_3. Then the announced partial characterization of 4-critical graphs reads as follows.

Theorem 4.5 *A graph $G \in \mathcal{C}$ is 4-χ_ρ-critical if and only if G is one of the following graphs:*

- $G = C_n$, $n \geq 5$, $n \not\equiv 0 \pmod 4$,
- *G is the net graph,*
- *G is obtained by attaching a single leaf to two adjacent vertices of C_4.*
- *G is obtained by attaching a single leaf to two vertices at distance 3 on C_8.*

Proof. Let $G \in \mathcal{C}$, let C be the unique cycle in G, and assume C has order n. If $n \not\equiv 0 \pmod 4$ and $n \geq 5$, then G is not 4-critical by Lemma 4.2 if G has at least one leaf. On the other hand it is straightforward to see that C_n ($n \not\equiv 0 \pmod 4$ and $n \geq 5$) is 4-critical. If $n = 3$, then it is also equally straightforward to verify that the net graph is the only graph from $\mathcal{C}$ that contains a 3-cycle and is 4-critical. Hence we may assume in the rest that $n = 4k$ for some $k \geq 1$.

Suppose that G is 4-χ_ρ-critical. We first claim G has at most one leaf attached to each of the vertices of C. Suppose on the contrary that a vertex u of C is adjacent to two leaves x and y. Since G is 4-χ_ρ-critical, $\chi_\rho(G - x) = 3$. In a 3-coloring of $G - x$, the vertex u is not colored 1. But then the 3-coloring of $G - x$ can be extended to a 3-coloring of G by coloring x with 1, a contradiction which proves the claim.

Suppose first that each pair of support vertices of G is at even distance. Then we can color C with the pattern $1, 2, 1, 3, 1, 2, 1, 3, \ldots$ such that each support vertex receives color 2 or 3. But then $\chi_\rho(G) = 3$. Similarly, if G has only one support vertex, then we also have $\chi_\rho(G) = 3$. This implies that G is not 4-critical, which is a contradiction.

Therefore G has at least two support vertices at odd distance. Let u and v be two support vertices that are closest possible on C and let $d_G(u, v) = \ell$. Note that $\ell \leq 2k - 1$. Suppose that $k \geq 3$ and let x be a vertex that is not on a u, v-shortest path and $d_G(x, u) \geq 3$ and $d_G(x, v) \geq 3$. As $k \geq 3$, such a vertex x exists. Then $G - x$ contains a subgraph isomorphic to the graph $G_{\ell+5}$ from Lemma 4.4 which in turn implies that G is not 4-critical. Assume next that $k = 2$, that is, $C = C_8$. If $\ell = 1$, the same argument implies that G is not 4-critical. If $\ell = 3$, then Lemma 4.4 yields $\chi_\rho(G) = 4$. On the other hand it is straightforward to verify that $\chi_\rho(G - x) = 3$ for every $x \in V(G)$. Hence G is 4-critical. Finally, if $k = 1$, then we only need to check two graphs. Among them the one that is obtained by attaching a single leaf to two adjacent vertices of C_4 is 4-critical. □

To conclude the section we present another infinite family of 4-χ_ρ-critical graphs. Let H_{2k+1}, $k \geq 0$, be the graph obtained from the disjoint union of two 4-cycles by connecting a vertex of one 4-cycle with a vertex of the other 4-cycle with a path of length $2k+1$. Then applying Lemma 4.4 we see that $\chi_\rho(H_{2k+1}) = 4$. If x is a diametrical vertex of H_{2k+1}, then distinguishing the cases when $2k+1 \equiv 3 \pmod 4$ and $2k+1 \equiv 1 \pmod 4$ we see that $\chi_\rho(H_{2k+1} - x) = 3$, see Fig. 4.

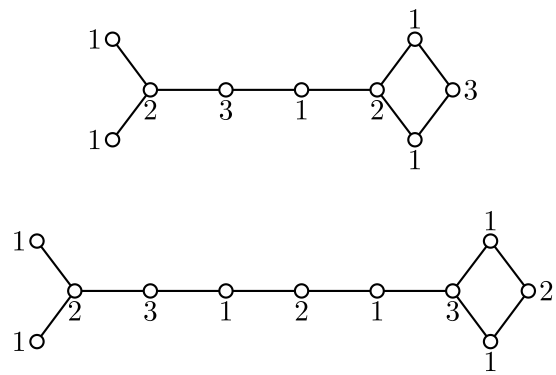

Figure 4: 3-χ_ρ-colorings of $H_3 - x$ and of $H_5 - x$.

For all the other vertices y of H_{2k+1} it is easy to see that $\chi_\rho(H_{2k+1} - y) = 3$. Hence H_{2k+1} is 4-critical for every $k \geq 0$. On the other hand, if in the construction of H_{2k+1} one or both 4-cycles are replaced with some longer cycle of order congruent to 0 modulo 4, then the obtained graph is not 4-critical as can be again deduced from Lemma 4.4.

5 On critical trees

We begin this section with the following interesting fact.

Proposition 5.1 *If $k \geq 2$, then there exists a k-χ_ρ-critical tree.*

Proof. Sloper [18] proved that the infinite 4-regular tree has no finite packing coloring. Hence there exists a finite tree T with $\chi_\rho(T) = k$. Indeed, by Sloper's result there exists a finite tree with packing chromatic number $s \geq k$. If $s > k$, then we repeatedly use Lemma 2.4 to arrive at the desired tree. Now, if T is critical, we are done. Otherwise there exists a vertex x of T such that $\chi_\rho(T - x) = \chi_\rho(T)$. It follows that $T - x$ contains a component T' with $\chi_\rho(T') = k$. If T' is critical, we are done, otherwise just continue the process until a k-χ_ρ-critical tree is found. □

In Fig. 5 five k-χ_ρ-critical trees are given, where $k \in \{3, 4, 5\}$.

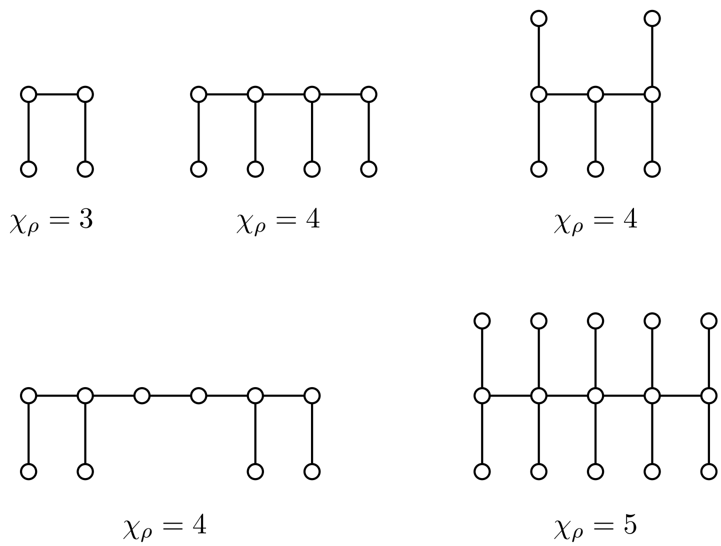

Figure 5: Some k-χ_ρ-critical caterpillars.

Note that all the critical trees from Fig. 5 are caterpillars. Hence it is natural to wonder for which k there exist k-χ_ρ-critical caterpillars.

Theorem 5.2 *A k-χ_ρ-critical caterpillar exists if and only if $k \leq 7$.*

Proof. Let P_∞ denote the one-way infinite path with vertices $v_1, v_2, \ldots$ such that v_1 has degree 1. By repeating the coloring pattern $2, 4, 3, 2, 5, 6, 2, 4, 3, 2, 5, 7$ (mentioned by Sloper in [18]) starting with v_1 one can see that P_∞ admits a $(2, 3, 4, 5, 6, 7)$-coloring. If T is a caterpillar, then using the described $(2, 3, 4, 5, 6, 7)$-coloring for its spine and color 1 for each of its leaves attached to the spine we infer that $\chi_\rho(T) \leq 7$ holds for an arbitrary caterpillar T.

Sloper also mentioned in [18] that no $(2, 3, 4, 5, 6)$-coloring of P_n exists if $n \geq 35$. (We have independently verified this fact using a backtracking algorithm.) Hence, if n is such, then attaching to every vertex of P_n precisely (or more) 6 leaves yields a caterpillar T with $\chi_\rho(T) = 7$. Iteratively applying Lemma 2.4 to the leaves of T we find a 7-χ_ρ-critical caterpillar. Continuing in this manner we then see that there also exists a k-χ_ρ-critical caterpillar for every $k \leq 6$. □

Note that the proof of Theorem 5.2 is not constructive. On the other hand, Fig. 5 shows k-χ_ρ-critical caterpillars for $k \leq 5$. In the next result we construct an explicit 6-χ_ρ-critical caterpillar.

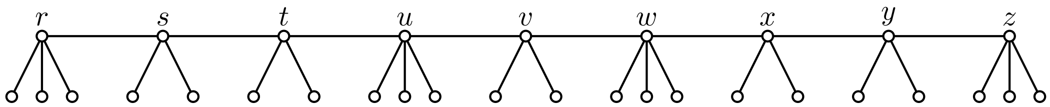

Figure 6: The caterpillar T.

Theorem 5.3 *The caterpillar T from Fig. 6 is 6-χ_ρ-critical.*

To prove Theorem 5.3 we first demonstrate the following two lemmas.

Lemma 5.4 *The path P_9 does not admit a $(2,3,4,5)$-coloring.*

Proof. Let the vertices of P_9 be denoted with $r, s, \ldots, z$ in the natural order. In a possible $(2,3,4,5)$-coloring of P_9, at most three vertices are colored 2, at most three vertices are colored 3, at most two vertices are colored 4, and at most two vertices are colored 5. Suppose there exists a $(2,3,4,5)$-coloring $c : V(P_9) \to \{2,3,4,5\}$. If $|c^{-1}(3)| = 3$, then $c^{-1}(3) = \{r, v, z\}$. However, this implies that $|c^{-1}(2)| = 2$ and $|c^{-1}(5)| = \{s, y\}$, which means at most one vertex from $\{t, u, w, x\}$ is colored 4. Thus no such c exists with $|c^{-1}(3)| = 3$.

Hence $|c^{-1}(2)| = 3$, $|c^{-1}(3)| = 2$, $|c^{-1}(4)| = 2 = |c^{-1}(5)|$. By symmetry, there are four cases to be considered.

Case 1. $c^{-1}(5) = \{r, x\}$.
The subpath s, t, u, v, w has at most one vertex colored 4 by c, which leads to a contradiction since the remaining four vertices cannot be colored using only colors 2 and 3.

Case 2. $c^{-1}(5) = \{r, y\}$.
In this case $|c^{-1}(2) \cap \{s, \ldots, x\}| \leq 2$, which implies $c(z) = 2$. This in turn forces $c^{-1}(4) = \{s, x\}$. But since $|c^{-1}(3) \cap \{t, u, v, w\}| \leq 1$, we get a contradiction.

Case 3. $c^{-1}(5) = \{r, z\}$.
This assumption together with $|c^{-1}(2)| = 3$ implies $c^{-1}(2) = \{s, v, y\}$, which forces $|c^{-1}(4) \cap \{t, u, w, x\}| = 2$, a contradiction.

Case 4. $c^{-1}(5) = \{s, y\}$.
Finally, $c^{-1}(5) = \{s, y\}$ implies that $c(r) = 4$ or $c(z) = 4$. Assume without loss of generality that $c(r) = 4$. If also $c(z) = 4$, then since $|c^{-1}(2) \cap \{t, u, v, w, x\}| \leq 2$, we get a contradiction. Hence $c(z) = 2$. Since $|c^{-1}(3)| = 2$, we must also have $c(t) = c(x) = 3$. But then the coloring can clearly not be completed. $\square$

Lemma 5.5 $\chi_\rho(T) = 6$.

Proof. Let the spine of T be the set $S = \{r, s, t, u, v, w, x, y, z\}$, see Fig. 6. For any vertex a in the spine S, if a is adjacent to k leaves, then for convenience we denote this set of leaves by $\{a_1, \dots, a_k\}$.

Coloring the vertices of the spine of T with colors $2, 3, 4, 2, 5, 3, 2, 4, 6$, and each of the leaves with 1 demonstrates that $\chi_\rho(T) \leq 6$. Thus it remains to prove that $\chi_\rho(T) \geq 6$.

Suppose for the sake of contradiction that $\chi_\rho(T) \leq 5$ and let c be a 5-packing coloring of T. Then, since P_9 does not admit a $(2, 3, 4, 5)$-coloring, it is the case that some vertex of S is colored 1. The neighbors of this vertex of S that is colored 1 receive pairwise different colors from $\{2, 3, 4, 5\}$. Consequently, the only vertices of S that can possibly be colored 1 are r, s, t, v, x, y and z. By symmetry we consider four cases, namely $c(r) = 1$, $c(s) = 1$, $c(t) = 1$ and $c(v) = 1$.

Case 1. $c(r) = 1$.
Since $c(\{r_1, r_2, r_3, s\}) = \{2, 3, 4, 5\}$ and the neighbors of t are at distance at most 4 from the vertices in $\{r_1, r_2, r_3, s\}$, it follows that $c(t) \neq 1$. This implies that $c(t) = 2$, that some vertex in $\{r_1, r_2, r_3\}$ is colored 2, and also that $c(u) \neq 1$. Considering the distance from u to the set $\{r_1, r_2, r_3, s\}$ we infer that $c(u) = 3$ and also that $3 \in c(\{r_1, r_2, r_3\})$. As a result either $c(v) = 1$ or $c(v) = 4$. If $c(v) = 1$, then $c(\{v_1, v_2, w\}) = \{2, 4, 5\}$, which implies that $5 \in c(\{r_1, r_2, r_3\})$ and thus $c(s) = 4$. However, this is a contradiction since the distance between s and any vertex in $\{v_1, v_2, w\}$ is 4. Therefore, $c(v) = 4$ and $4 \in c(\{r_1, r_2, r_3\})$. It now follows that $c(s) = 5$, $c(w) = 2$ and $c(x) = 1$. This is a contradiction since the distance between v and every vertex in $\{x_1, x_2, y\}$ is 3.

Case 2. $c(s) = 1$.
In this case $c(\{r, s_1, s_2, t\}) = \{2, 3, 4, 5\}$. This implies that $c(u) = 2$, which implies $c(v) = 3$ and $c(w) = 4$. (Note that $c(v) \neq 1$ since every neighbor of v is within distance 5 of $\{r, s_1, s_2, t\}$.) Now it follows that $c(t) = 5$ and $c(x) \neq 1$ since the distance between v and any neighbor of x is at most 3. We now infer that $c(x) = 2$, which forces $c(y) = 1$. But this is a contradiction since the distance between w and any vertex in $N(y)$ is at most 3.

Case 3. $c(t) = 1$.
It follows that $c(\{s, t_1, t_2, u\}) = \{2, 3, 4, 5\}$. By case 1 we may assume that $c(r) \neq 1$, which implies that in fact $c(r) = 2$, for otherwise the color $c(r) \in \{3, 4, 5\}$ could not be used on either of the vertices s, t_1, t_2, u. If $c(u) = 2$, then because $c(\{s, t_1, t_2, u\}) = \{2, 3, 4, 5\}$ we must have $c(v) = 1$. But in this case one of v_1, v_2 or w is colored 5, which contradicts the fact that some neighbor of vertex t is also colored 5. If $c(u) \neq 2$, then $c(v) \in \{1, 2\}$. Suppose first that $c(v) = 2$. Then $c(w) \in \{1, 3\}$, where the possibility $c(w) = 1$ is impossible because of the degree of w. So $c(w) = 3$. Then either $c(s) = 5$ and $c(u) = 4$, or $c(s) = 4$ and $c(u) = 5$. The first case forces $c(x) = 1$, but then one of the neighbors of x should have been colored 4, but this is not possible because $c(u) = 4$. And if $c(s) = 4$ and $c(u) = 5$, then $c(x) \neq 1$ because every neighbor of x is of distance at most 4 from u, which is colored 5. Hence $c(x) = 4$. But then

$\{c(y), c(z)\} = \{1, 2\}$ which readily leads to a contradiction since by symmetry and Cases 1 and 2 we may assume that $c(y) \neq 1$ and $c(z) \neq 1$. On the other hand, if $c(v) = 1$, then $\max\{c(v_1), c(v_2), c(w)\} \geq 4$, which again is in conflict with the fact that the distance between any of v_1, v_2 or w and any vertex in $\{s, t_1, t_2, u\}$ is at most 4.

Case 4. $c(v) = 1$.
We have shown in the first three cases that $1 \notin c(\{r, s, t\})$ and by symmetry that $1 \notin c(\{x, y, z\})$. Furthermore, $c(u) \neq 1$ and $c(w) \neq 1$ since they are neighbors of v. Hence, $c(\{r, s, t, u, v_1, v_2, w, x, y, z\}) = \{2, 3, 4, 5\}$. Since $c(\{u, w, t_1, t_2\}) = \{2, 3, 4, 5\}$, at most three of the vertices in $\{r, s, t, u, v_1, v_2, w, x, y, z\}$ can be colored 2, at most three can be colored 3, at most three can be colored 4 and at most one can be colored 5. The only way that three of these ten vertices can be colored 4 is if $c(r) = 4 = c(z)$ and one of v_1, v_2 is colored 4. But in this case it is easy to see that $|c^{-1}(\{2, 3\})| \leq 5$. This gives the contradiction that $|c^{-1}(\{2, 3, 4, 5\})| \leq 9$. However, if $|c^{-1}(4)| \leq 2$, then again we arrive at $|c^{-1}(\{2, 3, 4, 5\})| \leq 9$. □

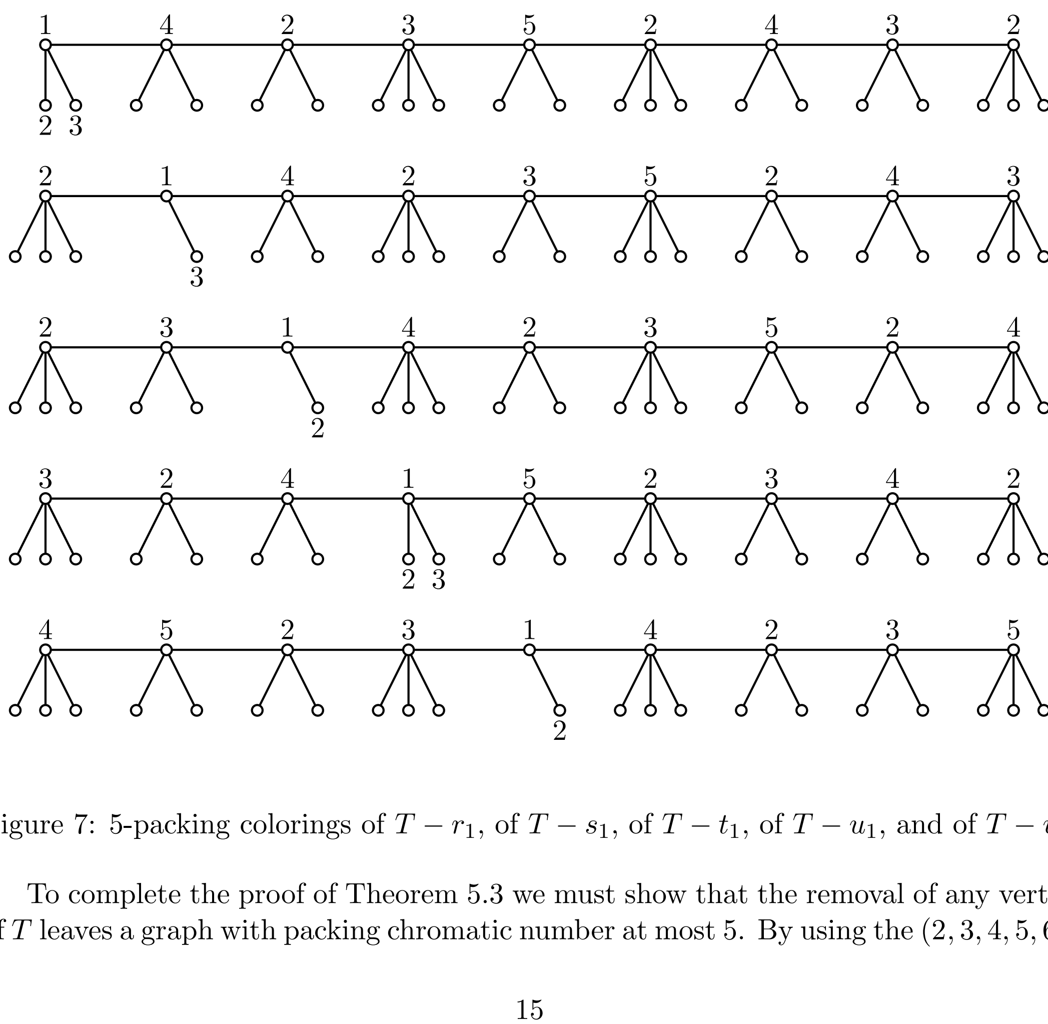

Figure 7: 5-packing colorings of $T - r_1$, of $T - s_1$, of $T - t_1$, of $T - u_1$, and of $T - v_1$.

To complete the proof of Theorem 5.3 we must show that the removal of any vertex of T leaves a graph with packing chromatic number at most 5. By using the $(2, 3, 4, 5, 6)$-

coloring of P_9 given above it follows that any caterpillar whose spine has order at most 8 has packing chromatic number at most 5. Consequently, we only need to show that there is a 5-packing coloring of the caterpillar that remains when one of the leaves of T is deleted. Using the symmetry of T, such packing colorings are shown in a sequence of five figures. See Fig. 7. In each case the leaves that are not labeled are colored 1. Theorem 5.3 is thus proved.

6 Critical Cartesian products

The Cartesian product $G \,\square\, H$ of graphs G and H has $V(G) \times V(H)$ as the vertex set, vertices (g, h) and (g', h') being adjacent if either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$, see [14].

Recall that a graph G is vertex-critical for the chromatic number, if $\chi(G - u) = \chi(G) - 1$ holds for every $u \in V(G)$. If the factors of a Cartesian product are both non-trivial, then $G \,\square\, H$ is not vertex-critical for the chromatic number. Indeed, recall that $\chi(G \,\square\, H) = \max\{\chi(G), \chi(H)\}$, see [14, Theorem 8.1]). Hence, since a vertex-deleted subgraph of $G \,\square\, H$ contains a subgraph isomorphic to G and a subgraph isomorphic to H, the vertex-deleted subgraph has the same chromatic number as $G \,\square\, H$. In this section we show that on the other hand the Cartesian product is a very rich source of χ_ρ-critical graphs. For additional investigation of χ_ρ on Cartesian products see [5, 15].

We begin with the following sporadic, but very illustrative example. Let $G = K_{1,3} \,\square\, P_3$ with the vertices labeled as in Fig. 8. We claim that G is 6-χ_ρ-critical even though neither $K_{1,3}$ nor P_3 is critical. First note that the independence number of G is 7, and that G has a unique independent set of cardinality 7, namely $I = \{u_1, u_2, u_3, v, w_1, w_2, w_3\}$. Furthermore, $\mathrm{diam}(G) = 4$ and so in any packing coloring of G any color 4 or more can be assigned at most once.

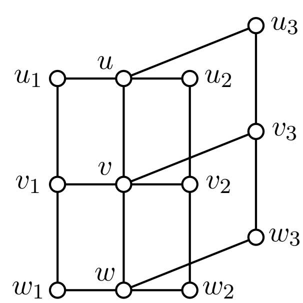

Figure 8: $K_{1,3} \,\square\, P_3$.

The packing coloring in which each vertex of I is colored 1 and the five neighbors of v are assigned distinct colors from the set $\{2, 3, 4, 5, 6\}$ shows that $\chi_\rho(G) \le 6$. To prove that $\chi_\rho(G) \ge 6$ we assume that $c : V(G) \longrightarrow [n]$ is a packing coloring with n as small

x	u	u_1	u_2	u_3	v	v_1	v_2	v_3	w	w_1	w_2	w_3
$c(x)$		3	1	2	4	1	2	1	1	2		3
$f_u(x)$		3	3	2	5	1	1	1	1	2	4	3
$f_{u_3}(x)$	1	3	2		5	1	1	1	1	2	3	4
$f_{v_3}(x)$	2	1	1	1	1	3	5		4	1	1	1
$f_v(x)$	1	3	5	2		1	1	1	1	2	4	3

Table 1: Some partial packing colorings of $K_{1,3} \,\square\, P_3$.

as possible. If $c(v) = 1$, then all of the vertices in $N(v)$ have distinct values under c, and this implies that $n \geq 6$. If $c(v) = 2$, then since the eccentricity of v is 2, it follows that $c^{-1}(2) = \{v\}$. This implies that $|c^{-1}(1)| \leq 6$. In this case if $|c^{-1}(3)| \geq 2$, then $3 \in c(I - \{v\})$. Consequently, $|c^{-1}(\{1,3\})| \leq 7$ and hence in this case $n \geq 7$. Similarly, if $c(v) = 3$, then $c^{-1}(3) = \{v\}$ and $|c^{-1}(\{1,2\})| \leq 7$, which again means $n \geq 7$. Finally, if $c(v) \in \{4,5\}$, then one can easily show that $|c^{-1}(\{1,2,3\})| \leq 9$, which leads to $n \geq 3 + (12 - 9) = 6$. (One such partial packing coloring c with $|c^{-1}(\{1,2,3\})| = 9$ is shown in Table 1.)

Finally to show that G is critical, we observe that by symmetry we need to show that $\chi_\rho(G - t) < 6$ for each vertex $t \in \{u, v, u_3, v_3\}$. In the table there is a packing coloring, $f_t : V(G - t) \longrightarrow [5]$ for each such vertex t. This proves that G is 6-critical.

Note that P_3 and $K_{1,3}$ are not critical. Hence the above example shows that even the Cartesian product of non-critical graphs can be critical.

To establish a greater variety of χ_ρ-critical Cartesian products, we recall an earlier result and prove a new lemma that might be of independent interest.

Theorem 6.1 [5, Theorem 1] *If G and H are connected graphs on at least two vertices, then*

$$\chi_\rho(G \,\square\, H) \geq (\chi_\rho(G) + 1)n(H) - \mathrm{diam}(G \,\square\, H)(n(H) - 1) - 1\,.$$

Lemma 6.2 *If G is a vertex-transitive graph with* $\mathrm{diam}(G) \leq \chi_\rho(G)$, *then G is χ_ρ-critical.*

Proof. Let x be any vertex of G. Since G is vertex-transitive, there is a $\chi_\rho(G)$-packing coloring c of G such that $c(x) = \chi_\rho(G)$. By Observation 2.2, x is the only vertex assigned the color $\chi_\rho(G)$ by c. The restriction of c to $V(G - x)$ is a packing coloring, which shows that $\chi_\rho(G - x) < \chi_\rho(G)$. $\square$

Consider the hypercubes Q_n, $n \geq 1$. Indeed, $\chi_\rho(Q_n)$ was determined exactly for $n \leq 5$ in [11] and for $6 \leq n \leq 8$ in [19]. Moreover, $\chi_\rho(Q_n)$ asymptotically grows as $\left(\frac{1}{2} - O(\frac{1}{n})\right) 2^n$. Since $\mathrm{diam}(Q_n) = n$, we conclude from Lemma 6.2 that Q_n is critical for every $n \geq 1$.

The announced result now reads as follows.

Theorem 6.3 *If G and H are connected, vertex-transitive graphs on at least two vertices and* $\mathrm{diam}(G) + \mathrm{diam}(H) \le \chi_\rho(G)$*, then $G \,\square\, H$ is χ_ρ-critical.*

Proof. Since $\mathrm{diam}(G)+\mathrm{diam}(H) \le \chi_\rho(G)$ we have $\chi_\rho(G)-\mathrm{diam}(G)-\mathrm{diam}(H)+1 \ge 1$, from which we get that

$$n(H)(\chi_\rho(G) - \mathrm{diam}(G) - \mathrm{diam}(H) + 1) \ge 1\,. \tag{2}$$

Using Theorem 6.1 and the well-known fact that $\mathrm{diam}(G \,\square\, H) = \mathrm{diam}(G) + \mathrm{diam}(H)$ (see [14, p.102]), we can estimate as follows:

$$\begin{aligned}\chi_\rho(G \,\square\, H) &\ge (\chi_\rho(G) + 1)n(H) - \mathrm{diam}(G \,\square\, H)(n(H) - 1) - 1\\ &= n(H)(\chi_\rho(G) - \mathrm{diam}(G) - \mathrm{diam}(H) + 1) + \mathrm{diam}(G) + \mathrm{diam}(H) - 1\\ &\ge \mathrm{diam}(G) + \mathrm{diam}(H)\,,\end{aligned}$$

where the last inequality follows by (2).

It is also well-known (see [13, Proposition 6.16]) that a Cartesian product of connected graphs is vertex-transitive if and only if the factors are such. Hence, $G \,\square\, H$ is a vertex-transitive graph. Moreover, by the above,

$$\mathrm{diam}(G \,\square\, H) = \mathrm{diam}(G) + \mathrm{diam}(H) \le \chi_\rho(G \,\square\, H)\,.$$

The result now follows by Lemma 6.2. □

Let $k \ge 3$ and let H be a vertex-transitive graph with $\mathrm{diam}(H) \le k - 1$. Then Theorem 6.3 implies that $K_k \,\square\, H$ is χ_ρ-critical. This fact in particular implies that every vertex-transitive graph is an induced (actually convex) subgraph of a χ_ρ-critical graph.

For a particular example consider $H = C_{4\ell}$, where $2\ell \le k - 1$. Then $K_k \,\square\, C_{4\ell}$ is χ_ρ-critical graph, where one factor, namely $C_{4\ell}$, is not χ_ρ-critical.

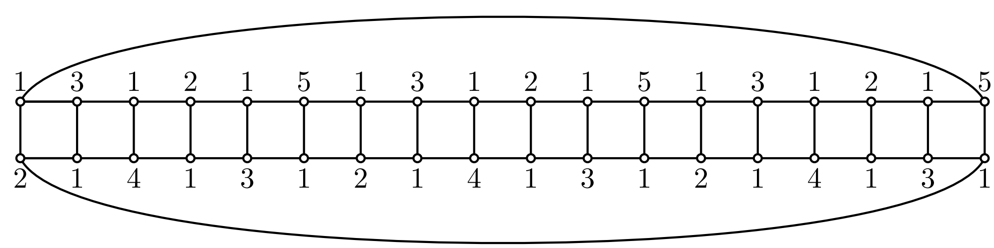

Figure 9: χ_ρ-coloring of $C_{18} \,\square\, P_2$

Although many pairs of vertex-transitive graphs that do not satisfy the diameter condition in Theorem 6.3 still produce a critical Cartesian product, it is not always the case. See Fig. 9 where a coloring of the product $C_{18} \,\Box\, P_2$ is shown. From [11, Proposition 6.1] we know that the $2 \times k$ grid has packing chromatic number 5 for every $k \geq 6$. We infer that $\chi_\rho(C_{18} \,\Box\, P_2) \geq 5$ since $P_{18} \,\Box\, P_2$ is a subgraph of $C_{18} \,\Box\, P_2$. It is easy to check that the coloring in Fig. 9 is indeed a packing coloring, which in turn shows that $\chi_\rho(C_{18} \,\Box\, P_2) = 5$. For any vertex x of $C_{18} \,\Box\, P_2$, the 2×17 grid is a subgraph of $(C_{18} \,\Box\, P_2) - x$. Since $C_{18} \,\Box\, P_2$ is vertex-transitive, this shows that $C_{18} \,\Box\, P_2$ is not χ_ρ-critical.

7 Concluding remarks

In this paper we have investigated χ_ρ-critical graphs. These graphs are vertex critical for the packing chromatic number. An equally legal concept would be edge critical graphs, that is, graphs G for which $\chi_\rho(G - e) < \chi_\rho(G)$ holds for every edge e of G. Their investigation seems quite different from the vertex-criticality studied in this paper though.

It would be interesting to classify vertex-transitive, χ_ρ-critical graphs. Examples include cycles (only those whose order is not congruent to 0 modulo 4), complete graphs, complete multipartite graphs with parts of equal size, hypercubes, the Petersen graph P, as well as the Cartesian product of those that satisfy the additional condition of Theorem 6.3. A sporadic example from the latter family of graphs is $P \,\Box\, P$.

We did not give a complete list of 4-χ_ρ-critical graphs. A different approach to this problem would be to use the structure of graphs G with $\chi_\rho(G) = 3$ as described in [11]. In particular, for the 2-connected case we have the following characterization.

Proposition 7.1 [11, Proposition 3.2] *Let G be a 2-connected graph. Then $\chi_\rho(G) = 3$ if and only if G is either the subdivision of a bipartite multigraph or the join of K_2 and an independent set.*

Based on this proposition we say that a graph G is *join reducible* if there exists at least one vertex x of G such that $G - x$ is the join of K_2 and an independent set. Then we can prove the following result, the proof of which is a tedious case analysis and hence is omitted.

Proposition 7.2 *If G is a 4-χ_ρ-critical graph, then G is join reducible if and only if G is one of the five graphs in Fig. 10.*

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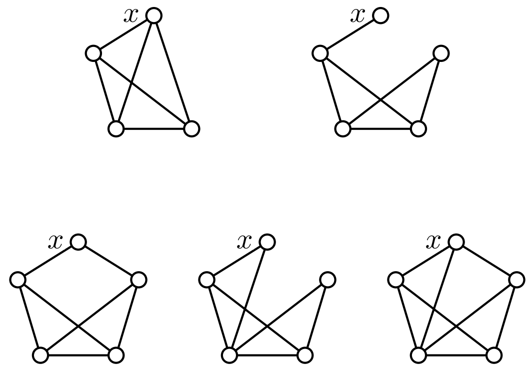

Figure 10: Some 4-χ_ρ-critical graphs.

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