Total version of the domination game

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Abstract

In this paper, we continue the study of the domination game in graphs introduced by Brešar, Klavžar, and Rall [SIAM J. Discrete Math. 24 (2010) 979–991]. We study the total version of the domination game and show that these two versions differ significantly. We present a key lemma, known as the Total Continuation Principle, to compare the Dominator-start total domination game and the Staller-start total domination game. Relationships between the game total domination number and the total domination number, as well as between the game total domination number and the domination number, are established.

Key words: Domination game; Total domination number.

AMS Subj. Class: 05C57, 91A43, 05C69

1 Introduction

In this paper, we continue the study of the domination game in graphs introduced in [2]. Before saying more about the game, we briefly recall basic concepts needed throughout the paper.

For notation and graph theory terminology not defined herein, we in general follow [8]. Let $G = (V, E)$ be a graph with no isolated vertex. A *dominating set* of G is a set S of vertices of G such that every vertex in $V \setminus S$ is adjacent to a vertex in S. Thus a set $S \subseteq V$ is a dominating set in G if $N[S] = V$, where $N[S]$ is the closed neighborhood of S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. If X and Y are subsets of vertices in G, then the set X *dominates* the set Y in G if $Y \subseteq N[X]$. In particular, if X dominates V, then X is a dominating set in G. A *total dominating set*, abbreviated TD-set, of G is a set S of vertices of G such that every vertex is adjacent to a vertex in S. Thus a set $S \subseteq V$ is a TD-set in G if $N(S) = V$, where $N(S)$ is the open neighborhood of S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A vertex u *totally dominates* a vertex v if $v \in N(u)$. If X and Y are subsets of vertices in G, then the set X *totally dominates* the set Y in G if $Y \subseteq N(X)$. In particular, if X totally dominates V, then X is a TD-set in G. For more information on the total domination in graphs see the recent book [9]. Since an isolated vertex in a graph cannot be totally dominated by definition, *all graphs considered will be without isolated vertices.*

The domination game is played on a graph G by players, *Dominator* and *Staller*, who take turns choosing a vertex from G. Each vertex chosen must dominate at least one vertex not dominated by the vertices previously chosen. The game ends when the set of vertices chosen becomes a dominating set in G. Dominator wishes to dominate the graph as fast as possible and Staller wishes to delay the process as much as possible. The *game domination number* (resp. *Staller-start game domination number*), $\gamma_g(G)$ (resp. $\gamma_g'(G)$), of G is the number of vertices chosen when Dominator (resp. Staller) starts the game and both players play optimally.

Let us recall some results proved about this game. First of all, it was proved in [2, 10] that the game domination number and the Staller-start game domination number are always close together; more precisely, $|\gamma_g(G) - \gamma_g'(G)| \le 1$ holds for every graph G. A key lemma needed to give a short proof of this result is the so-called *Continuation Principle* proved in [10]. We will not state it here but instead will prove a completely parallel assertion, named the *Total Continuation Principle*, for the total domination game; see Lemma 2.1. Call a pair of integers (k, ℓ) *realizable* if there exists a graph G with $\gamma_g(G) = k$ and $\gamma_g'(G) = \ell$. Realizable pairs were studied in [2, 3, 13]; for the complete answer (with relatively simple families of graphs) see [12]. Several exact values for the game domination number were established in [11], including paths, cycles (the problem is non-trivial even on these graphs!) and several grid-like graphs. The main message of the paper [3] is that the game domination number can behave intrinsically different from the usual domination number. Kinnersley, West, and Zamani [10] conjectured that if G is an isolate-free forest of order n or an isolate-free graph of order n, then $\gamma_g(G) \le 3n/5$. Progress on these two 3/5-*conjectures* was made in [4] by constructing large families of trees that attain the conjectured 3/5-bound and by finding all extremal trees on up to 20 vertices. Bujtás [5] proved the 3/5-*conjecture* for the class of forests in which no two leaves are at distance 4 apart. In [6], which is an extended version of [5], she also proved that $\gamma_g(G) \le 5n/8$ holds for every isolate-free forest G. Finally, in [1], the effect of removing or adding an edge was studied and proved that if $e \in E(G)$, then $|\gamma_g(G) - \gamma_g(G-e)| \le 2$ and $|\gamma_g'(G) - \gamma_g'(G-e)| \le 2$, and that each of the possibilities here is realizable by connected graphs G for all values of $\gamma_g(G)$ and $\gamma_g'(G)$ larger than 5.

In this paper we introduce and study the total version of the domination game and in particular compare it with the usual domination game. The *total domination game*, played on a graph G again consists of two players called *Dominator* and *Staller* who take turns choosing a vertex from G. In this version of the game, each vertex chosen must totally dominate at least one vertex not totally dominated by the vertices previously chosen. We say that a move v in the total domination game is *legal* if the played vertex v totally dominates at least one new vertex. The game ends when the set of vertices chosen becomes a total dominating set in G. Dominator wishes to totally dominate the graph as fast as possible and Staller wishes to delay the process as much as possible. The *game total domination number*, $\gamma_{tg}(G)$, of G is the number of vertices chosen when Dominator starts the game and both players play optimally. The *Staller-start game total domination number*, $\gamma_{tg}'(G)$, of G is the number of vertices chosen when Staller starts the game and both players play optimally. For simplicity, we shall refer to the Dominator-start total domination game and the Staller-start total domination game as *Game 1* and *Game 2*, respectively.

We proceed as follows. In Section 2, we compare the total domination Game 1 and Game 2 and present our key lemma, named the *Total Continuation Principle*, which shows that the number of moves in Game 1 and Game 2 when played optimally can differ by at most one. A relationship between the game total domination number and the total domination number is established in Section 3, as is a relation between the game domination number and the game total domination number. Corollary 3.2 and the examples afterwards demonstrate that the latter two invariants can differ significantly. We close in Section 4 by established a relationship between the game total domination number and the domination number.

2 Game 1 versus Game 2

In this section we prove that just as for the usual domination game, also for the total domination game the number of moves in Game 1 and Game 2 when played optimally can differ by at most one. To do so, we will mimic the idea of the proof of the Continuation Principle from [10], for which we need the following concept.

A *partially totally dominated graph* is a graph together with a declaration that some vertices are already totally dominated; that is, they need not be totally dominated in the rest of the game. If G is a graph and $A \subseteq V(G)$ is such a set, we will denote with G_A this partially totally dominated graph. Moreover, $\gamma_{tg}(G_A)$ and $\gamma_{tg}'(G_A)$ are the minimum number of moves needed to finish the game on G_A when Dominator or Staller starts, respectively.

Lemma 2.1 (Total Continuation Principle) *Let G be a graph, $A, B \subseteq V(G)$, and let G_A and G_B be the corresponding partially totally dominated graphs. If $B \subseteq A$,* *then* $\gamma_{tg}(G_A) \le \gamma_{tg}(G_B)$ *and* $\gamma'_{tg}(G_A) \le \gamma'_{tg}(G_B)$.

Proof. Two games will be played, Game A on the graph G_A and Game B on the graph G_B. The first of these will be the real game, while Game B will only be imagined by Dominator. In Game A, Staller will play optimally while in Game B, Dominator will play optimally. More precisely, in Game B, Dominator will copy each move of Staller played in Game A, imagine that Staller played this move in game B, and then reply with an optimal move. Dominator will then play this move, which was optimal in Game B, also in Game A if it is a legal move in Game A even though his move need not be optimal.

We claim that in each stage of the games, the set of vertices that are totally dominated in Game B is a subset of the vertices that are totally dominated in Game A. This is clearly true at the start of the games. Suppose now that Staller has (optimally) selected vertex u in Game A. Then by the induction assumption, vertex u is a legal move in Game B because a new vertex v that was totally dominated by u in Game A, is not yet totally dominated in Game B. Then Dominator imagines that Staller played the vertex u in Game B and replies with an optimal move (in Game B). If this move is legal in Game A, Dominator plays it in Game A as well. Otherwise, if the game is not yet over, Dominator plays any other legal move in Game A. In both cases the claim assumption is preserved, which by induction also proves the claim.

We have thus proved that Game A finishes no later than Game B. Suppose thus that Game B lasted r moves. Because Dominator was playing optimally in Game B, it follows that $r \le \gamma_{tg}(G_B)$. On the other hand, because Staller was playing optimally in Game A and Dominator has a strategy to finish the game in r moves, we infer that $\gamma_{tg}(G_A) \le r$. Therefore,

$$\gamma_{tg}(G_A) \le r \le \gamma_{tg}(G_B),$$

and we are done if Dominator is the first to play. Note that in the above arguments we did not assume who starts first, hence in both cases Game A will finish no later than Game B. Hence the conclusion holds for γ'_{tg} as well. □

As a consequence of the Total Continuation Principle whenever x and y are legal moves for Dominator and $N(x) \subseteq N(y)$, then Dominator will play y instead of x. We remark that the proof of Lemma 2.1 could be modified to work on other (but not all) variants of possible domination games, but since we do not wish to initiate an inflation of such games, we stated the result for the total version only.

Lemma 2.1 leads to the following fundamental property:

Theorem 2.2 *For any graph* G, *we have* $|\gamma_{tg}(G) - \gamma'_{tg}(G)| \le 1$.

Proof. Consider Game 1 and let v be the first move of Dominator. Let $A = N(v)$ and consider the partially totally dominated graph G_A. Set in addition $B = \emptyset$ and note that $G_B = G$. Note first that $\gamma_{tg}(G) = 1 + \gamma'_{tg}(G_A)$. By the Total Continuation Principle, $\gamma'_{tg}(G_A) \le \gamma'_{tg}(G_B) = \gamma'_{tg}(G)$. Therefore,

$$\gamma_{tg}(G) \le \gamma'_{tg}(G_A) + 1 \le \gamma'_{tg}(G) + 1\,.$$

By a parallel argument, $\gamma'_{tg}(G) \le \gamma_{tg}(G) + 1$. □

One can check directly that $\gamma_{tg}(P_4) = \gamma'_{tg}(P_4) = 3$, $\gamma_{tg}(P_5) = 3 = \gamma'_{tg}(P_5) - 1$, and $\gamma_{tg}(C_8) = 5 = \gamma'_{tg}(C_8) + 1$. For instance, in Game 2 on C_8, an optimal first move of Dominator is the vertex opposite to the vertex played by Staller in the first move of the game. The second move of Staller must be adjacent to one of the two vertices already played, and then Dominator can finish the game by picking the vertex that is opposite to the second move of Staller. So there are graphs G_1, G_2 and G_3 such that $\gamma_{tg}(G_1) = \gamma'_{tg}(G_1)$, $\gamma_{tg}(G_2) = \gamma'_{tg}(G_2) + 1$, and $\gamma_{tg}(G_3) = \gamma'_{tg}(G_3) - 1$. Infinite families of such examples can be obtained using the next result. It involves graphs G with $\gamma_{tg}(G) = \gamma'_{tg}(G) = 2$. Examples of such graphs are complete multipartite graphs $K_{n_1,\ldots,n_r}$ where $r \ge 2$, and all graphs that contain a universal vertex. (Recall that a *universal vertex* is a vertex adjacent to every other vertex.)

With $\cup_i G_i$ we denote the disjoint union of the graphs G_i.

Proposition 2.3 *Let G_0 be a graph, r a positive integer, and let G_i, $1 \le i \le r$, be graphs with $\gamma_{tg}(G_i) = \gamma'_{tg}(G_i) = 2$. Then*

$$\gamma_{tg}(\cup_{i=0}^r G_i) = \gamma_{tg}(G_0) + 2r \quad \textit{and} \quad \gamma'_{tg}(\cup_{i=0}^r G_i) = \gamma'_{tg}(G_0) + 2r\,.$$

Proof. Set $G = \cup_{i=0}^r G_i$. Consider first Game 1 and the following strategy of Dominator. He starts with an optimal (with respect to the game restricted to G_0) move in G_0. In each later move he follows Staller in G_0 as long as possible.

In this way Dominator ensures that the game restricted to G_0 is Game 1 so that he can guarantee that at most $\gamma_{tg}(G_0)$ moves will be played on G_0. Since $\gamma_{tg}(G_i) = \gamma'_{tg}(G_i) = 2$, exactly two moves will be played on each G_i, $i \ge 1$. Consequently, $\gamma_{tg}(G) \le \gamma_{tg}(G_0) + 2r$. On the other hand, a similar strategy of Staller, namely that she follows a move of Dominator in G_0 if possible, guarantees her that at least $\gamma_{tg}(G_0)$ moves will be played on G_0, and consequently $\gamma_{tg}(G) \ge \gamma_{tg}(G_0) + 2r$. This establishes the first assertion.

The second assertion follows by analogous strategies by Staller and Dominator, respectively. □

To conclude the section we remark that recently a detailed study of the (usual) domination game played on the disjoint union of two graphs was done in [7].

3 Game total domination versus total domination

The game total domination number can be bounded by the total domination number as follows.

Theorem 3.1 *If G is a graph on at least two vertices, then*

$$\gamma_t(G) \le \gamma_{tg}(G) \le 2\gamma_t(G) - 1\,.$$

Moreover, given any integer $n \ge 2$ and any $0 \le \ell \le n-1$ there exists a connected graph H such that $\gamma_t(H) = n$ and $\gamma_{tg}(H) = n + \ell$.

Proof. At the end of Game 1 played on a graph G the vertices played form a TD-set of G, hence $\gamma_{tg}(G) \ge \gamma_t(G)$. Moreover, if D is a minimum TD-set of G, then the strategy of Dominator to consecutively play vertices from D if possible, and otherwise playing some other vertex, guarantees that the game ends in no more than $2\gamma_t(G) - 1$ moves. This proves the claimed bounds. For the rest of the proof fix a positive integer $n \ge 2$.

Let G_n be the graph obtained from the disjoint union of the path $P^{(0)} = P_n$ on n vertices and n copies of P_{n+1}, denoted $P^{(i)}$, $1 \le i \le n$, by joining the ith vertex of $P^{(0)}$ $(1 \le i \le n)$ to all vertices of $P^{(i)}$. Then it is straightforward to see that $\gamma_t(G_n) = n$. Hence by the already proved upper bound, $\gamma_{tg}(G_n) \le 2n - 1$. On the other hand, Staller has a strategy to play at least $n-1$ vertices of the subgraphs $P^{(i)}$, $1 \le i \le n$, which then implies that $\gamma_{tg}(G_n) \ge 2n-1$. Consequently, $\gamma_{tg}(G_n) = 2n - 1$. This settles the theorem's case $\ell = n - 1$.

Let k be a positive integer and let ℓ be a non-negative integer such that $n - 1 = k + \ell$. Let X_i, $1 \le i \le k$, be the complete graph K_2 with $V(X_i) = \{x_i, y_i\}$. Let Y_i, $1 \le i \le \ell$, be the graph with $V(Y_i) = \{u_i\} \cup \{a_{i,1}, \ldots, a_{i,\ell}, b_{i,1}, \ldots, b_{i,\ell}\}$, in which u_i is a universal vertex (that is, adjacent to all other vertices), the remaining edges being $a_{i,1}b_{i,1}, \ldots, a_{i,\ell}b_{i,\ell}$. (In other words, Y_i is obtained from ℓ disjoint triangles by identifying a vertex in each of them.) Let ${}_kG_\ell$ be the graph constructed from the disjoint union of $X_1, \ldots, X_k$, $Y_1, \ldots, Y_\ell$, and a vertex w by adding the edges $wx_1, \ldots, wx_k$ and $wu_1, \ldots, wu_\ell$.

Consider the following strategy of Dominator when the total domination game is played on ${}_kG_\ell$. Dominator will play first at vertex w. Note that this move prevents any y_i to be played in the rest of the game. Regardless of how the game proceeds, all of the vertices in the set $\{x_1, \ldots, x_k\}$ will be chosen. Whenever Staller plays a vertex in some Y_i, Dominator responds by playing u_i. This is always possible as it is clearly not optimal for Staller to select u_i as the first vertex to be played in Y_i. Using this strategy of Dominator not more than two vertices will be played in each Y_i. Consequently,

$$\gamma_{tg}({}_kG_\ell) \le 1 + k + 2\ell\,.$$

Consider next Staller's strategy. Suppose first that Dominator plays w as his first move. As in the previous paragraph where we described Dominator's strategy, no vertex from $\{y_1, \ldots, y_k\}$ can be played and every vertex from $\{x_1, \ldots, x_k\}$ must be played at some point in the game. When Dominator plays on any Y_i he will play vertex u_i by the Total Continuation Principle. The strategy of Staller is to first play, say, $a_{1,1}$, and then reply to a move u_i of Dominator by playing on some Y_j on which Dominator did not play yet. There she selects any legal move different from u_j. Note that in this way Staller always has a legal move as long as at least one u_i has not yet been played. As Staller was the first to play on some Y_i this means that she will be able to play ℓ vertices of degree 2 in the subgraphs Y_i. Finally, if u_j is the last vertex not yet played in the Y_i's, Dominator must play u_j to finish the game on Y_j. In this way Staller forces the game to last at least $1 + k + 2\ell$ moves, because at least one move will be played on each X_i.

Suppose next that w is not the first move of Dominator. Assume that Dominator first played u_i. Then Staller replies with a move in Y_i, say $a_{i,1}$. As long as Dominator plays vertices u_j, Staller accordingly replies in Y_j. In this way in all subgraphs Y_j that were played, two vertices were selected. Suppose that at some stage of the game, Dominator plays w. In that case, Staller follows the same strategy on the remaining subgraphs Y_j as she was playing in the case when Dominator started the game on w. This will ensure that the game will last at least $1+k+2\ell$ moves. Assume next that at some stage of the game Dominator played in some X_j before he played w. In that case Staller replies with a move in X_j and consequently at least $k+1$ moves will be played in all the X_i's. Assume finally that Dominator will never play the vertex w. Then, as soon as he plays in some X_i, Staller replies with a move in the same X_i. In this way (at least) 2ℓ vertices will be played on the subgraphs Y_i and (at least) $k+1$ on the X_i's. (Note that here we essentially need the assumption that $k \geq 1$.) Hence also in this case at least $1+k+2\ell$ moves will be played, therefore

$$\gamma_{tg}({}_kG_\ell) \geq 1 + k + 2\ell = n + \ell\,.$$

Since this holds for any $0 \leq \ell \leq n-2$ and because it is clear that

$$\gamma_t({}_kG_\ell) = k + \ell + 1 = n\,,$$

we are done. □

The game domination number and the game total domination number are related as follows:

Corollary 3.2 *If G is a graph on at least two vertices, then $\gamma_g(G) \leq 2\gamma_{tg}(G) - 1$.*

Proof. It was proved in [2] that $\gamma_g(G) \leq 2\gamma(G) - 1$, hence

$$\begin{aligned} \gamma_g(G) &\leq 2\gamma(G) - 1 \quad ([2, \text{Theorem } 1]) \\ &\leq 2\gamma_t(G) - 1 \\ &\leq 2\gamma_{tg}(G) - 1 \quad (\text{Theorem } 3.1)\,. \end{aligned}$$

□

To see that Corollary 3.2 is close to being optimal consider the following examples. For any $n \geq 2$, let G_n be the graph obtained from the complete graph on n vertices by attaching n leaves to each of its vertices. We claim that

$$\gamma_{tg}(G_n) = n + 1 \quad \text{and} \quad \gamma_g(G_n) = 2n - 1\,.$$

Consider first the total domination game and let Dominator start the game in a vertex u of the n-clique. (By the following it is clear that playing a leaf in the first move is not optimal for Dominator.) An optimal reply of Staller is to play a leaf attached to u, because otherwise she could play no leaf in due course. After the first two moves both players must alternatively play all the $n-1$ remaining vertices of the n-clique. Therefore, $\gamma_{tg}(G_n) = n + 1$.

Consider next the usual domination game. From the Continuation Principle it follows that an optimal first move for Dominator is a vertex of the n-clique. Since each vertex of it has n leafs attached, Staller will be able to play a leaf as long as Dominator has not played all the vertices from the clique. So $\gamma_g(G_n) \geq 2n - 1$. On the other hand, $\gamma_g(G_n) \leq 2\gamma(G_n) - 1 = 2n - 1$. Consequently, $\gamma_g(G_n) = 2n - 1$.

4 Game total domination versus domination

The game total domination number can be bounded by the domination number as follows.

Theorem 4.1 *If G is a graph such that $\gamma(G) \geq 2$, then $\gamma(G) \leq \gamma_{tg}(G) \leq 3\gamma(G) - 2$. Moreover, the bounds are sharp.*

Proof. The lower bound follows from the inequality chain $\gamma(G) \leq \gamma_t(G) \leq \gamma_{tg}(G)$. Note first that $\gamma(K_{2,n}) = \gamma_{tg}(K_{2,n}) = 2$, $n \geq 3$. Let u be a vertex of degree 2 in $K_{2,n}$ and append to it a path of length 2. Call this graph H_n. Then $\gamma(H_n) = 3$ and hence also $\gamma_{tg}(H_n) \geq 3$. Suppose now that Dominator plays u. Then the only legal move for Staller is one of the three neighbors of u in H_n. But then Dominator can finish the game in the next move, thus $\gamma_{tg}(H_n) \leq 3$. Consequently $\gamma(H_n) = \gamma_{tg}(H_n) = 3$. To get all other possible values that attain the lower bound apply Proposition 2.3.

To prove the upper bound, let D be an arbitrary $\gamma(G)$-set. Dominator's strategy is to select vertices in D sequentially whenever such a move is legal. Once Dominator has played all allowable vertices in D, at most $2|D| - 1 = 2\gamma(G) - 1$ moves have been made. At this point of the game all vertices in $N(D)$ are totally dominated.

Case 1. No vertex in D is currently totally dominated.
In this case D is an independent set and both Dominator and Staller only played vertices from D. That is, exactly $|D| = \gamma(G)$ moves have been made. The only

remaining legal moves are those that totally dominate vertices in D, implying that at most $|D|$ additional moves are required to complete the game. Hence the total number of moves played is at most $2|D| = 2\gamma(G) \le 3\gamma(G) - 2$.

Case 2. At least one vertex in D is currently totally dominated.
Now the only legal moves remaining in the game are those that totally dominate vertices in D if any are not yet totally dominated. This implies that at most $|D|-1$ additional moves are required to complete the game. Hence the total number of moves is at most $(2|D|-1) + (|D|-1) = 3\gamma(G) - 2$.

This proves the upper bound. In order to show its sharpness, let B_k, $k \ge 2$, be the graph constructed as follows. For $i = 1, 2, \ldots, k^2$ let Q_i be a complete graph of order k with the vertex set $\{y_1^{(i)}, y_2^{(i)}, \ldots, y_k^{(i)}\}$. Then take the disjoint union of these cliques, add vertices $x_1, x_2, \ldots, x_k$, and for $i = 1, 2, \ldots, k$, join x_i to the k^2 vertices $y_i^{(1)}, y_i^{(2)}, \ldots, y_i^{(k^2)}$. Finally, add a pendant edge to each vertex x_i and call the resulting leaf w_i. See Fig. 1 for B_k. For further reference let $X = \{x_1, x_2, \ldots, x_k\}$, and for $i = 1, 2, \ldots, k$ let $Y_i = \{y_i^{(j)} :\ j = 1, 2, \ldots, k^2\}$.

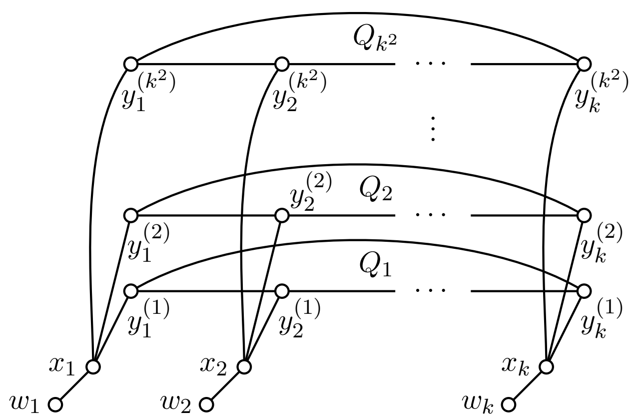

Figure 1: The graph B_k

Note first that $\gamma(B_k) = k$ and in fact, X is a unique minimum dominating set of B_k. Since we have already proved that $\gamma_{tg}(B_k) \le 3k-2$, it remains to show that Staller has a strategy that will ensure the game to last $3k-2$ moves. Her strategy is to play vertices from Y_1 as long as possible and to never play a vertex from X. Then at least $k-1$ moves are played on vertices from Y_1, since as long as not all vertices from X are played, Staller can play inside Y_1. Note next that during the game all vertices from X must be played in order to totally dominate vertices w_i. In order to totally dominate the vertices from $X' = X \setminus \{x_1\}$, an additional $k-1$ moves are needed since the vertices from X' are pairwise at distance 3. Hence at least $(k-1) + k + (k-1) = 3k-2$ moves are needed to complete the game. □

As an immediate consequence of Theorem 4.1 and the fact that $\gamma(G) \leq \gamma_g(G)$ for all graphs G, we have the following upper bound on the game total domination number in terms of the game domination number.

Corollary 4.2 *If G is a graph such that $\gamma(G) \geq 2$, then $\gamma_{tg}(G) \leq 3\gamma_g(G) - 2$.*

We remark that if $G = 2K_2$, then $\gamma_{tg}(G) = 4$ and $\gamma_g(G) = 2$, and so in this case $\gamma_{tg}(G) = 3\gamma_g(G) - 2$.

Acknowledgements

Research supported in part by the South African National Research Foundation and the University of Johannesburg, by the Ministry of Science of Slovenia under the grants P1-0297, and by a grant from the Simons Foundation (#209654 to Douglas Rall) and by the Wylie Enrichment Fund of Furman University. We thank the referees for several useful suggestions.

References

[1] B. Brešar, P. Dorbec, S. Klavžar, G. Košmrlj, Domination game: effect of edge- and vertex-removal, Discrete Math. 330 (2014) 1–10.

[2] B. Brešar, S. Klavžar, D. F. Rall, Domination game and an imagination strategy, SIAM J. Discrete Math. 24 (2010) 979–991.

[3] B. Brešar, S. Klavžar, D. F. Rall, Domination game played on trees and spanning subgraphs, Discrete Math. 313 (2013) 915–923.

[4] B. Brešar, S. Klavžar, G. Košmrlj, D. F. Rall, Domination game: extremal families of graphs for the 3/5-conjectures, Discrete Appl. Math. 161 (2013) 1308–1316.

[5] Cs. Bujtás, Domination game on trees without leaves at distance four, Proceedings of the 8th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications (A. Frank, A. Recski, G. Wiener, eds.), June 4–7, 2013, Veszprém, Hungary, 73–78.

[6] Cs. Bujtás, Domination game on forests, arXiv:1404.1382 (2014).

[7] P. Dorbec, G. Košmrlj, G. Renault, The domination game played on unions of graphs, manuscript, 2013.

[8] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[9] M. A. Henning, A. Yeo, *Total Domination in Graphs.* Springer Monographs in Mathematics, ISBN-13: 978-1461465249 (2013).

[10] W. B. Kinnersley, D. B. West, R. Zamani, Extremal problems for game domination number, SIAM J. Discrete Math. 27 (2013) 2090–2107.

[11] W. B. Kinnersley, D. B. West, R. Zamani, Game domination for grid-like graphs, manuscript, 2012.

[12] G. Košmrlj, Realizations of the game domination number, J. Comb. Optim. 28 (2014) 447–461.

[13] R. Zamani, Hamiltonian cycles through specified edges in bipartite graphs, domination game, and the game of revolutionaries and spies, Ph. D. Thesis, University of Illinois at Urbana-Champaign. Pro-Quest/UMI, Ann Arbor (Publication No. AAT 3496787)