Progress Towards the Total Domination Game $\frac{3}{4}$-Conjecture

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Abstract

In this paper, we continue the study of the total domination game in graphs introduced in [Graphs Combin. 31(5) (2015), 1453–1462], where the players Dominator and Staller alternately select vertices of G. Each vertex chosen must strictly increase the number of vertices totally dominated, where a vertex totally dominates another vertex if they are neighbors. This process eventually produces a total dominating set S of G in which every vertex is totally dominated by a vertex in S. Dominator wishes to minimize the number of vertices chosen, while Staller wishes to maximize it. The game total domination number, $\gamma_{\rm tg}(G)$, of G is the number of vertices chosen when Dominator starts the game and both players play optimally. Henning, Klavžar and Rall [Combinatorica, to appear] posted the $\frac{3}{4}$-Game Total Domination Conjecture that states that if G is a graph on n vertices in which every component contains at least three vertices, then $\gamma_{\rm tg}(G) \le \frac{3}{4}n$. In this paper, we prove this conjecture over the class of graphs G that satisfy both the condition that the degree sum of adjacent vertices in G is at least 4 and the condition that no two vertices of degree 1 are at distance 4 apart in G. In particular, we prove that by adopting a greedy strategy, Dominator can complete the total domination game played in a graph with minimum degree at least 2 in at most $3n/4$ moves.

Keywords: Total domination game; Game total domination number; 3/4-Conjecture
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1 Introduction

The domination game in graphs was first introduced by Brešar, Klavžar, and Rall [2] and extensively studied afterwards in [1, 3, 4, 5, 6, 8, 9, 11, 15, 17, 18] and elsewhere. A vertex *dominates* itself and its neighbors. A *dominating set* of G is a set S of vertices of G such that every vertex in G is dominated by a vertex in S. The *domination game* played on a graph G consists of two players, *Dominator* and *Staller*, who take turns choosing a vertex from G. Each vertex chosen must dominate at least one vertex not dominated by the vertices previously chosen. The game ends when the set of vertices chosen becomes a dominating set in G. Dominator wishes to minimize the number of vertices chosen, while Staller wishes to end the game with as many vertices chosen as possible. The *game domination number*, $\gamma_g(G)$, of G is the number of vertices chosen when Dominator starts the game and both players play optimally.

Much interest in the domination game arose from the 3/5-Game Domination Conjecture posted by Kinnersley, West, and Zamani in [17], which states that if G is an isolate-free forest on n vertices, then $\gamma_g(G) \le \frac{3}{5}n$. This conjecture remains open, although to date it is shown to be true for graphs with minimum degree at least 2 (see, [12]), and for isolate-free forests in which no two leaves are at distance 4 apart (see, [6]).

Recently, the total version of the domination game was investigated in [13], where it was demonstrated that these two versions differ significantly. A vertex *totally dominates* another vertex if they are neighbors. A *total dominating set* of a graph G is a set S of vertices such that every vertex of G is totally dominated by a vertex in S. The *total domination game* consists of two players called *Dominator* and *Staller*, who take turns choosing a vertex from G. Each vertex chosen must totally dominate at least one vertex not totally dominated by the set of vertices previously chosen. Following the notation of [13], we call such a chosen vertex a *legal move* or a *playable vertex* in the total domination game. The game ends when the set of vertices chosen is a total dominating set in G. Dominator's objective is to minimize the number of vertices chosen, while Staller's is to end the game with as many vertices chosen as possible.

The *game total domination number*, $\gamma_{\rm tg}(G)$, of G is the number of vertices chosen when Dominator starts the game and both players employ a strategy that achieves their objective. If Staller starts the game, the resulting number of vertices chosen is the *Staller-start game total domination number*, $\gamma'_{\rm tg}(G)$, of G.

A *partially total dominated graph* is a graph together with a declaration that some vertices are already totally dominated; that is, they need not be totally dominated in the rest of the game. In [13], the authors present a key lemma, named the *Total Continuation Principle*, which in particular implies that when the game is played on a partially total dominated graph G, the numbers $\gamma_{\rm tg}(G)$ and $\gamma'_{\rm tg}(G)$ can differ by at most 1.

Determining the exact value of $\gamma_{\rm tg}(G)$ and $\gamma'_{\rm tg}(G)$ is a challenging problem, and is currently known only for paths and cycles [10]. Much attention has therefore focused on obtaining upper bounds on the game total domination number in terms of the order of the graph. The best general upper bound to date on the game total domination number for general graphs is established in [14].

Theorem 1 ([14]) *If G is a graph on n vertices in which every component contains at least three vertices, then $\gamma_{\rm tg}(G) \le \frac{4}{5}n$.*

Our focus in the present paper is the following conjecture posted by Henning, Klavžar and Rall [14].

$\frac{3}{4}$-Game Total Domination Conjecture ([14]) *If G is a graph on n vertices in which every component contains at least three vertices, then $\gamma_{\rm tg}(G) \le \frac{3}{4}n$.*

As remarked in [14], if the $\frac{3}{4}$-Game Total Domination Conjecture is true, then the upper bound is tight as may be seen by taking, for example, $G \cong k_1P_4 \cup k_2P_8$ where $k_1, k_2 \ge 0$ and $k_1 + k_2 \ge 1$. Since $\gamma_{\rm tg}(P_4) = \gamma'_{\rm tg}(P_4) = 3$ and $\gamma_{\rm tg}(P_8) = \gamma'_{\rm tg}(P_8) = 6$, the optimal strategy of Staller is whenever Dominator starts playing on a component of G, she plays on that component and adopts her optimal strategy on the component. This shows that $\gamma_{\rm tg}(G) = 3k_1 + 6k_2 = 3n/4$, where $n = 4k_1 + 8k_2$ is the number of vertices in G.

Bujtás, Henning, and Tuza [7] recently proved the $\frac{3}{4}$-Conjecture over the class of graphs with minimum degree at least 2. To do this, they raise the problem to a higher level by introducing a transversal game in hypergraphs, and establish a tight upper bound on the game transversal number of a hypergraph with all edges of size at least 2 in terms of its order and size. As an application of this result, they prove that if G is a graph on n vertices with minimum degree at least 2, then $\gamma_{\rm tg}(G) \le \frac{8}{11}n$, which validates the $\frac{3}{4}$-Game Total Domination Conjecture on graphs with minimum degree at least 2.

For notation and graph theory terminology not defined herein, we in general follow [16]. We denote the *degree* of a vertex v in a graph G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. The minimum degree among the vertices of G is denoted by $\delta(G)$. A vertex of degree 1 is called a *leaf* and its neighbor a *support vertex*. If X and Y are subsets of vertices in a graph G, then the set X *totally dominates* the set Y in G if every vertex of Y is adjacent to at least one vertex of X. In particular, if X totally dominates the vertex set of G, then X is a total dominating set in G. For more information on total domination in graphs see the recent book [16]. Since an isolated vertex in a graph cannot be totally dominated by definition, all graphs considered will be without isolated vertices. We also use the standard notation $[k] = \{1, \ldots, k\}$.

2 Main Result

In this paper we prove the following result. Its proof is given in Section 3.

Theorem 2 *The $\frac{3}{4}$-Game Total Domination Conjecture is true over the class of graphs G that satisfy both conditions* (a) *and* (b) *below:*
(a) *The degree sum of adjacent vertices in G is at least* 4*.*
(b) *No two leaves are at distance exactly* 4 *apart in G.*

As a special case of Theorem 2, the $\frac{3}{4}$-Game Total Domination Conjecture is valid on graphs with minimum degree at least 2.

Corollary 1 ([7]) *The $\frac{3}{4}$-Game Total Domination Conjecture is true over the class of graphs with minimum degree at least* 2*.*

3 Proof of Main Result

In this section, we give a proof of our main theorem, namely Theorem 2. For this purpose, we adopt the approach of the authors in [14] and color the vertices of a graph with four colors that reflect four different types of vertices. More precisely, at any stage of the game, if D denotes the set of vertices played to date where initially $D = \emptyset$, we define as in [14] a *colored-graph* with respect to the played vertices in the set D as a graph in which every vertex is colored with one of four colors, namely white, green, blue, or red, according to the following rules.

- A vertex is colored *white* if it is not totally dominated by D and does not belong to D.
- A vertex is colored *green* if it is not totally dominated by D but belongs to D.
- A vertex is colored *blue* if it is totally dominated by D but has a neighbor not totally dominated by D.
- A vertex is colored *red* if it and all its neighbors are totally dominated by D.

As remarked in [14], in a partially total dominated graph the only playable vertices are those that have a white or green neighbor since a played vertex must totally dominate at least one new vertex. In particular, no red or green vertex is playable. Further, as observed in [14], once a vertex is colored red it plays no role in the remainder of the game, and edges joining two blue vertices play no role in the game. Therefore, we may assume a partially total dominated graph contains no red vertices and has no edge joining two blue vertices. The resulting graph is called a *residual graph.* We note that the degree of a white or green vertex in the residual graph remains unchanged from its degree in the original graph.

Where our approach in the current paper differs from that in [14] is twofold. First, we define two new colors in the colored-graph that may possibly be introduced as the game is played. Second, our assignment of weights to vertices of each color differs from the assignment in [14]. Here, we associate a weight with every vertex in the residual graph as follows:

Color of vertex	*Weight of vertex*
white	3
green	2
blue	1
red	0

Table 1. The weights of vertices according to their color.

We denote the *weight* of a vertex v in the residual graph G by $\mathrm{w}(v)$. For a subset $S \subseteq V(G)$ of vertices of G, the *weight* of S is the sum of the weights of the vertices in S, denoted w(S). The *weight* of G, denoted w(G), is the sum of the weights of the vertices in G; that is, $\mathrm{w}(G) = \mathrm{w}(V(G))$. We define the *value* of a playable vertex as the decrease in weight resulting from playing that vertex.

We say that Dominator can *achieve his* 4*-target* if he can play a sequence of moves guaranteeing that on average the weight decrease resulting from each played vertex in the game is at least 4. In order to achieve his 4-target, Dominator must guarantee that a sequence of moves $m_1, \ldots, m_k$ are played, starting with his first move m_1, and with moves alternating between Dominator and Staller such that if w_i denotes the decrease in weight after move m_i is played, then

$$\sum_{i=1}^{k} \mathrm{w}_i \geq 4k\,, \tag{1}$$

and the game is completed after move m_k. In the discussion that follows, we analyse how Dominator can achieve his 4-target. For this purpose, we describe a move that we call a greedy move.

- A *greedy move* is a move that decreases the weight by as much as possible. We say that Dominator follows a *greedy strategy* if he plays a greedy move on each turn.

We are now in a position to prove our main result, namely Theorem 2. Recall its statement.

Theorem 2. *The* $\frac{3}{4}$*-Game Total Domination Conjecture is true over the class of graphs* G *that satisfy both conditions* (a) *and* (b) *below:*

(a) *The degree sum of adjacent vertices in* G *is at least* 4*.*
(b) *No two leaves are at distance exactly* 4 *apart in* G*.*

Proof. Let G be a graph that satisfies both conditions (a) and (b) in the statement of the theorem. Coloring the vertices of G with the color white we produce a colored-graph in which every vertex is colored white. In particular, we note that G has n white vertices and has weight $\mathrm{w}(G) = 3n$. Before any move of Dominator, the game is in one of the following two phases.

- *Phase* 1, if there exists a legal move of value at least 5.
- *Phase* 2, if every legal move has value at most 4.

We proceed with the following claims.

Claim 2.1 *Every legal move in a residual graph decreases the total weight by at least* 3*.*

Proof. Every legal move in a colored-graph is a white vertex with at least one white neighbor or a blue vertex with at least one white or green neighbor. Let v be a legal move in a residual graph. Suppose that v is a white vertex, and so v has at least one white neighbor. When v is played, the vertex v is recolored green while each white neighbor of v is recolored blue, implying that the weight decrease resulting from playing v is at least 3. Suppose that v is a blue vertex, and so each neighbor of v is colored white or green. Playing the vertex v recolors each white neighbor of v blue or red and recolors each green neighbor of v red. The weight of each neighbor of the blue vertex v is therefore decreased by at least 2 when v is played, while the vertex v itself is recolored red and its weight decreases by 1. Hence, the total weight decrease resulting from playing v is at least 3. (□)

Claim 2.2 *Let R be the residual graph and suppose the game is in Phase 2 and C is an arbitrary component of R. If the minimum degree in G is at least 2, then one of the following holds.*

(a) *$C \cong P_4$, with both leaves colored blue and both internal vertices colored white.*

(b) *$C \cong P_3$, with both leaves colored blue and with the central vertex colored green.*

Proof. Since the degree of a white or green vertex in the residual graph R remains unchanged from its degree in the original graph G, the minimum degree at least 2 requirement in G implies that R contains no white or green leaf. We show first that every white vertex has at most one white neighbor in the residual graph R. Suppose, to the contrary, that a white vertex v has at least two white neighbors. When v is played the weight decreases by at least $1 + 2 \cdot 2 = 5$, since the vertex v is recolored green while each white neighbor of v is recolored blue. This contradicts the fact that every legal move decreases the weight by at most 4.

We show next that every blue vertex has degree 1 in the residual graph R. Suppose, to the contrary, that a blue vertex v has degree at least 2 in R. Playing the vertex v recolors each white neighbor of v blue or red and recolors each green neighbor of v red. Thus, playing the vertex v decreases the weight of each of its neighbors by at least 2. In addition, the vertex v is recolored red, and so its weight decreases by 1. Hence, the weight decrease resulting from playing v is at least 5, a contradiction.

Suppose that R contains a green vertex, v. Each neighbor of v is colored blue, and, by our earlier observations, is therefore a blue leaf. As noted earlier, the vertex v has at least two neighbors in R. If v has at least three neighbors in R, then since every neighbor of v is a blue leaf, the weight decrease resulting from playing an arbitrary neighbor of v is at least 5, noting that such a move recolors v and all its neighbors red. This produces a contradiction. Therefore, v has exactly two neighbors in R, implying that the component containing v is a path isomorphic to P_3 with both leaves colored blue and with the central vertex, namely v, colored green, and therefore satisfies condition (b) in the statement of the claim. Hence, we may assume that there is no green vertex, for otherwise the desired result holds.

Suppose that there is a white vertex, u, in the residual graph R. As noted earlier, the vertex u has at least two neighbors in R. Suppose that u has no white neighbor. By our earlier observations, every neighbor of u is a blue leaf. Playing a neighbor of u therefore recolors all the neighbors of u from blue to red, and decreases the weight by at least $3 + d_R(u) \geq 5$, a contradiction. Therefore, the vertex u has exactly one white neighbor. Let x be the white neighbor of u. Every neighbor of u different from x is a blue leaf, and every neighbor of x different from u is a blue leaf. Since neither u nor x is a leaf, both u and x have at least one blue leaf neighbor. Suppose that u or x, say u, has degree at least 3. Playing the vertex x

recolors x from white to green, recolors u from white to blue, and recolors each neighbor of u different from x from blue to red, implying that the total weight decrease resulting from playing x is at least 5, a contradiction. Therefore, both u and x have degree 2. Thus, the component containing u and x is a path isomorphic to P_4 with both leaves colored blue and both internal vertices colored white, and therefore satisfies condition (a) in the statement of the claim. This completes the proof of Claim 2.2. $(\Box)$

By Claim 2.2, once the game enters Phase 2 the residual graph is determined and each component satisfies one of the conditions (a)–(d) in the statement of the claim.

Claim 2.3 *If the minimum degree in G is at least 2, then Dominator can achieve his 4-target by following a greedy strategy.*

Proof. Suppose that $\delta(G) \geq 2$ and Dominator follows a greedy strategy. Thus, at each stage of the game, Dominator plays a (greedy) move that decreases the weight by as much as possible. By Claim 2.1, every move of Staller decreases the weight by at least 3. Hence, whenever Dominator plays a vertex that decreases the weight by at least 5, his move, together with Staller's response, decreases the weight by at least 8. Therefore, we may assume that at some stage the game enters Phase 2, for otherwise Inequality (1) is satisfied upon completion of the game and Dominator can achieve his 4-target.

Suppose that the first ℓ moves of Dominator each decrease the weight by at least 5, and that his $(\ell+1)$st move decreases the weight by at most 4. Thus, $\mathrm{w}(m_{2i-1}) + \mathrm{w}(m_{2i}) \geq 8$ for $i \in [\ell]$, and $\mathrm{w}(m_{2\ell+1}) \leq 4$. Thus,

$$\sum_{i=1}^{2\ell} \mathrm{w}_i = \sum_{i=1}^{\ell} (\mathrm{w}(m_{2i-1}) + \mathrm{w}(m_{2i})) \geq 8 \cdot \ell = 4 \cdot (2\ell).$$

Let R denote the residual graph immediately after Staller plays her ℓth move, namely the move $m_{2\ell}$. By Claim 2.2, every component C of R satisfies $C \cong P_4$, with both leaves colored blue and both internal vertices colored white, or $C \cong P_3$, with both leaves colored blue and with the central vertex colored green. If $C \cong P_4$, then $\mathrm{w}(V(C)) = 8$ and exactly two additional moves are required to totally dominate the vertices $V(C)$, while if $C \cong P_3$, then $\mathrm{w}(V(C)) = 4$ and exactly one move is played in C to totally dominate the vertices $V(C)$. Suppose that R has t components isomorphic to P_4 and s components isomorphic to P_3. Thus, $2t + s$ additional moves are needed to complete the game once it enters Phase 2. Further, these remaining $2t + s$ moves satisfy

$$\sum_{i=2\ell+1}^{2\ell+2t+s} \mathrm{w}_i = 4 \cdot (2t + s).$$

Hence,

$$\sum_{i=1}^{2\ell+2t+s} \mathrm{w}_i = \sum_{i=1}^{2\ell} \mathrm{w}_i + \sum_{i=2\ell+1}^{2\ell+2t+s} \mathrm{w}_i \geq 4 \cdot (2\ell + 2t + s),$$

and so Inequality (1) is satisfied upon completion of the game. Thus, Dominator can achieve his 4-target simply by following a greedy strategy. This completes the proof of Claim 2.3. (□)

We now return to the proof of Theorem 2. By Claim 2.3, we may assume that G contains at least one leaf, for otherwise Dominator can achieve his 4-target (and he can do so by following a greedy strategy).

As the game is played, we introduce a new color, namely *purple*, which we use to recolor certain white support vertices. A purple vertex will have the same properties as a white vertex, except that the weight of a purple vertex is 4. The idea behind the re-coloring is that the additional weight of 1 assigned to a purple vertex will represent a *surplus weight* that we can "bank" and withdraw later. To formally define the recoloring procedure, we introduce additional terminology.

Consider a residual graph R that arises during the course of the game. Suppose that $uvwx$ is an induced path in R, where v, w and x are all white vertices, and where w is a support vertex and x a leaf in R. We note that the vertex u is colored white or blue. Such a vertex u turns out to be problematic for Dominator, and we call such a vertex a *problematic vertex*. Further, we call the path $uvwx$ a *problematic path* associated with u, and we call w a *support vertex associated with* u.

Suppose that Staller plays a problematic vertex, u. Suppose that there are exactly k support vertices, say $w_1, \dots, w_k$, associated with u. We note that $k \geq 1$. For $i \in [k]$, let $uv_iw_ix_i$ be a problematic path associated with u that contains w_i. Thus, v_i, w_i and x_i are all white vertices, w_i is a support vertex, and x_i a leaf in R. Since no two leaves are at distance 4 apart in G, we note that if $k \geq 2$, then $v_i \neq v_j$ for $1 \leq i, j \leq k$ and $i \neq j$.

Suppose first that $k \geq 2$. In this case, playing the problematic vertex, u, decreases the total weight by at least $2k + 1$, since u is recolored from blue to red or from white to green, while each neighbor v_i, $i \in [k]$, of u is recolored from white to blue. Thus, the current value of u is at least $2k + 1$. We now discharge the value of u as follows. We discharge a weight of k from the value of u and add a weight of 1 to every support vertex w_i, $i \in [k]$. Thus, by playing u the resulting decrease in total weight is the value of u in R minus k, which is at least $k + 1 \geq 3$. Further, the weight of each (white) support vertex w_i, $i \in [k]$, increases from 3 to 4. We now re-color each support vertex w_i, $i \in [k]$, from white to purple.

Suppose secondly that $k = 1$ and the value of u is at least 4. In this case, we proceed exactly as before: We discharge a weight of $k = 1$ from the value of u, add a weight of 1 to the support vertex w_1, and re-color w_1 from white to purple. Thus, by playing u the resulting decrease in total weight is at least 3.

In both cases, we note that the new weight of w_i, $i \in [k]$, is 4. Thus, every newly created purple vertex is a support vertex and has weight 4. We define a purple vertex to have the identical properties of a white vertex, except that its weight is 4. Thus, a purple vertex is not totally dominated by the vertices played to date and has not yet been played.

We note that if Staller plays a problematic vertex, u, whose current value is at least 4, then the above discharging argument recolors every support vertex associated with u from white to purple. Further, by playing u the resulting decrease in total weight is at least 3, and the weight of each newly created purple vertex is 4. We state this formally as follows.

Claim 2.4 *If Staller plays a problematic vertex whose current value is at least* 4*, then the resulting decrease in total weight is at least* 3*.*

We note further that if Staller plays a problematic vertex, u, whose current value is exactly 3, the two internal vertices of the problematic path associated with u are unique. In particular, the support vertex associated with u is unique.

We now introduce an additional new color, namely *indigo*, which we use to recolor certain purple vertices. An indigo vertex will have the same properties of a blue vertex, except that the weight of an indigo vertex is 2 (while the weight of a blue vertex is 1). The idea behind the re-coloring is that the additional weight of 1 assigned to an indigo vertex will represent a *surplus weight* that, as before, we can "bank" and withdraw later.

More formally, suppose that a white leaf, say z, adjacent to a purple vertex, say x, is played. When the leaf z is played, it changes color from white to green and its support neighbor, x, changes color from purple to blue (noting that a purple vertex has the same properties as a white vertex). Thus, when the leaf z is played, the weight of z decreased by 1 and the weight of x decreased by 3, implying that the value of z is at least 4. However, when z is played we discharge a weight of 1 from the value of z and add a weight of 1 to the vertex x, thereby increasing its weight to 2. Thus, by playing z the resulting decrease in total weight is one less than the value of z, and is therefore at least $4 - 1 = 3$. Further, the weight of the resulting support vertex x increases from 1 to 2. We now re-color the vertex x from blue to *indigo*. We note the following.

Claim 2.5 *If Staller plays a white leaf adjacent to a purple support vertex, then the resulting decrease in total weight is at least* 3*.*

We establish next the structure of a component in the residual graph when the game is in Phase 2.

Claim 2.6 *Let R be the residual graph. If the game is in Phase 2 and if C is an arbitrary component of R, then one of the following holds.*
(a) *$C \cong P_4$, with both leaves colored blue and both internal vertices colored white.*
(b) *$C \cong P_3$, with both leaves colored blue and with the central vertex colored green.*
(c) *$C \cong P_2$, with one leaf colored blue and the other colored green.*
(d) *$C \cong P_2$, with one leaf colored blue and the other colored white.*
(e) *$C \cong P_2$, with one leaf colored indigo and the other colored green.*

Proof. Suppose the game is in Phase 2. As remarked earlier, a purple vertex has the same properties as a white vertex, except that the weight of a purple vertex is 4. Further, an indigo vertex has the same properties of a blue vertex, except that the weight of an indigo vertex is 2. Analogously as in the proof of Claim 2.2, every white or purple vertex has at most one white neighbor in the residual graph R, and every blue or indigo vertex has degree 1 in R.

Suppose that R contains an indigo vertex, w. Necessarily, every indigo vertex in R is adjacent to a green leaf in R. Thus, the indigo vertex w is a leaf in R adjacent to a green leaf in R, and therefore satisfies condition (e) in the statement of the claim. Hence, we may assume that no vertex in R is colored indigo, for otherwise the desired result holds.

Suppose that R contains a green vertex, v. If v is not a leaf, then as in the proof of Claim 2.2, the green vertex v is the central vertex of a path component isomorphic to P_3 with both leaves colored blue and therefore satisfies condition (b) in the statement of the claim. Hence, we may assume that v is a leaf. By assumption, there are no indigo vertices, implying that the component containing v is a path isomorphic to P_2 with one leaf colored blue and the other colored green, and therefore satisfies condition (c) in the statement of the claim. Hence, we may assume that there is no green vertex, for otherwise the desired result holds.

Suppose that R contains a purple vertex, z. Necessarily, every purple vertex in R is adjacent to a white leaf in R. Let y be the white leaf-neighbor of the purple vertex z. We note that the degree of a purple vertex in the residual graph R remains unchanged from its degree in the original graph G. Since the degree sum of adjacent vertices in G is at least 4, and since y is a white leaf in both G and R, the vertex z has degree at least 3. Playing the vertex y recolors y from white to green, recolors z from purple to indigo, and recolors all neighbors of z different from y from blue to red. Hence, playing y decreases the total weight by at least 5, a contradiction. Hence, there is no purple vertex in R.

Suppose that there is a white vertex, u, in R. Suppose that u has no white neighbor. By our earlier observations, every neighbor of u is a blue leaf. Playing a neighbor of u therefore recolors all the neighbors of u from blue to red. If u is not a leaf, then playing a neighbor of u decreases the weight by at least 5, a contradiction. Hence, u is a leaf, and the component containing u is a path isomorphic to P_2 with one leaf colored blue and the other colored white. We may therefore assume that the vertex u has exactly one white neighbor, for otherwise the component containing u satisfies condition (d) in the statement of the claim.

Let x be the white neighbor of u. Every neighbor of u different from x is a blue leaf, and every neighbor of x different from u is a blue leaf. Suppose that u or x, say u, is a leaf. Since the degree sum of adjacent vertices in G is at least 4, and $d_G(u) = d_R(u)$, the vertex x has degree at least 3. Playing the vertex u recolors u from white to green, recolors x from white to blue, and recolors all neighbors of x different from u from blue to red. Hence, playing u decreases the total weight by at least 5, a contradiction. Therefore, neither u nor x is a leaf. Proceeding now analogously as in the proof of Claim 2.2, the component containing u and x is a path isomorphic to P_4 with both leaves colored blue and both internal vertices colored white, and therefore satisfies condition (a) in the statement of the claim. This completes the proof of Claim 2.6. $(\Box)$

A white support vertex with a white leaf neighbor in R we call a *targeted support vertex* in R. Since no two leaves are at distance 4 apart in G, every pair of targeted support vertices in R are either adjacent or at distance at least 3 apart in R.

Dominator henceforth applies the following rules.

Dominator's strategy:

(R1) *Whenever Staller plays a white leaf adjacent to a targeted support vertex, Dominator immediately responds by playing on the resulting (blue) support vertex.*

(R2) *Whenever Staller plays a problematic vertex, u, whose current value is exactly 3, Dominator immediately responds by playing the unique (targeted) support vertex associated with u.*

(R3) *If Dominator cannot play according to (R1) and (R2), he plays a targeted support vertex of maximum value.*

(R4) *If Dominator cannot play according to (R1), (R2) and (R3), he plays a greedy move.*

We remark that if Dominator cannot play according to (R1) and (R2), then on her previous move Staller played neither a white leaf adjacent to a targeted support vertex nor a problematic vertex with current value exactly 3. It remains for us to show that Dominator's strategy which applies rules (R1), (R2), (R3) and (R4) above, does indeed guarantee that on average the weight decrease resulting from each played vertex in the game is at least 4. We note that Dominator's strategy when playing according to (R1), (R2) and (R3) is to play a targeted support vertex or a blue support vertex with a green leaf neighbor. However, the order in which he plays such support vertices is important. Recall that by our earlier assumptions, G contains at least one leaf. The following claim will prove to be useful.

Claim 2.7 *While Dominator plays according to rule* (R1)*,* (R2) *and* (R3)*, the following three statements hold.*

(a) *After each move of Dominator, every targeted support vertex has at least three white neighbors.*
(b) *After each move of Dominator, there is no green leaf adjacent to a blue vertex.*
(c) *Each move that Dominator plays has value at least* 5*.*

Proof. We proceed by induction on the number, $m \geq 1$, of moves played by Dominator whenever he plays according to rule (R1), (R2) and (R3). Further, we recall that the degree sum of adjacent vertices in G is at least 4 and the degree of a white vertex in the residual graph remains unchanged from its degree in the original graph. We note that every targeted support vertex has degree at least 3 in the residual graph. As observed earlier, every pair of targeted support vertices in R are either adjacent or at distance at least 3 apart in R. When Dominator plays a targeted support vertex, v say, then the neighbors of v that are themselves targeted support vertices change color from white to blue, and are therefore no longer targeted support vertices. Further, since no two targeted support vertices are at distance 2 apart in R, when Dominator plays the targeted support vertex v, the white neighbors of every *remaining* targeted support vertex (that was not a neighbor of v) retain their color.

On Dominator's first move of the game, he plays a targeted support vertex of maximum value according to rule (R3). Such a (white) support vertex has degree at least 3 and all its neighbors are white, and therefore playing his first move decreases the weight by at least 7 and no green leaf is created. This establishes the base case when $m = 1$. Suppose that $m \geq 2$ and that Dominator plays according to rule (R1), (R2) and (R3), and assume that after the first $m-1$ moves, every targeted support vertex has at least three white neighbors, there is no green leaf adjacent to a blue vertex, and each of his first $m-1$ moves has value at least 5. We show that after Dominator's mth move, the three properties (a), (b) and (c) hold.

Suppose that Staller's $(m-1)$st move plays a white leaf x adjacent to a targeted support vertex y. Her move recolors x from white to green, and recolors y from white to blue. By the inductive hypothesis, before Staller played her move, the vertex y had at least three white neighbors. According to rule (R1), Dominator immediately responds to Staller's $(m-1)$st move by playing on the resulting (blue) support vertex, y. Since the support vertex y has at least two white neighbors after Staller played her $(m-1)$st move, his move decreases the weight by at least 7. Further, since the white neighbors of every remaining targeted support vertex retain their color, after Dominator's mth move the induction hypothesis implies that every targeted support vertex has at least three white neighbors and there is no green leaf adjacent to a blue vertex.

Suppose that Staller's $(m-1)$st move plays neither a white leaf adjacent to a targeted support vertex nor a problematic vertex. In this case, the white neighbors of every remaining targeted support vertex retain their color after Staller's move. If there remains a targeted support vertex, then, according to rule (R3), Dominator's mth move plays a targeted support vertex. By induction, such a support vertex has at least three white neighbors, and therefore has value at least 7. Thus, as before, the desired properties (a), (b) and (c) follow by induction after Dominator's mth move.

Suppose that Staller's $(m-1)$st move plays a problematic vertex, u, whose current value is at least 4. Applying our discharging arguments, every targeted support vertex associated with u is recolored from white to purple. The only targeted support vertices affected by Staller's move, in the sense that it or at least one of its white neighbors changes color, are targeted support vertices associated with u or adjacent to u. Thus, as before, the desired properties (a), (b) and (c) follow by induction after Dominator's mth move.

Suppose, finally, that Staller's $(m-1)$st move plays a problematic vertex, u, whose current value is 3. In this case, either u is a blue leaf with a white neighbor or u is a white vertex with exactly one white neighbor. Further, the two internal vertices of a problematic path associated with u are unique. Let $uvwx$ be such a problematic path associated with u, and so v and w are unique. In fact, v is the only white neighbor of u. By the inductive hypothesis, immediately before Staller played her $(m-1)$st move, the targeted support vertex w has at least three white neighbors. Since the vertex v is the only such white neighbor of w that is adjacent to u, after Staller plays u, the (white) support vertex w has at least two white neighbors, including the white leaf neighbor x. According to rule (R2), Dominator immediately responds by playing as his mth move this unique targeted support vertex, w, associated with u. Since w has at least two white neighbors, it has value at least 5. As observed earlier, the white neighbors of every remaining targeted support vertex retain their color after Dominator's move. Therefore, after Dominator's mth move, the desired properties (a), (b) and (c) hold. (□)

By Claim 2.7, while Dominator plays according to rule (R1), (R2) and (R3), each move he plays has value at least 5. By Claim 2.1, Claim 2.4 and Claim 2.5, each move of Staller's decreases the weight by at least 3. Hence, each move Dominator plays during this stage of the game, together with Staller's response, decreases the weight by at least 8. Therefore, we may assume that at some stage of the game, Dominator cannot play according to (R1), (R2) and (R3), for otherwise Inequality (1) is satisfied upon completion of the game and Dominator can achieve his 4-target. We note that at this stage of the game, there no longer exists a targeted support vertex. Further, there is no green leaf adjacent to a blue vertex. This implies that no green leaf adjacent to a blue vertex can be created in the remainder of the game.

According to rule (R4), Dominator now plays a greedy move and he continues to do so until the game is complete. We may assume that at some stage the game enters Phase 2, for otherwise once again Inequality (1) is satisfied upon completion of the game and Dominator can achieve his 4-target. By Claim 2.6 and our observation that there is no green leaf adjacent to a blue vertex when the game is in Phase 2, if C is an arbitrary component of the residual graph R at this stage of the game when Dominator cannot play according to (R1), (R2) and (R3), then $C \not\cong P_2$ with one blue and one green vertex. That is, C satisfies one of (a), (b), (d) or (e) in the statement of Claim 2.6.

If $C \cong P_4$, then C satisfies statement (a) of Claim 2.6, implying that $\mathrm{w}(V(C)) = 8$ and exactly two additional moves are required to totally dominate the vertices $V(C)$. If $C \cong P_3$ or if $C \cong P_2$, then C satisfies statement (b), (d) or (e) of Claim 2.6, implying that $\mathrm{w}(V(C)) = 4$ and exactly one move is played in C to totally dominate the vertices $V(C)$. Analogously as in the proof of Claim 2.3, this implies that Dominator can achieve his 4-target. Thus, since G has n white vertices and has weight $\mathrm{w}(G) = 3n$, Dominator can make sure that the average decrease in the weight of the residual graph resulting from each played vertex in the game is at least 4. Thus, in the colored-graph G, $\gamma_{\rm tg}(G) \le w(G)/4 = 3n/4$. □

As an immediate consequence of the proof of Theorem 2 (see Claim 2.3), we have the following result.

Corollary 2 *If G is a colored-graph with $\delta(G) \ge 2$ and Dominator follows a greedy strategy, then he can achieve his 4-target.*

Corollary 2 in turn implies Corollary 1. Recall its statement.

Corollary 1 ([7]). *The $\frac{3}{4}$-Game Total Domination Conjecture is true over the class of graphs with minimum degree at least 2.*

Proof. Let G be a graph with $\delta(G) \geq 2$. Coloring the vertices of G with the color white we produce a colored-graph in which every vertex is colored white. In particular, we note that G has n white vertices and has weight $\mathrm{w}(G) = 3n$. By Corollary 2, Dominator can achieve his 4-target by following a greedy strategy. Thus, Dominator can make sure that the average decrease in the weight of the residual graph resulting from each played vertex in the game is at least 4. Thus, in the colored-graph G, $\gamma_{\rm tg}(G) \leq w(G)/4 = 3n/4$. $\Box$

4 Summary

As remarked earlier, the authors in [7] prove a stronger result than Corollary 1 by showing, using game transversals in hypergraphs, that if G is a graph on n vertices with minimum degree at least 2, then $\gamma_{\rm tg}(G) \leq \frac{8}{11}n$. However, our result, namely Corollary 2, is surprising in that Dominator can complete the total domination game played in a graph with minimum degree at least 2 in at most $\frac{3}{4}n$ moves by simply following a greedy strategy in the associated colored-graph in which every vertex is initially colored white. Our main result, namely Theorem 2, shows that the $\frac{3}{4}$-Game Total Domination Conjecture holds in a general graph G (with no isolated vertex) if we remove the minimum degree at least 2 condition, but impose the weaker condition that the degree sum of adjacent vertices in G is at least 4 and the requirement that no two leaves are at distance 4 apart in G.

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References

[1] B. Brešar, P. Dorbec, S. Klavžar, and G. Košmrlj, Domination game: effect of edge- and vertex-removal. *Discrete Math.* **330** (2014), 1–10.

[2] B. Brešar, S. Klavžar, and D. F. Rall, Domination game and an imagination strategy. *SIAM J. Discrete Math.* **24** (2010), 979–991.

[3] B. Brešar, S. Klavžar, and D. F. Rall, Domination game played on trees and spanning subgraphs. *Discrete Math.* **313** (2013), 915–923.

[4] B. Brešar, S. Klavžar, G. Košmrlj, and D. F. Rall, Domination game: extremal families of graphs for the 3/5-conjectures. *Discrete Appl. Math.* **161** (2013), 1308–1316.

[5] B. Brešar, S. Klavžar, and D. Rall, Domination game played on trees and spanning subgraphs. *Discrete Math.* **313** (2013), 915–923.

[6] Cs. Bujtás, Domination game on trees without leaves at distance four, Proceedings of the 8th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications (A. Frank, A. Recski, G. Wiener, eds.), June 4–7, 2013, Veszprém, Hungary, 73–78.

[7] Cs. Bujtás, M. A. Henning, and Z. Tuza, Total domination game: A proof of the $\frac{3}{4}$-Conjecture for graphs with minimum degree at least two, manuscript.

[8] Cs. Bujtás and Zs. Tuza, The Disjoint domination game. *Discrete Math.*, to appear.

[9] Cs. Bujtás, S. Klavžar, and G. Košmrlj, Domination game critical graphs. *Discuss. Math. Graph Theory*, to appear.

[10] P. Dorbec and M. A. Henning, Game total domination for cycles and paths. *Discrete Applied Math.*, to appear.

[11] P. Dorbec, G. Košmrlj, and G. Renault, The domination game played on unions of graphs. *Discrete Math.* **338** (2015), 71–79.

[12] M. A. Henning and W. B. Kinnersley, Domination Game: A proof of the 3/5-Conjecture for graphs with minimum degree at least two. *SIAM J. Discrete Math.* **30**(1) (2016), 20–35.

[13] M. A. Henning, S. Klavžar, and D. F. Rall, Total version of the domination game. *Graphs Combin.* **31**(5) (2015), 1453–1462.

[14] M. A. Henning, S. Klavžar, and D. F. Rall, The 4/5 upper bound on the game total domination number. *Combinatorica*, to appear.

[15] M. A. Henning and C. Löwenstein, Domination game: Extremal families for the 3/5-conjecture for forests. *Discuss. Math. Graph Theory*, to appear.

[16] M. A. Henning and A. Yeo, *Total domination in graphs (Springer Monographs in Mathematics)* 2013. ISBN: 978-1-4614-6524-9 (Print) 978-1-4614-6525-6 (Online).

[17] W. B. Kinnersley, D. B. West, and R. Zamani, Extremal problems for game domination number. *SIAM J. Discrete Math.* **27** (2013), 2090–2107.

[18] G. Košmrlj, Realizations of the game domination number. *J. Combin. Opt.* **28** (2014), 447–461.