Dominating sequences in graphs

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Abstract

A sequence of vertices in a graph G is called a legal dominating sequence if every vertex in the sequence dominates at least one vertex not dominated by those vertices that precede it, and at the end all vertices of G are dominated. While the length of a shortest such sequence is the domination number of G, in this paper we investigate legal dominating sequences of maximum length, which we call the Grundy domination number of G. We prove that every graph has a legal dominating sequence of each possible length between its domination number and its Grundy domination number, and characterize the graphs for which the domination number and Grundy domination number are both equal to k, for $k \leq 3$. We also prove that the decision version of the Grundy domination number is NP-complete, even when restricted to the class of chordal graphs, and present linear time algorithms for determining this number in trees, cographs and split graphs. Several results are extended to the context of edge covers in hypergraphs.

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1 Introduction

Given a hypergraph $\mathcal{H} = (X, \mathcal{E})$ with no isolated vertices an *edge cover* of $\mathcal{H}$ is a set of hyperedges from $\mathcal{E}$ that cover all vertices of X, i.e., the union of hyperedges from an edge cover is the ground set X. The minimum number of hyperedges in an edge cover of $\mathcal{H}$ is

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called the *(edge) covering number* of $\mathcal{H}$ and is denoted by $\rho(\mathcal{H})$, cf. [1]. When a greedy algorithm is applied aiming to obtain an edge cover, hyperedges from $\mathcal{H}$ are picked one by one, resulting in a sequence $\mathcal{C} = (C_1, \dots, C_r)$, where $C_i \in \mathcal{E}$. In each step i, $1 \le i \le r$, C_i is picked in such a way that it covers some vertex not captured by previous steps, i.e., $C_i \setminus (\cup_{j<i} C_j) \ne \emptyset$; we call this a *legal* choice of the hyperedge, and a sequence is *legal* if all hyperedges in the sequence are legally chosen. If the algorithm happens to produce an optimal solution, the set $\widehat{\mathcal{C}} = \{C_1, \dots, C_r\}$ is a minimum edge cover of cardinality $\rho(\mathcal{H})$, but in general $r \ge \rho(\mathcal{H})$. How large can this r be? We call the maximum length r of a sequence of hyperedges $(C_1, \dots, C_r)$ of $\mathcal{H}$, such that $C_i \setminus (\cup_{j<i} C_j) \ne \emptyset$ for all i, the *Grundy covering number* of $\mathcal{H}$, and denote it by $\rho_{gr}(\mathcal{H})$. (Note that the name was inspired by the Grundy coloring number, which is the largest number of colors obtainable by a greedy coloring algorithm.) Can each value between $\rho(\mathcal{H})$ and $\rho_{gr}(\mathcal{H})$ be realized as the length of some legal sequence of hyperedges in $\mathcal{H}$? When can the order of hyperedges in a legal sequence be changed so that the condition on enlargement of the set of covered vertices in each step is not destroyed? These and related questions concerning such sequences will be considered in this paper.

Our main focus and motivation are the so-called dominating sequences. In the above context they arise from the hypergraph of the closed neighborhoods of vertices in a graph G, i.e., $\mathcal{H} = (V(G), \mathcal{N}(G))$, where $\mathcal{N}(G)$ denotes the set of all closed neighborhoods of vertices in G. More precisely, we study arbitrary sequences of distinct vertices from a graph G, such that each vertex u in the sequence dominates at least one vertex that is not dominated by the vertices preceding u. The idea for studying these sequences first came from the domination game, as introduced in [2] and studied in several papers (see, e.g., [10, 11]). In this game vertices are chosen, one at a time, by two players Dominator and Staller, and each chosen vertex must enlarge the set of vertices of G dominated to that point in the game. While the aims of players are opposite (Dominator wants as few moves as possible in the game, while Staller wants to maximize the number of moves) the outcome of the game is a sequence of vertices with the property that each chosen vertex is *legal* in the sense of enlargement of the set of dominated vertices. Next, we formalize the discussion and fix terminology.

Let $S = (v_1, \dots, v_k)$ be a sequence of distinct vertices of a graph G. The corresponding set $\{v_1, \dots, v_k\}$ of vertices from the sequence S will be denoted by $\widehat{S}$. A sequence $S = (v_1, \dots, v_k)$, where $v_i \in V(G)$, is a *dominating sequence* if $\widehat{S}$ is a dominating set of G, and S is called a *legal (dominating) sequence* if, in addition, v_i is a legal choice for each i; that is,

$$N[v_i] \setminus \cup_{j=1}^{i-1} N[v_j] \ne \emptyset.$$

Adopting the notation from domination theory, each vertex $u \in N[v_i] \setminus \cup_{j=1}^{i-1} N[v_j]$ is called a *private neighbor* of v_i with respect to $\{v_1, \dots, v_i\}$. We will also use a more suggestive term by saying that v_i *footprints* the vertices from $N[v_i] \setminus \cup_{j=1}^{i-1} N[v_j]$, and that v_i is the *footprinter* of any $u \in N[v_i] \setminus \cup_{j=1}^{i-1} N[v_j]$. (The name comes from the fact that in each step the vertex in the sequence must leave some "evidence" of its presence – a footprint that has not been seen before.) For a legal sequence S any vertex in $V(G)$ has a unique footprinter in $\widehat{S}$. Thus the function $f_S : V(G) \to \widehat{S}$ that maps each vertex to its footprinter is well defined.

Clearly the length k of a legal sequence $S = (v_1, \dots, v_k)$ is bounded from below by the domination number $\gamma(G)$ of a graph G. For the upper bound, we introduce the notion of a Grundy domination number, following a similar approach as in the general hypergraph context. Namely, given a finite graph G, the maximum length of a legal dominating sequence in G will be called the *Grundy domination number* of a graph G and will be denoted by $\gamma_{gr}(G)$. The corresponding sequence is called a *Grundy dominating sequence* of G.

The paper is organized as follows. In the next section we give some basic observations about Grundy dominating sequences and numbers in standard classes of graphs. We present the general upper bound, $\gamma_{gr}(G) \leq n - \delta(G)$, and show its sharpness by infinite families of graphs. Next, some sharp lower and upper bounds for this parameter in trees, split graphs and cographs are proven. In Section 3 we start from a more general perspective of legal covering sequences in hypergraphs, and first consider such sequences, which remain legal under any permutation. Using this we prove that an arbitrary hypergraph $\mathcal{H}$ has a legal covering sequence of any length between $\rho(\mathcal{H})$ and $\rho_{gr}(\mathcal{H})$. Then we again restrict to dominating sequences in graphs, by considering the graphs in which all legal dominating sequences are of the same length, and characterize the graphs where this length is k, for $k \leq 3$. In particular, for $k = 3$ there are no such connected graphs. In Section 4 we prove that the decision version of the problem of computing the Grundy covering number in hypergraphs is NP-complete, and then strengthen this result by showing that the decision version of computing the Grundy domination number is NP-complete, even when restricted to chordal graphs. Finally, in Section 5 we present a linear time algorithm for computing the Grundy domination number of a tree.

2 Grundy domination number

While the lower bound $\gamma_{gr}(G) \geq \gamma(G)$ is obvious, we will focus on the question, in which graphs G the equality is attained, in the next section. In particular, we will characterize the graphs in which $\gamma_{gr}(G) = \gamma(G) = k$ for $k \in \{1, 2, 3\}$. We start this section by presenting a natural upper bound for $\gamma_{gr}(G)$ in arbitrary graphs G. Recall that $\delta(G)$ stands for the minimum degree of a vertex in G.

Proposition 2.1 *For an arbitrary graph G, $\gamma_{gr}(G) \leq |V(G)| - \delta(G)$.*

Proof. Let $S = (s_1, \ldots, s_k)$ be a Grundy dominating sequence of G. Let u be a vertex footprinted in the last step, that is, $u \in f_S^{-1}(s_k)$. Since u is not dominated before the last step, we have $N[u] \cap \{s_1, \ldots, s_{k-1}\} = \emptyset$, and so

$$|\{s_1, \ldots, s_{k-1}\}| = k - 1 \leq |V(G)| - (\deg(u) + 1).$$

Thus, $\gamma_{gr}(G) = k \leq |V(G)| - \delta(G)$. $\square$

Note that the upper bound in Proposition 2.1 can be easily generalized to the hypergraph case, i.e. $\rho_{gr}(\mathcal{H}) \leq |\mathcal{E}| - \delta(\mathcal{H}) + 1$, where $\delta(\mathcal{H})$ stands for the minimum degree of a vertex in $\mathcal{H}$. The bound in Proposition 2.1 is attained by several infinite families of graphs, such as complete graphs, cycles ($\gamma_{gr}(C_n) = n - 2$), and complete bipartite graphs ($\gamma_{gr}(K_{r,s}) = s$ if $r \leq s$). In addition, it is also easy to see that $\gamma_{gr}(T) = |V(T)| - 1$, if T is a caterpillar (recall that caterpillar is a tree in which removal of all its leaves results in a path).

On the other hand, the bound in Proposition 2.1 can be far from the exact value. For instance, in the corona K_n^* of the complete graph (i.e., to each vertex of K_n a leaf is attached), the upper bound is $2n - 1$, but $\gamma_{gr}(K_n^*) = n + 1$. Similarly, if a leaf is attached to just one vertex of K_n, then the Grundy domination number of this graph is 2, but the upper bound from the proposition is n.

To set the newly introduced concept in the context of domination theory, let us recall some definitions. If the condition of being a dominating set is not enforced on a subset D but we only require each vertex of D to have a private neighbor with respect to D, then D is an *irredundant set*. The smallest and largest cardinalities of a maximal irredundant set are denoted $ir(G)$ and $IR(G)$ respectively. These two invariants together with the independent domination number, $i(G)$, and the vertex independence number, $\alpha(G)$, are related by the well-known chain of inequalities:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G) .$$

For more information on this sequence of invariants see [8]. It is easy to see that $\gamma_{gr}(G) \geq IR(G)$. Indeed, suppose that $\{v_1, v_2, \ldots, v_m\}$ is an irredundant set of order $m = IR(G)$. If $2 \leq i \leq m$, vertex v_i has a private neighbor with respect to $\{v_1, v_2, \ldots, v_m\}$ and so also has a private neighbor with respect to $\{v_1, v_2, \ldots, v_i\}$. This implies that either $(v_1, v_2, \ldots, v_m)$ is a legal dominating sequence (if $\{v_1, v_2, \ldots, v_m\}$ is also a dominating set), or vertices can be appended to the end of this sequence to form a legal dominating sequence. Thus $\gamma_{gr}(G)$ can indeed be put to the right-hand side of the above chain of inequalities.

Note that there are graphs G with $IR(G) = 2$ and $\gamma_{gr}(G)$ arbitrarily large. These are for example the *balanced co-chain graphs*, which are special co-bipartite graphs, obtainable as follows: take the disjoint union of two cliques C, C' of size n, and add edges between them in such a way that the neighborhoods of the n vertices in C in the other clique form a strictly increasing chain (with respect to inclusion) of sets of respective sizes $1, \ldots, n$. It is easy to see that there is a legal dominating sequence of length n in this graph, yet all its irredundant sets are of cardinality at most 2.

In the rest of this section we study the Grundy domination number in some standard classes of graphs, starting with trees.

2.1 Trees

We present two simple formulas of similar flavor, one expressing a lower and the other an upper bound on the Grundy domination number of a tree.

Let T be an arbitrary tree. A vertex $v \in V(T)$, adjacent to a leaf (a vertex of degree 1) of T is called a *support vertex* of T. Given a vertex $v \in V(T)$, we denote by $deg'(v)$ the number of neighbors of v in T that are not leaves. We say that a support vertex $v \in V(T)$ is an *end support vertex* if $deg'(v) \leq 1$. Note that a support vertex is an end support vertex if and only if it does not lie on a path between two other support vertices. Let us denote by $ES(T)$ the set of all end support vertices of T. Note that $|ES(T)| \geq 2$ for any tree T that is not a star.

Now, we present an algorithm that yields a legal dominating sequence S of size at least $|V(T)| - |ES(T)| + 1$. It is based on a breadth-first search of a tree which is rooted in an arbitrarily chosen end support vertex x. The main part of the algorithm is a recursive procedure called Branching(v), where v corresponds to the root of a branch (i.e., a rooted subtree, whose root is a vertex v, which is not a leaf, together with all of its descendants). The recursive procedure starts by determining the leaves adjacent to the root v, the set of which is denoted by *Leaves(v)*, and by determining the set *Branches*(v), consisting of all the branches rooted at the other children of v. Then in the procedure we distinguish two cases, based on whether the parent of v is dominated or not. In each of the cases we determine the order of marking vertices (i.e., appending them to the sequence S) and calling the procedure Branching.

Proposition 2.2 *Algorithm 1 determines a legal dominating sequence of length at least* $|V(T)| - |ES(T)| + 1$.

Algorithm 1: Dominating Sequence

Input: A tree T on n vertices which is not a star
Output: A dominating sequence S of length at least $|V(T)| - |ES(T)| + 1$.

Let $x \in ES(T)$
Branching(x)
Procedure Branching(v)
Let Leaves(v) $= \{u_1, \dots, u_s\}$; Branches(v) $= \{B_1(v_1), \dots, B_t(v_t)\}$
if $v \neq x$ *and parent(v) not dominated* **then**
  Branching(v_1), ..., Branching(v_t), Mark(u_1), ...,Mark(u_s),Mark(v)
else if $t > 0$ **then**
  Branching(v_1), ..., Branching(v_{t-1}),
  Mark(u_1), ...,Mark(u_s),Mark(v),Branching(v_t)
else
  Mark(u_1), ...,Mark(u_s)

Proof. First let us prove that whenever *Mark* is performed by the algorithm, the corresponding vertex has a private neighbor with respect to the set of vertices that were marked before it. Since in all cases leaves are marked before v, this is clear for leaves. In the case when parent(v) is not dominated, v footprints *parent(v)*. In the case when parent(v) is dominated, v will be marked before the last subbranch, if any, is visited. Hence v footprints the root v_t of the subbranch $B_t(v_t)$.

Note that x (the end support vertex chosen as the root) is in S. To conclude the proof, we must show that all vertices of $V(T) \setminus ES(T)$ will be marked, that is, they will be in S. Clearly, all leaves will be marked. Note that for every vertex v that is not an end support vertex nor a leaf *Branches*(v) is nonempty which means that when v is processed by the algorithm, t (the number of subbranches) will be positive. Hence v will be marked at one point in the algorithm. □

Corollary 2.3 *If T is a tree that is not a star, then $\gamma_{gr}(T) \geq |V(T)| - |ES(T)| + 1$.*

Any caterpillar that is not a star, and any subdivided star (all edges subdivided at least once) achieves the lower bound of Corollary 2.3. On the other hand, for the tree T_{10} on 10 vertices, obtained from the two copies of P_5 by adding an edge between the two central vertices of the five-vertex paths, the bound is not sharp. Note that $\gamma_{gr}(T_{10}) = 8 > 7 = |V(T_{10})| - |ES(T_{10})| + 1$. Algorithm 1, depending on the choice of the order of the branches in which the procedure Branching is executed, in one case marks 8 and in another case marks 7 vertices. The algorithm works as well if T is a star $K_{1,n}$, in which case it marks all its leaves, yielding a Grundy dominating sequence of length $|V(T)| - |ES(T)|$.

Now, we focus on the upper bound for the Grundy domination number of trees.

Lemma 2.4 *Let T be a tree, v a vertex of degree at least 2, and S a legal dominating sequence in T. If C is a component of $T - v$ such that all vertices of C belong to $\widehat{S}$, then the neighbor of v in C is the last vertex in C to appear in S.*

Proof. Let w be the last vertex from C appearing in the sequence S. Clearly w does not footprint itself nor any other vertex from C, hence it must footprint a vertex not in C. Hence the only possibility is that w is the neighbor of v from C. □

We introduce the following relation on the set $ES(T)$ of end support vertices of a tree T. Let $u, v \in ES(T)$. We say that $u \sim v$ if the path between u and v has at most one vertex z such that $deg'(z) > 2$. Clearly $\sim$ is reflexive and symmetric. To see that it is also transitive consider any three vertices $u, v, w \in ES(T)$ such that $u \sim v$ and $v \sim w$. Note that there exists a unique vertex z that lies on a path between every pair from u, v, w. Clearly z is not a leaf and is not one of u, v, w since they are end support vertices. We find that $deg'(z) \geq 3$, hence z is the only vertex on the path between u (resp. v, w) and z with deg' at least 3. Thus $u \sim w$. In addition to transitivity we deduce that z lies between any two end support vertices of this equivalence class. We say that z is *the delegate* of the particular equivalence class. We denote by $\widetilde{T}$ the set of all equivalence classes of this relation.

Proposition 2.5 *Let T be a tree. Then $\gamma_{gr}(T) \leq |V(T)| - |ES(T)| + |\widetilde{T}|$*

Proof. Let S be an arbitrary dominating sequence in T. Consider an arbitrary equivalence class from $\widetilde{T}$ and let z be its delegate. Consider the components $C_1, \ldots, C_k$ of $G - z$ that contain an end support vertex of this class. Each of the components C_i contains exactly one end support vertex. Now, suppose that a component C_j has all vertices in $\widehat{S}$. Then by Lemma 2.4 the neighbor x of z from C_j is the last vertex from C_j that is appended to S. In particular, at the time x is being added, z is still undominated to ensure legality. Hence there can be only one component from $C_1, \ldots, C_k$ such that all its vertices are in S. Thus in this equivalence class from $\widetilde{T}$ at least $k - 1$ vertices are not in S. The formula of the proposition follows. $\square$

Note that the bound is sharp for any non-star tree T such that $|\tilde{T}| = 1$, e.g., for (non-star) caterpillars, and for proper subdivisions of stars. For a different example, consider the tree T_{10} on 10 vertices mentioned above, and note that there are four end support vertices in T_{10}, while $\widetilde{T_{10}}$ has two elements. Thus, the upper bound from Proposition 2.5 is equal to $\gamma_{gr}(T_{10})$, which is equal to 8.

In Section 5 we will present an efficient (in fact, linear) yet rather involved algorithm for computing the Grundy domination number of an arbitrary tree.

2.2 Split graphs and cographs

Recall that a graph is *split* if its vertex set can be partitioned into a clique and an independent set. A *split partition* of a split graph G is a pair (K, I) such that K is a clique, I is an independent set, $K \cup I = V(G)$ and $K \cap I = \emptyset$. In the next theorem, we establish a close relationship between the Grundy domination number and the independence number of a split graph.

Theorem 2.6 *Let G be a split graph with a maximum independent set I, and let $K = V \setminus I$. Then,*

$$\gamma_{gr}(G) = \begin{cases} \alpha(G), & \text{if every two vertices in } K \text{ have a common neighbor in } I; \\ \alpha(G) + 1, & \text{otherwise.} \end{cases}$$

In particular, there exists a polynomial time algorithm for computing the Grundy domination number of a split graph.

Proof. Given a split graph G, let (K, I) be a split partition of G such that I is a maximal (in fact, maximum) independent set in G. Such a partition can be found in linear time [7].

Recall that the inequality $\gamma_{gr}(G) \geq \alpha(G)$ holds for general graphs.

To show that $\gamma_{gr}(G) \leq \alpha(G) + 1$, we will prove that there exists a Grundy dominating sequence for G that contains at most one vertex from K. Let S be the Grundy dominating sequence $(v_1, \ldots, v_k)$ and assume that it maximizes the number of elements from I. Suppose that there exist two indices $i, j \in [k]$ with $i < j$ such that $v_i, v_j \in K$. Just before v_j is added to the sequence, all vertices in K are already dominated (by v_i). Therefore, v_j footprints a vertex w from I, and consequently w does not appear in S. But then, we could replace v_j with w in S to obtain an equally long legal dominating sequence with one more element from I, contradicting the choice of S. This shows that S contains at most one vertex from K, which implies $\gamma_{gr}(G) \leq \alpha(G) + 1$.

Suppose that every two vertices in K have a common neighbor in I. Again, let S be a Grundy dominating sequence for G that maximizes the number of elements from I, say $S = (v_1, \ldots, v_k)$. We want to show that $k = \alpha(G)$, which will follow if we show that s contains no vertex from K. Suppose for a contradiction that S contains a (unique) vertex from K, say $v_i \in K$. By the choice of S, all vertices in $N(v_i) \cap I$ appear in S before v_i. Therefore, v_i footprints another vertex from K, say w. However, since v_i and w have a common neighbor in I, w is already dominated by some vertex in $\{v_1, \ldots, v_{i-1}\}$. This is in a contradiction with the fact that S is a legal sequence.

It only remains to show that $\gamma_{gr}(G) \geq \alpha(G) + 1$ in the case when not every two vertices in K have a common neighbor in I. Let $x, y \in K$ be two vertices with $N(x) \cap N(y) \cap I = \emptyset$. A Grundy dominating sequence S of length $\alpha(G) + 1$ can be obtained as follows: $S = (v_1, \ldots, v_{\alpha(G)+1})$, where $N(x) \cap I = \{v_1, \ldots, v_p\}$, $v_{p+1} = x$ and $\{v_{p+2}, \ldots, v_{\alpha(G)+1}\} = I \setminus N(x)$. It is easy to see that this is indeed a legal sequence since all the vertices in S that are from I footprint themselves, and v_{p+1} footprints y. □

Another class of perfect graphs where the Grundy domination number is closely related with the independence number, and can be efficiently computed, is the class of P_4-free graphs also known as *cographs*. Cographs are characterized as the graphs that can be constructed by an iterative application of applying operations of disjoint union and join starting from the one-vertex graphs. (Recall that the *join* of two graphs G and H is the graph obtained from the disjoint union of G and H by adding all edges connecting a vertex of G with a vertex of H.) This implies that every cograph with at least two vertices is either disconnected, or its complement is disconnected [3, 5, 6, 12].

Theorem 2.7 *For every cograph G, we have $\gamma_{gr}(G) = \alpha(G)$.*

Proof. We use induction on $n = |V(G)|$. For $n = 1$, we have $\gamma_{gr}(G) = \alpha(G) = 1$. For $n > 1$, graph G is either a disjoint union or a join of two smaller cographs, say G_1 and G_2. If G is the disjoint union of G_1 and G_2, then clearly $\gamma_{gr}(G) = \gamma_{gr}(G_1) + \gamma_{gr}(G_2) = \alpha(G_1) + \alpha(G_2) = \alpha(G)$, where the second equality holds by the induction hypothesis.

Suppose now that G is the join of G_1 and G_2. Then $\alpha(G) = \max\{\alpha(G_1), \alpha(G_2)\}$, so it suffices to show that $\gamma_{gr}(G) = \max\{\gamma_{gr}(G_1), \gamma_{gr}(G_2)\}$ and apply induction. Since every Grundy dominating sequence for G_1 or G_2 is a legal dominating sequence for G, we have $\gamma_{gr}(G) \geq \max\{\gamma_{gr}(G_1), \gamma_{gr}(G_2)\}$. We may assume that G is not complete (since otherwise $\gamma_{gr}(G) = \alpha(G) = 1$ and we are done). Since G is the join of G_1 and G_2, every legal dominating sequence of G that contains a vertex from G_1 and a vertex from G_2 is of length 2. Since G is not complete, G contains a legal dominating sequence of length 2 containing vertices from only one of G_1 and G_2. Consequently, G has a Grundy dominating sequence containing vertices from only one of G_1 and G_2, say from G_1. Such a sequence is a legal dominating sequence for G_1, which implies $\max\{\gamma_{gr}(G_1), \gamma_{gr}(G_2)\} \geq \gamma_{gr}(G_1) \geq \gamma_{gr}(G)$. $\square$

The above result implies that the Grundy domination number can be computed in linear time for cographs. Moreover, since the four-vertex path P_4 is not a cograph and $\gamma_{gr}(P_4) > \alpha(P_4)$, the class of all graphs G such that $\gamma_{gr}(H) = \alpha(H)$ for every induced subgraph H of G is exactly the class of cographs.

3 Lengths of legal covering and dominating sequences

Recall that in a hypergraph $\mathcal{H} = (X, \mathcal{E})$ with no isolated vertices a set $\mathcal{C}$ of hyperedges from $\mathcal{E}$ such that $\cup_{A\in\mathcal{C}} A = X$, is called an *edge cover* of $\mathcal{H}$. The minimum cardinality of an edge cover of $\mathcal{H}$ is denoted by $\rho(\mathcal{H})$. A sequence $(B_1, \ldots, B_k)$ of hyperedges in $\mathcal{H}$ is a *legal sequence* if whenever $1 < i \leq k$, $B_i \setminus (B_1 \cup \cdots \cup B_{i-1}) \neq \emptyset$. If in addition $X = B_1 \cup \cdots \cup B_k$, then the sequence $(B_1, \ldots, B_k)$ is called a *legal covering sequence*. Similarly as for dominating sequences, given a covering sequence $\mathcal{S} = (B_1, \ldots, B_k)$, we denote by $\widehat{\mathcal{S}}$ the set $\{B_1, \ldots, B_k\}$.

An edge cover $\mathcal{C}$ of $\mathcal{H}$ is a *minimal* edge cover if for every $A \in \mathcal{C}$, $\mathcal{C} \setminus \{A\}$ is not an edge cover of $\mathcal{H}$. This is equivalent to requiring that for every $A \in \mathcal{C}$, there exists a vertex $v \in X$ such that

$$v \in A \setminus \bigcup_{B\in\mathcal{C}, B\neq A} B\,.$$

Any such vertex v is called a *private member* of A with respect to $\mathcal{C}$. Clearly any edge cover of cardinality $\rho(\mathcal{H})$ is a minimal edge cover, but there may exists minimal edge covers of $\mathcal{H}$ of greater cardinalities.

Let $\mathcal{S} = (B_1, \ldots, B_k)$ where $B_i \in \mathcal{E}$ for each i, $1 \leq i \leq k$. Suppose $\mathcal{S}$ is a legal covering sequence of $\mathcal{H}$. We say that $\mathcal{S}$ is *commutative* if for any permutation π of $\mathcal{S}$, the sequence $\pi(\mathcal{S})$, defined by $(\pi(B_1), \ldots, \pi(B_k))$, is also a legal covering sequence of $\mathcal{H}$.

Lemma 3.1 *A legal covering sequence $\mathcal{S}$ in a hypergraph $\mathcal{H}$ is commutative if and only if $\widehat{\mathcal{S}}$ is a minimal edge cover of $\mathcal{H}$.*

Proof. Let $\mathcal{S} = (B_1, \ldots, B_k)$, and assume that $\mathcal{S}$ is a legal covering sequence in $\mathcal{H}$. Suppose first that $\mathcal{S}$ is commutative. For any hyperedge B_i in $\widehat{\mathcal{S}}$, let π be a permutation such that $\pi(B_k) = B_i$; that is, B_i is the last hyperedge in the sequence $\pi(\mathcal{S})$. Since $\mathcal{S}$ is commutative, it follows that B_i has a private member with respect to $\{B_1, \ldots, B_k\}$. This implies that $\widehat{\mathcal{S}}$ is a minimal edge cover of $\mathcal{H}$.

If $\widehat{\mathcal{S}}$ is a minimal edge cover of $\mathcal{H}$, then each B_i has a private member with respect to $\widehat{\mathcal{S}}$. In particular, this implies that each B_i has a private member with respect to $\{U_1, \ldots, U_j, B_i\}$, where $\{U_1, \ldots, U_j\}$ is an arbitrary subset of $\widehat{\mathcal{S}} \setminus \{B_i\}$. Hence, the sequence $(U_1, \ldots, U_j, B_i)$ is a legal covering sequence. Consequently, any permutation π of $(B_1, \ldots, B_k)$ is a legal covering sequence, which implies that $\mathcal{S}$ is commutative. $\square$

It follows directly from the definitions that if $(B_1, \ldots, B_k)$ is a legal covering sequence of $\mathcal{H}$ that is not commutative and $1 < i \leq k$ such that B_i has no private member with respect to $\{B_1, \ldots, B_k\}$, then $(B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_k)$ is also a legal covering sequence of $\mathcal{H}$. We are now able to prove the following "interpolation" result for legal covering sequences in a hypergraph.

Theorem 3.2 *Let $\mathcal{H}$ be a hypergraph. For any number ℓ such that $\rho_{gr}(\mathcal{H}) \leq \ell \leq \rho_{gr}(\mathcal{H})$ there is a legal covering sequence of $\mathcal{H}$ having length ℓ.*

Proof. Let $\mathcal{T}$ be a legal covering sequence in $\mathcal{H}$ that has length $\rho_{gr}(\mathcal{H})$. Suppose that $\widehat{\mathcal{T}}$ is not a minimal edge cover of $\mathcal{H}$. Using the statement above we can remove hyperedges from $\mathcal{T}$ one by one until a legal covering sequence $\mathcal{S}$ is reached such that $\widehat{\mathcal{S}}$ is a minimal edge cover. Hence all lengths of legal covering sequences from $|\widehat{\mathcal{S}}|$ up to $\rho_{gr}(\mathcal{H})$ are realized in $\mathcal{H}$. To conclude the proof we will now show that also all lengths of legal covering sequences from $\rho(\mathcal{H})$ up to $|\widehat{\mathcal{S}}|$ are realized in $\mathcal{H}$.

Let $\mathcal{D}$ be a minimum cardinality edge cover of $\mathcal{H}$; that is, $|\mathcal{D}| = \rho(\mathcal{H})$. If $|\widehat{\mathcal{S}}| = |\mathcal{D}|$ then the proof is done. Otherwise $\widehat{\mathcal{S}}$ is not a minimum edge cover, and there is a hyperedge A that lies in $\widehat{\mathcal{S}}$ but not in $\mathcal{D}$. Since $\widehat{\mathcal{S}}$ is a minimal edge cover, we may assume, by applying Lemma 3.1, that A is the last hyperedge in $\mathcal{S}$. Let $\mathcal{S}'$ be the sequence obtained from $\mathcal{S}$ by removing A. Since A has a private member with respect to $\widehat{\mathcal{S}}$, we conclude that $\mathcal{S}'$ is not a covering sequence (the private members of A with respect to $\widehat{\mathcal{S}}$ do not belong to $\cup_{F \in \mathcal{S}'} F$). Now, let $\mathcal{D}' \subseteq \mathcal{D} \setminus \mathcal{S}$ be a smallest set of hyperedges whose union contains the private members of A with respect to $\widehat{\mathcal{S}}$. By adding hyperedges from $\mathcal{D}'$ at the end of $\mathcal{S}'$ we obtain a sequence $\mathcal{S}''$ which is a legal covering sequence of $\mathcal{H}$. Note that $\widehat{\mathcal{S}''}$ may not be a minimal edge cover. (However, each hyperedge of $\mathcal{D}'$ that was added to $\mathcal{S}'$ at the end, has a private member with respect to $\widehat{\mathcal{S}''}$, by the choice of $\mathcal{D}'$.)

If $\widehat{\mathcal{S}''}$ is not a minimal edge cover, we remove a hyperedge from $\mathcal{S}''$ that has no private member with respect to $\widehat{\mathcal{S}''}$. This results in a legal covering sequence. If necessary, continue removing such hyperedges one by one until we reach a legal covering sequence $\mathcal{S}'''$ which is commutative. Thus we obtain all values of lengths of legal covering sequences from $|\widehat{\mathcal{S}''}|$ to $|\widehat{\mathcal{S}'''}|$. Note that at this point it is not necessary that the length of $\mathcal{S}'''$ is less than the length of $\mathcal{S}$. However, after this exchange, the resulting sequence $\mathcal{S}'''$ has fewer hyperedges from $\widehat{\mathcal{S}} \setminus \mathcal{D}$ than there were such hyperedges in $\widehat{\mathcal{S}}$. By repeating this procedure, we obtain a list of legal covering sequences whose number of hyperedges from $\widehat{\mathcal{S}} \setminus \mathcal{D}$ is decreasing. Since $\widehat{\mathcal{S}} \setminus \mathcal{D}$ is finite, we obtain after a certain number of steps a legal covering sequence whose hyperedges are from $\mathcal{D}$. By the construction all lengths of sequences from $|\widehat{\mathcal{S}}|$ down to $|\mathcal{D}|$ are realized in $\mathcal{H}$. The proof is complete. $\square$

For any graph G, let $(V(G), \mathcal{N}(G))$ be the hypergraph where $\mathcal{N}(G) = \{N[v] \mid v \in V(G)\}$. By applying Theorem 3.2 to this hypergraph we get the following corollary.

Corollary 3.3 *Let G be a graph. For any number ℓ such that $\gamma(G) \leq \ell \leq \gamma_{gr}(G)$ there is a legal dominating sequence of G having length ℓ.*

In the rest of the section we consider the graphs for which all legal dominating sequences have the same length, say k. We call such a graph a *k-uniform dominating sequence* graph, or simply a *k-uniform* graph for ease of reference in this paper. In case we are not interested in the particular value of k in the above, we simply refer to G as a uniform dominating sequence graph. Similarly, we will speak about *k-uniform covering sequence hypergraph* if all legal covering sequences in such a hypergraph have the same length.

The following result follows from definitions.

Proposition 3.4 *Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph such that two copies of the hyperedge $B \subseteq X$ appear in $\mathcal{E}$, and let $\mathcal{H}'$ be the hypergraph obtained from $\mathcal{H}$ by removing one copy of B from $\mathcal{E}$. Then $\mathcal{H}$ is k-uniform if and only if $\mathcal{H}'$ is k-uniform.*

Two distinct nodes x and y in a graph are called *twins* if $N[x] = N[y]$. A graph G is said to be *twin-free* if no two of its distinct nodes are twins. By Proposition 3.4 we derive that if G is a graph with twins x and y, then G is k-uniform if and only if $G - x$ is k-uniform. Hence to characterize k-uniform graphs it suffices to concentrate only on twin-free graphs.

Lemma 3.5 *Let G be a twin-free uniform length dominating sequence graph and let $x, y \in V(G)$. If $N[x] \subseteq N[y]$, then $x = y$.*

Proof. Suppose that $N[x] \subseteq N[y]$. If $N[x] = N[y]$ then we must have $x = y$ since G is twin-free. Hence, we may assume that there exists a vertex $z \in N(y) \setminus N(x)$. Since G is k-uniform for some k, the sequence $S = (x, y)$ can be extended to a legal dominating sequence S' of G of length k, say $S' = (x, y, v_3, \ldots, v_k)$. But then, $(y, v_3, \ldots, v_k)$ is a legal dominating sequence of G of length $k - 1$, contradicting the k-uniformity assumption. $\square$

In Theorem 3.6 below, we characterize twin-free k-uniform graphs for $k \in \{1, 2, 3\}$. For graphs G and H and a positive integer n, we denote by $\overline{G}$ the complement of G, by $G + H$ the disjoint union of G and H, and by nG the disjoint union of n copies of G. By abuse of language, a disjoint union may also refer to only one graph, in particular $1G$ is G. In the proof we will use some well-known properties of the class of cographs.

Theorem 3.6 *If G is a graph, then*

(i) G is 1-uniform if and only if G is a complete graph,

(ii) G is 2-uniform if and only if its complement $\overline{G}$ is the disjoint union of one or more complete bipartite graphs.

(iii) G is 3-uniform if and only if G is the disjoint union of a 1-uniform and a 2-uniform graph.

Proof. It is easy to verify that every complete graph is 1-uniform, that the complement of the disjoint union of one or more complete bipartite graphs is 2-uniform, and that the disjoint union of a 1-uniform and a 2-uniform graph is a 3-uniform graph.

If G is 1-uniform, then $\alpha(G) = 1$, hence G is a complete graph, as desired.

Suppose that G is 2-uniform. It follows that $|V(G)| \geq 2$. Moreover, G is P_4-free, since otherwise, assuming that vertices a, b, c, d induce a P_4 with edges ab, bc, cd, the sequence (a, b, c) is a legal sequence (not necessarily dominating) of G of length 3, contrary to $\gamma_{gr}(G) = 2$. Suppose first that G is disconnected. Since $\alpha(G) = 2$, every connected component of G is complete. Thus in this case G is the disjoint union $K_r + K_s$ for some positive integers r and s, and so $\overline{G}$ is the complete bipartite graph $K_{r,s}$. Suppose now that G is connected. In this case $\overline{G}$ is disconnected with $n \geq 2$ components, that is, G is the join of $C_1, \ldots, C_n$ with each C_i either disconnected or K_n. We infer that each C_i must be a 2-uniform graph. Thus, each C_i is disconnected, and by the above case, each C_i is isomorphic to the disjoint union of two complete graphs. Thus, $\overline{G}$ is isomorphic to the disjoint union of n complete bipartite graphs.

Suppose that G is 3-uniform. By Proposition 3.4 we may assume that G is twin-free. (Note that the only twin-free complete graph is K_1, and the only 2-uniform twin-free graphs are $n\overline{K_2}$ for some $n \geq 1$.) We will first show that every two distinct vertices of G have exactly one common non-neighbor. Let x and y be two distinct vertices of G. By Lemma 3.5, there exists a vertex $x' \in N[x] \setminus N[y]$, and a vertex $y' \in N[y] \setminus N[x]$. Since $\gamma(G) = 3$, the set $\{x, y\}$

is not a dominating set in G, hence the set $R := V(G) \setminus (N[x] \cup N[y])$ is nonempty. Notice that R is a clique, since otherwise, assuming u and v are two non-adjacent vertices in R, the sequence $S = (x, y, u, v)$ would be extendable to a legal dominating sequence of G, yielding $\gamma_{gr}(G) \geq 4$. Suppose for a contradiction that $|R| \geq 2$, and let $u, v \in R$ with $u \neq v$. By Lemma 3.5, there exists a vertex $u' \in N(u) \setminus N(v)$. But now, the sequence $S = (x, y, u', u)$ can be extended to a legal dominating sequence of G (note that x footprints x', y footprints y', u' footprints u, and u footprints v).

Since every two distinct vertices in G have a unique common non-neighbor, in the complementary graph $\overline{G}$ every two distinct vertices have a unique common neighbor. By a theorem of Erdős, Rényi and Sós [4], $\overline{G}$ is isomorphic to a friendship graph, that is, a graph obtained from nK_2 (the disjoint union of n copies of K_2, where $n \geq 1$) by adding to it a dominating vertex. This completes the proof. □

4 NP-completeness

Recall that the Grundy domination number $\gamma_{gr}(G)$ of a graph G is defined as the maximum length of a legal dominating sequence of G. It is natural to ask about the computational complexity of the following related problem:

Grundy Domination Number	
Input:	A graph $G = (V, E)$, and an integer k.
Question:	Is $\gamma_{gr}(G) \geq k$?

In Theorem 4.1 below, we prove hardness of this problem. Recall that a graph G is *chordal* if it does not contain any induced cycle of order at least 4.

Theorem 4.1 Grundy Domination Number *is NP-complete, even for chordal graphs.*

In order to obtain the result, we first prove NP-completeness of the decision version of the Grundy covering problem in hypergraphs. For a positive integer k, we denote by $[k]$ the set $\{1, \ldots, k\}$.

Grundy Covering Number in Hypergraphs	
Input:	A hypergraph $\mathcal{H} = (X, \mathcal{E})$, and an integer k.
Question:	Is $\rho_{gr}(\mathcal{H}) \geq k$?

Theorem 4.2 Grundy Covering Number in Hypergraphs *is NP-complete.*

Proof. Membership in NP is trivial. To show hardness, we reduce Feedback Arc Set to Grundy Covering Number in Hypergraphs. In Feedback Arc Set, we are given a directed graph $D = (V, A)$ and an integer k, and are asked to determine whether there exists an ordering $v_1, \ldots, v_n$ of all the vertices in V such that at most k arcs $(v_i, v_j) \in A$ are directed backward (that is, have $j < i$). Karp showed in 1972 that the Feedback Arc Set problem is NP-complete [9].

Given an instance $(D = (V, A), k)$ to Feedback Arc Set, we construct an instance $(X, \mathcal{E}, k')$ of Grundy Covering Number in Hypergraphs, as follows: For every vertex $v \in V$, we introduce a ground-set element a_v; also, for every arc $(u, v) \in A$, we introduce a ground-set element $x_{(u,v)}$. Thus $X = \{a_v \ : \ v \in V\} \cup \{x_{(u,v)} \ : \ (u, v) \in A\}$. The family $\mathcal{E}$ of hyperedges is constructed as follows:

- for every vertex $v \in V$, we place in $\mathcal{E}$ the set $E^v := \{a_v\} \cup \{x_{(u,v)} : (u,v) \in A, u \in V\}$;
- for every arc $(u,v) \in A$, we place in $\mathcal{E}$ the set $E^{(u,v)} := E^u \cup \{x_{(u,v)}\}$.

Finally, set $k' = n + |A| - k$.

Lemma 4.3 *If there exists an ordering $v_1, \ldots, v_n$ of the vertices in V such that at least t arcs (v_i, v_j) in A are directed forward (i.e., have $i < j$), then there exists a sequence $E_1, \ldots, E_{n+t}$ of $n+t$ sets in $\mathcal{E}$ such that $E_i \setminus \cup_{j<i} E_j \neq \emptyset$ for all $i \in [n+t]$.*

Proof. The sequence is made of n consecutive substrings, where the i-th substring is produced as follows: take E^{v_i}, and then, in any order, take all the sets $E^{(v_i,v_j)}$ such that $(v_i, v_j) \in A$ and with $j > i$.

Notice that E^{v_i} will be the first set to cover a_{v_i}, and each set of the form $E^{(v_i,v_j)}$ will be the first set to cover the element $x_{(v_i,v_j)}$ since $j > i$. □

Lemma 4.4 *If there exists a sequence $E_1, \ldots, E_{n+t}$ of $n+t$ sets in $\mathcal{E}$ such that $E_i \setminus \cup_{j<i} E_j \neq \emptyset$ for all $i \in [n+t]$, then there exists an ordering $v_1, \ldots, v_n$ of the vertices in V such that at least t arcs (v_i, v_j) in A are directed forward (i.e., have $i < j$).*

Proof. Notice first that, for every vertex $v \in V$, the sets built by means of our reduction respect the following properties:

(i) the sets containing the element a_v are precisely the set E^v, and the sets $E^{(v,z)}$ where (v,z) is an arc exiting v in D;

(ii) $E^v \subseteq E^{(v,z)}$ for every arc $(v,z) \in A$;

(iii) $E^{(v,z)} \setminus E^v = \{x_{(v,z)}\}$ for every arc $(v,z) \in A$.

From these properties we can enforce a stronger structure on the sequence, by making some modifications if necessary. Thus the sequence $E_1, \ldots, E_{n+t}$ will be in *standard form* if, for every v, we have the following:

(1) The set E^v belongs to the sequence, and is the first set in the sequence covering the element a_v.

Note that the element a_v is covered by some set in the sequence. Suppose that the set E^v does not belong to the sequence. Let E_i be the first set in the sequence covering the element a_v. Then the sequence can be modified by inserting $E_i := E^v$. Note that feasibility is maintained, and E_i will still be the first set in the sequence covering the element a_v, using the above properties.

(2) The sets of the form $E^{(v,z)}$, if present in the sequence, are all (in some order) just after the set E^v.

Indeed, we already know they will all occur after E^v by (1). If they appear right after E^v, then since $x_{(v,z)}$ is the only element first covered by $E^{(v,z)}$ (see (iii)), this does not change the feasibility of all other sets in the sequence.

As we have seen, it is always possible to assume (by performing local modifications which do not shorten the sequence nor invalidate its feasibility) that the sequence is in standard form. The ordering of the vertices is then obtained from the order in which the members of $\{E^v \mid v \in V\}$ appear in the standard form. By the construction it follows that all of the t sets of the form $E^{(v,z)}$ that belong to the sequence correspond to arcs from A that are directed forward. $\square$

Lemmas 4.3 and 4.4 imply that there exists an ordering $v_1, \ldots, v_n$ of the vertices in V such that at most k arcs in A are directed backward if and only if there exists a sequence $E_1, \ldots, E_{k'}$ of $k' = n + |A| - k$ hyperedges in $\mathcal{E}$ such that $E_i \setminus \cup_{j<i} E_j \neq \emptyset$ for all $i \in [k']$. Therefore, the Grundy Covering Number in Hypergraphs problem is NP-hard, which completes the proof of Theorem 4.2. $\square$

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Membership in NP is trivial. To show hardness, we reduce Grundy Covering Number in Hypergraphs to Grundy Domination Number. Assume we are given an instance of Grundy Covering Number in Hypergraphs, that is, a collection $\mathcal{E}$ of sets over a finite ground set $X = \{x_1, \ldots, x_n\}$, and an integer k. Our task is to determine whether there exists a sequence $B_1, \ldots, B_k$ of k sets in $\mathcal{E}$ such that $B_i \setminus \cup_{j<i} B_j \neq \emptyset$ for every $i \in [k]$.

We construct an instance (G, k') of Grundy Domination Number where G is a chordal graph, as follows.

Let $\widetilde{X} := X \cup \{x_0\}$. The vertex set V contains each element in $\widetilde{X}$, plus, for every set $B \in \mathcal{E}$, we also have $\overline{B}(0)$, $\overline{B}(1)$ and $\overline{B}$ as vertices. As far as the edges are concerned, we have in E all the edges with both endpoints in $\widetilde{X}$, plus, for every $B \in \mathcal{E}$, we have the two edges $\overline{B}(0)\overline{B}(1)$ and $\overline{B}(1)\overline{B}$ plus the edges $\overline{B}x$ for every $x \in B$. Finally, let $k' = 2|\mathcal{E}| + k + 1$.

Lemma 4.5 *If there exists a sequence $B_1, \ldots, B_k$ of k sets in $\mathcal{E}$ such that $B_i \setminus \cup_{j<i} B_j \neq \emptyset$ for all $i \in [k]$, then there exists a legal dominating sequence of $G = (V, E)$ of length $2|\mathcal{E}| + k + 1$.*

Proof. The sequence is constructed as follows. The first $|\mathcal{E}|$ vertices in the sequence are the vertices of the form $\overline{B}(0)$, with $B \in \mathcal{E}$. The next $|\mathcal{E}|$ vertices are the vertices of the form $\overline{B}(1)$, with $B \in \mathcal{E}$. Complete the sequence by appending $\overline{B_1}, \ldots, \overline{B_k}, x_0$ to obtain the resulting legal dominating sequence of length $2|\mathcal{E}| + k + 1$. $\square$

For the proof of the next lemma, we need some preparation. First recall that a vertex a in a graph G is called *simplicial* if $N[a]$ is a clique. Let S be a legal dominating sequence in a graph G. If v is a simplicial vertex without a twin in G, then there exists a legal dominating sequence S' with $|\widehat{S'}| \geq |\widehat{S}|$ such that $f_{S'}(v) = v$. Indeed, if $f_S(v) \neq v$, then v does not occur in S since $N[v] \subset N[f_S(v)]$. Hence, if we replace $f_S(v)$ by v in S we obtain a legal sequence of the same length as S, which may not be dominating. Finally, we obtain a legal dominating sequence S' by adding $f_S(v)$ at the end if necessary. In the proof of the following lemma this transformation will be used, implying the assumption for legal dominating sequences that each simplicial vertex without a twin is its own footprinter.

Lemma 4.6 *If there exists a legal dominating sequence S of $G = (V, E)$ of length $2|\mathcal{E}| + k + 1$, then there exists a sequence $B_1, \ldots, B_k$ of k hyperedges in $\mathcal{E}$ such that $B_i \setminus \cup_{j<i} B_j \neq \emptyset$ for all $i \in [k]$.*

Proof. We will prove that the following properties may be assumed for a given legal dominating sequence S of $G = (V, E)$ of length $2|\mathcal{E}| + k + 1$.

(1) x_0 is the first vertex in $\widetilde{X} = N[x_0]$ to occur in the sequence;

(2) for every $B \in \mathcal{E}$, $\overline{B}(0)$ occurs in the sequence;

(3) if vertex $\overline{B}(1)$ occurs in the sequence for some $B \in \mathcal{E}$, then it occurs after $\overline{B}(0)$;

(4) for every $B \in \mathcal{E}$, $f_S(\overline{B}) = \overline{B}(1)$;

(5) no vertex of X occurs in the sequence;

(6) the first $|\mathcal{E}|$ vertices in the sequence are the vertices of the form $\overline{B}(0)$, with $B \in \mathcal{E}$; the next $|\mathcal{E}|$ vertices in the sequence are the vertices of the form $\overline{B}(1)$, with $B \in \mathcal{E}$; then some (at least k) vertices of the form $\overline{B}$, and finally x_0.

We may assume that properties (1), (2) and (3) hold by using the remark preceding the statement of the lemma. To prove (4) assume there exists an $B \in \mathcal{E}$ such that $f_S(\overline{B}) \neq \overline{B}(1)$. By (3) we derive that $\overline{B}(1)$ does not occur in the sequence. By replacing $f_S(\overline{B})$ by $\overline{B}(1)$ (and, if it is not already the case, move $\overline{B}(0)$ in front of $\overline{B}(1)$), the resulting sequence remains legal and has the same length as S. Hence, we can assume that there is a legal dominating sequence of this length satisfying (4).

By (4) and (1), no vertex of X can appear in the sequence before all vertices of the form $\overline{B}$ are dominated. Since x_0 is in the sequence before any vertex of X, (5) follows.

By (4), a vertex of the form $\overline{B}$ can be a footprinter only of a vertex in X. This immediately implies that x_0 appears in S after all vertices of the form of the form $\overline{B}$. We may also assume, without loss of legality of the sequence, that vertices of the form $\overline{B}$ all appear after the vertices of the form $\overline{B}(1)$ (by (4)), and these appear after those of the form $\overline{B}(0)$. This proves (6).

Since S is of length $2|\mathcal{E}| + k + 1$, it follows that at least k vertices in S are of the form $\overline{B}$. The corresponding sets B give us the required (legal edge covering) sequence of sets in $\mathcal{E}$. □

Lemmas 4.5 and 4.6 imply that there exists a legal dominating sequence of $G = (V, E)$ of length k' if and only if there exists a sequence $B_1, \ldots, B_k$ of k sets in $\mathcal{E}$ such that $B_i \setminus \cup_{j<i} B_j \neq \emptyset$ for all $i \in [k]$. Therefore, the Grundy Domination Number problem is NP-hard on chordal graphs, completing the proof of Theorem 4.2. □

5 A linear time algorithm for Grundy domination number of trees

We will use the following notation throughout this section. Let T be a tree. Arbitrarily choose a vertex v, making T a rooted tree with root v. Let $v_1, \ldots, v_r$ be the children of v in T. Furthermore let $T_1, \ldots, T_r$ be connected components of $T \setminus \{v\}$, where every T_i is a rooted tree with root v_i.

We need the following different legal sequence parameters:

- $\gamma_{gr}(T) = \max\{k : (\exists(a_1 \ldots, a_k) \in V(T)^k)\ (\forall i)(N[a_i] \setminus (\bigcup_{j<i} N[a_j]) \neq \emptyset)\}$;
  This is the usual Grundy domination number.

- $\gamma^{+}_{gr}(T) = \max\{k :\ (\exists(a_1 \ldots, a_k) \in V(T)^k)\ (\forall i)(N[a_i] \setminus (\bigcup_{j<i} N[a_j]) \neq \emptyset) \wedge (\exists i)(a_i = v)\}$;

  In words, the parameter $\gamma^{+}_{gr}(T)$ is the maximum length of a legal sequence in T that contains the root v.

- $\gamma^{-}_{gr}(T) = \max\{k :\ (\exists(a_1 \ldots, a_k) \in V(T)^k)\ (\forall i)(N[a_i] \setminus (\bigcup_{j<i} N[a_j]) \neq \emptyset) \wedge (\forall i)(a_i \neq v)\}$;

  In words, the parameter $\gamma^{-}_{gr}(T)$ is the maximum length of a legal sequence in T that does not contain the root v.

- $\gamma^{'}_{gr}(T) = \max\{k :\ (\exists(a_1 \ldots, a_k) \in V(T)^k)\ (\forall i)(N[a_i] \setminus (\bigcup_{j<i} N[a_j]) \neq \emptyset) \wedge (a_i = v \Rightarrow a_i \in \bigcup_{j<i} N(a_j))\}$;

  In words, the parameter $\gamma^{'}_{gr}(T)$ is the maximum length of a legal sequence in T, in which the root v may only appear after it was already dominated (i.e., v does not footprint itself).

- $\gamma^{''}_{gr}(T) = \max\{k :\ (\exists(a_1 \ldots, a_k) \in V(T)^k)\ (\forall i)(N[a_i] \setminus (\bigcup_{j<i} N[a_j]) \not\subseteq \{v\})\}$.

  In words, the parameter $\gamma^{''}_{gr}(T)$ is the maximum length of a legal (not necessarily dominating) sequence in T, in which the root v is not footprinted alone; i.e., if and when v is footprinted, some other vertex is footprinted as well.

We start with some obvious remarks.

Lemma 5.1 *The following inequalities hold:*

1. $\gamma_{gr}(T) \geq \gamma^{+}_{gr}(T)$;
2. $\gamma_{gr}(T) \geq \gamma^{''}_{gr}(T)$;
3. $\gamma_{gr}(T) \geq \gamma^{'}_{gr}(T) \geq \gamma^{-}_{gr}(T)$.

From the definitions of $\gamma^{-}_{gr}, \gamma^{+}_{gr}$ and γ_{gr} we get the following result.

Lemma 5.2 *For the rooted tree T,*

$$\gamma_{gr}(T) = \max\{\gamma^{-}_{gr}(T), \gamma^{+}_{gr}(T)\}.$$

Lemma 5.3 *For the rooted tree T,*

$$\gamma^{-}_{gr}(T) \geq \gamma_{gr}(T) - 1.$$

Proof. Let S be an optimal legal dominating sequence of length $\gamma_{gr}(T)$. If the root v of T is contained in this sequence, then the sequence obtained from S by the removal of v is legal for $\gamma^{-}_{gr}(T)$ and has length $\gamma_{gr}(T) - 1$. Thus $\gamma_{gr}(T) - 1 \leq \gamma^{-}_{gr}(T)$. On the other hand, if $v \notin \widehat{S}$, then the sequence S is also legal for $\gamma^{-}_{gr}(T)$. Thus $\gamma_{gr}(T) \leq \gamma^{-}_{gr}(T)$. □

Corollary 5.4 *For the rooted tree T,*

$$\gamma^{-}_{gr}(T) \geq \max\{\gamma^{+}_{gr}(T), \gamma^{'}_{gr}(T), \gamma^{''}_{gr}(T)\} - 1.$$

Lemma 5.5 *For the rooted tree T,*

$$\gamma^{''}_{gr}(T) \geq \gamma_{gr}(T) - 1.$$

Proof. Let $(a_1, \ldots, a_k)$ be an optimal legal sequence for $\gamma_{gr}(T)$. Since at most one vertex from the sequence footprints only v, we get $\gamma_{gr}^{''}(T) \geq \gamma_{gr}(T) - 1$. $\square$

Now we will calculate $\gamma_{gr}^{+}, \gamma_{gr}^{-}, \gamma_{gr}^{'}, \gamma_{gr}^{''}$.

Lemma 5.6 *For the rooted tree T,*

$$\gamma_{gr}^{-}(T) = \begin{cases} \max_{1\leq i\leq r} \{(\gamma_{gr}^{-}(T_i) + 1) + \sum_{k\neq i} \gamma_{gr}(T_k)\} & \text{if } |T| \neq 1, \\ 0 & \text{if } |T| = 1. \end{cases}$$

Proof. It is clear from the definition of γ_{gr}^{-} that $\gamma_{gr}^{-}(T) = 0$ if $|T| = 1$. Thus we may assume that $|T| > 1$. First we prove that $\gamma_{gr}^{-}(T) \geq A = \max_{1\leq i\leq r} \{(\gamma_{gr}^{-}(T_i) + 1) + \sum_{k\neq i} \gamma_{gr}(T_k)\}$. It is enough to find a legal dominating sequence of length A, which does not contain the root v of T. For every $i \in \{1, \ldots, r\}$ we construct the following sequence. First we put in a sequence as many vertices from T_i as possible, such that v_i is not in a sequence. Then we add v_i, which is allowed, because v is not dominated yet. The length of this part is $\gamma_{gr}^{-}(T_i) + 1$. Then we add as many vertices as possible from each subtree T_j, where $j \neq i$. The length of this is $\sum_{j\neq i} \gamma_{gr}(T_j)$. Thus for every $i \in \{1, \ldots, r\}$ there exists a legal dominating sequence without root v of length $(\gamma_{gr}^{-}(T_i) + 1) + \sum_{j\neq i} \gamma_{gr}(T_j)$ and hence $\gamma_{gr}^{-}(T) \geq A$.

For the converse let S be the sequence $(a_1, \ldots, a_k)$, optimal for $\gamma_{gr}^{-}(T)$. We need to prove that $k \leq A$. Let v_i be the first child of v that is contained in S (such v_i exists since $v \notin S$), that is, v_i footprints v. For every $j \in \{1, \ldots, r\}$, let $(a_1^j, \ldots, a_{k_j}^j)$ be the subsequence of S which contains all vertices from S that lie in T_j. Thus $k = k_1 + \ldots + k_r$. Then $k_i \leq \gamma_{gr}^{-}(T_i) + 1$ and $k_j \leq \gamma_{gr}(T_j)$ for $j \neq i$. Thus $\gamma_{gr}^{-}(T) = k = \sum_{1\leq i\leq r} k_i \leq \gamma_{gr}^{-}(T_i) + 1 + \sum_{j\neq i} \gamma_{gr}(T_j) \leq A$. $\square$

Lemma 5.7 *Let T be the rooted tree. If $|T| > 1$ then*

$$\gamma_{gr}^{''}(T) = \max\{\max_{1\leq i\leq r} \{\gamma_{gr}^{''}(T_i) + \sum_{j\neq i} \gamma_{gr}(T_j) + 1\}, \max_{1\leq i\leq r} \{\gamma_{gr}^{+}(T_i) + \sum_{j\neq i} \gamma_{gr}(T_j)\}\}.$$

If $|T| = 1$ then $\gamma_{gr}^{''}(T) = 0$.

Proof. It is clear from the definition of $\gamma_{gr}^{''}$ that $\gamma_{gr}^{''}(T) = 0$ if $|T| = 1$. Thus we may assume that $|T| > 1$. First we prove that $\gamma_{gr}^{''}(T) \geq \max\{A, B\}$, where $A = \max_{1\leq i\leq r} \{\gamma_{gr}^{''}(T_i) + \sum_{j\neq i} \gamma_{gr}(T_j) + 1\}$ and $B = \max_{1\leq i\leq r} \{\gamma_{gr}^{+}(T_i) + \sum_{j\neq i} \gamma_{gr}(T_j)\}$. To prove this we will construct two legal dominating sequences for $\gamma_{gr}^{''}(T)$, one of length A and one of length B. Let i be an arbitrary index from $\{1, \ldots, r\}$.

We construct the first sequence in the following way. For every $j \neq i$ we put in the sequence as many vertices from T_j as possible (with respect to T_j), which means that every vertex in the sequence dominates at least one new vertex from T_j, at the time it is added. This gives a sequence of length $\sum_{j\neq i} \gamma_{gr}(T_j)$. Then we add the root v of T and finally we add as many vertices from T_i as possible. Since v_i is already dominated with v, no vertex from T_i is allowed to dominate just v_i at the time it is added. The length of this part is $1 + \gamma_{gr}^{''}(T_i)$. Since index i was arbitrary, this implies $\gamma_{gr}^{''}(T) \geq A$.

For the second sequence we first add as many vertices from T_i as possible (with respect to T_i, which means that every vertex, at the time it is added, dominates at least one vertex from T_i), such that the root v_i of T_i will be contained in this sequence. The length of this part is $\gamma^+_{gr}(T_i)$. Then we add, for each $j \neq i$, to the sequence as many vertices from T_j as possible. Therefore we added $\sum_{j\neq i}\gamma_{gr}(T_j)$ new vertices to the sequence. By construction, every vertex from the sequence dominates a new vertex different from v at the time it is added. Thus $\gamma^{''}_{gr}(T) \geq B$.

For the converse let S be the sequence $(a_1, \ldots, a_k)$, optimal for $\gamma^{''}_{gr}(T)$. We need to prove that $k \leq \max\{A, B\}$. For every $i \in \{1, \ldots, r\}$, let $(a^i_1, \ldots, a^i_{k_i})$ be the subsequence of S which contains all vertices from S that lie in T_i. We distinguish two cases.

Case 1: The root v of T is contained in S. Then $k = 1 + k_1 + \ldots + k_r$. Since S is a legal sequence for $\gamma^{''}_{gr}(T)$, v footprints at least one of its children. First, we argue that we may assume without loss of generality that at the time v is added, it footprints exactly one of its children. Suppose this is not the case, and let $C = \{v_{i_1}, \ldots, v_{i_p}\}$ (where $p \geq 2$) denote the set of children footprinted by v. Consider the sequence S' obtained from S by removing v, and let $C' \subseteq C$ be the set of those members of C that are footprinted by some vertex in S'. Clearly, vertices in C' are footprinted by distinct vertices of S'. Hence, we may assume without loss of generality that $C' = \{v_{i_1}, \ldots, v_{i_q}\}$ (for some $0 \leq q \leq p$) where $(v_{i_1}, \ldots, v_{i_q})$ denotes the order in which the vertices of C' are footprinted by a member of S'. Then $q \geq p-1$ since otherwise we could extend S' by appending to it the sequence $(v_{i_{q+1}}, \ldots, v_{i_{p-1}}, v)$, obtaining thus a legal sequence for $\gamma^{''}_{gr}(T)$ longer than S, which contradicts the choice of S. If $q = p-1$ then a legal sequence for $\gamma^{''}_{gr}(T)$ of the same length as S may be obtained by appending v at the end of S'. If $q = p$ then such a sequence may be obtained by reinserting v into S' just before w, where w denotes the vertex of S' that footprints v_{i_p}. Hence, in each case we can transform S to an optimal sequence such that at the time v footprints exactly one of its children.

Let v_i be the unique child of v that is footprinted by v. If v_i or any of its children is contained in S, then it is added after v. Therefore $\gamma^{''}_{gr}(T_i) \geq k_i$. Now let $j \neq i$. By the choice of i, sequence $(a^j_1, \ldots, a^j_{k_j})$ is legal for $\gamma_{gr}(T_j)$, and hence $\gamma_{gr}(T_j) \geq k_j$. Together we get $k = \sum_{i=1}^r k_i + 1 \leq \gamma^{''}_{gr}(T_i) + \sum_{j\neq i}\gamma_{gr}(T_j) + 1 \leq A \leq \max\{A, B\}$.

Case 2: The root v of T is not contained in S. We consider two subcases.

Case 2.1: No child of v is in $\widehat{S}$. Since $v \notin \widehat{S}$, all children of v are footprinted by some member of S. In particular, for every $i \in \{1, \ldots, r\}$, there is a child w_i of v_i in S which footprints v_i. Let w denote the vertex from this set $\{w_1, \ldots, w_r\}$ that footprints a v_i. Replacing w with v in S produces a legal sequence for $\gamma^{''}_{gr}(T)$ of the same length as S that contains v. Hence, we have $k \leq A \leq \max\{A, B\}$ by the analysis above.

Case 2.2: Some child of v is in S. Let v_i denote the first child of v added to S. Since S is optimal for $\gamma^{''}_{gr}(T)$, v_i footprints at least one vertex from T_i. Thus $k_i \leq \gamma^+_{gr}(T_i)$. Now let $j \neq i$. Since v is already footprinted by v_i, every vertex from T_j, that is in S, footprints a vertex from T_j. Therefore $k_j \leq \gamma_{gr}(T_j)$ and thus $k = \sum_{i=1}^r k_i = \gamma^+_{gr}(T_i) + \sum_{j\neq i}\gamma_{gr}(T_j) \leq B \leq \max\{A, B\}$. □

The following result follows from Lemmas 5.1, 5.5 and 5.7.

Corollary 5.8 *If T is the rooted tree with $|T| > 1$, then*

$$\gamma^{''}_{gr}(T) = \max_{1\leq i\leq r}\{\gamma^{''}_{gr}(T_i) + \sum_{j\neq i}\gamma_{gr}(T_j) + 1\}.$$

Corollary 5.9 *If T is a rooted tree, then $\gamma^{''}_{gr}(T) = \gamma^-_{gr}(T)$.*

Proof. The proof is by induction on the number of vertices of T. If T is a tree with $|T| = 1$, then $\gamma_{gr}^{''}(T) = \gamma_{gr}^{-}(T) = 0$. Suppose that $\gamma_{gr}^{''}(T') = \gamma_{gr}^{-}(T')$ for every rooted tree T', with $|T'| < |T|$. Then

$$\begin{aligned}\gamma_{gr}^{''}(T) &= \max_{1\le i\le r}\{(\gamma_{gr}^{''}(T_i)+1)+\textstyle\sum_{j\neq i}\gamma_{gr}(T_j)\}\\ &= \max_{1\le i\le r}\{(\gamma_{gr}^{-}(T_i)+1)+\textstyle\sum_{j\neq i}\gamma_{gr}(T_j)\}\\ &= \gamma_{gr}^{-}(T),\end{aligned}$$

where the middle equality follows from the inductive assumption. □

Lemma 5.10 *For the rooted tree* T,

$$\gamma_{gr}^{'}(T) = \max\{\gamma_{gr}^{-}(T), \max_{i\neq j}\{\gamma_{gr}^{-}(T_i)+1+\sum_{k\neq i,j}\gamma_{gr}(T_k)+1+\gamma_{gr}^{''}(T_j)\}\},$$

where the second term in the outer maximum appears only if $r \ge 2$.

Proof. First we prove that $\gamma_{gr}^{'}(T) \ge \max\{A, B\}$, where $A = \gamma_{gr}^{-}(T)$ and $B = \max_{i\neq j}\{\gamma_{gr}^{-}(T_i)$ $+1+\sum_{k\neq i,j}\gamma_{gr}(T_k)+1+\gamma_{gr}^{''}(T_j)\}$ (or $B = A$ if $r < 2$). The inequality $\gamma_{gr}^{'}(T) \ge A$ has already been established in Lemma 5.1. To prove $\gamma_{gr}^{'}(T) \ge B$, we will construct a legal dominating sequence for $\gamma_{gr}^{'}(T)$ of length B. Assume that $r \ge 2$ and let $i \neq j$ be arbitrary indices from $\{1, \dots, r\}$. We start the sequence with an optimal sequence for $\gamma_{gr}^{-}(T_i)$. Then we add v_i, which is allowed, since it footprints v. The length of this part is $\gamma_{gr}^{-}(T_i)+1$. Then for every $k \neq i, j$ we append to the sequence an optimal sequence for $\gamma_{gr}^{-}(T_i)$. This extends the length of our sequence for $\sum_{k\neq i,j}\gamma_{gr}(T_k)$. Finally we add the root v and append an optimal sequence for $\gamma_{gr}^{''}(T_j)$. It is clear that we obtain a legal sequence for $\gamma_{gr}^{'}(T)$ of total length $\gamma_{gr}^{-}(T_i)+1+\sum_{k\neq i,j}\gamma_{gr}(T_k)+1+\gamma_{gr}^{''}(T_j)$. Since the choice of $i \neq j$ was arbitrary, this implies $\gamma_{gr}^{'}(T) \ge B$.

For the converse let S be the legal sequence $(a_1, \dots, a_k)$, optimal for $\gamma_{gr}^{'}(T)$. We need to prove that $k \le \max\{A, B\}$. Let $(a_1^i, \dots, a_{k_i}^i)$ be the subsequence of S which contains all vertices from S that lie in T_i. We distinguish two cases. If v is not contained in S then clearly $k \le \gamma_{gr}^{-}(T) \le \max\{A, B\}$. Thus let v be contained in S. Since at the time v is added to the sequence v is already dominated, at least one child of v is in S before v. Let v_i be the first such vertex, that is, if v_j with $j \neq i$ is contained in S, then v_i is added before v_j. Therefore $\gamma_{gr}^{-}(T_i) \ge k_i - 1$. Since v footprints at least one vertex, say v_j, v is in S before v_j and before any child of v_j. Thus no vertex from T_j, which is in S, footprints just v_j. Therefore $\gamma_{gr}^{''}(T_j) \ge k_j$. Since v is footprinted by v_i, $\gamma_{gr}(T_\ell) \ge k_\ell$ for every $\ell \neq i, j$. Together we get $k = \sum_{\ell=1}^{r} k_\ell + 1 \le \gamma_{gr}^{-}(T_i)+1+\sum_{\ell\neq i,j}\gamma_{gr}(T_\ell)+1+\gamma_{gr}^{''}(T_j) \le B \le \max\{A, B\}$. □

From Lemmas 5.5, 5.6 and 5.10 we get the following result.

Corollary 5.11 *If* T *is a rooted tree, then*

$$\gamma_{gr}^{'}(T) = \begin{cases} \max_{i\neq j}\{(\gamma_{gr}^{-}(T_i)+1)+\sum_{p\neq i,j}\gamma_{gr}(T_p)+1+\gamma_{gr}^{''}(T_j)\} & \text{if } r > 1,\\ \gamma_{gr}^{-}(T) & \text{if } r \le 1.\end{cases}$$

From Lemma 5.1 and Corollary 5.9 we immediately infer the following result.

Corollary 5.12 *If T is a rooted tree, then*

$$\gamma_{gr}^{''}(T) \leq \gamma_{gr}^{'}(T).$$

Lemma 5.13 *If T is a rooted tree, then*

$$\gamma_{gr}^{+}(T) = \max\left\{1 + \sum_{i=1}^{r} \gamma_{gr}^{'}(T_i), \max_{i \neq j}\left\{(\gamma_{gr}^{-}(T_i)+1) + \gamma_{gr}^{''}(T_j) + \sum_{k \neq i,j} \gamma_{gr}(T_k) + 1\right\}\right\},$$

where the second term in the outer maximum appears only if $r \geq 2$.

Proof. First we prove that $\gamma_{gr}^{+}(T) \geq \max\{A, B\}$, where $A = 1 + \sum_{i=1}^{r} \gamma_{gr}^{'}(T_i)$ and $B = \max_{i \neq j}\{(\gamma_{gr}^{-}(T_i)+1) + \gamma_{gr}^{''}(T_j) + \sum_{k \neq i,j} \gamma_{gr}(T_k) + 1\}$. It is enough to find a legal dominating sequence of length $\max\{A, B\}$ which contains v. First we construct such sequence of length A in the following way. For every $i \in \{1, \ldots, r\}$ let S_i be the legal sequence $(a_1^i, \ldots a_{k_i}^i)$, optimal for $\gamma_{gr}^{'}(T_i)$, that is, $k_i = \gamma_{gr}^{'}(T_i)$. Furthermore let $v_i = a_{s_i}^i$, where $s_i = k_i + 1$ if v_i is not in the sequence S_i. It is easy to see that

$$(a_1^1, \ldots, a_{s_1-1}^1, a_1^2, \ldots, a_{s_2-1}^2, \ldots, a_1^r, \ldots, a_{s_r-1}^r, v, a_{s_1}^1, \ldots, a_{k_1}^1, \ldots, a_{s_r}^r, \ldots, a_{k_r}^r)$$

is a legal dominating sequence for T of length A, containing v.

On the other hand, if $r > 1$, we can also find a legal dominating sequence containing v with length B. Let $i \neq j$ be two arbitrary indices from $\{1, \ldots, r\}$. We start the sequence with an optimal dominating sequence for T_i, not containing v_i, and then we add v_i. The length of this part is $\gamma_{gr}^{-}(T_i)+1$. Then we add as many as possible vertices from each subtree T_k, for $k \neq i, j$, which gives the length $\sum_{k \neq i,j} \gamma_{gr}(T_k)$. Finally we add the root v of T and an optimal sequence for $\gamma_{gr}^{'}(T_j)$. Notice that the sequence so constructed is a legal sequence for $\gamma^{+}(T)$ with length $\gamma_{gr}^{-}(T_i) + 1 + \gamma_{gr}^{''}(T_j) + \sum_{k \neq i,j} \gamma_{gr}(T_k) + 1$. Since the choice of $i \neq j$ was arbitrary, this implies $\gamma_{gr}^{+}(T) \geq B$. This shows that $\gamma_{gr}^{+}(T) \geq \max\{A, B\}$ for $r \geq 2$.

For the converse inequality, let $S = (a_1, \ldots, a_k)$ and assume that S is an optimal sequence for T, containing v (thus $k = \gamma_{gr}^{+}(T)$). We need to prove that $k \leq \max\{A, B\}$ if $r \geq 2$ and $k \leq A$ otherwise. For every $i \in \{1, \ldots, r\}$, let $(a_1^i, \ldots, a_{k_i}^i)$ be the subsequence of S which contains all vertices from S that lie in T_i. Thus $k = 1 + k_1 + \ldots + k_r$. We distinguish three cases.

Case 1: the vertex v footprints only itself. Therefore for every $i \in \{1, \ldots, r\}$, at least one child of v_i precedes v in S, and no child of v precedes v in S. Hence, if v_i is in S, then v_i footprints at least one vertex from $T_i \setminus \{v_i\}$, and it does not footprint itself, which means that $\gamma_{gr}^{'}(T_i) \geq k_i$. Thus $k = \sum_{1 \leq i \leq r} k_i + 1 \leq \sum_{1 \leq i \leq r} \gamma_{gr}^{'}(T_i) + 1 = A$.

Case 2: the vertex v footprints itself and at least one of its children. Without loss of generality let v footprint $v_1, \ldots, v_l$ for $1 \leq l \leq r$ and let $v_{l+1}, \ldots, v_r$ be already footprinted when v is added to S. Since v footprints itself, v precedes all of its children in S. Furthermore, for every $1 \leq i \leq l$, v precedes all children of v_i in S. Hence, $\gamma_{gr}^{''}(T_i) \geq k_i$ for every i such that $1 \leq i \leq l$. Now let $i \in \{l+1, \ldots, r\}$. If v_i is in S then it footprints at least one of its children but it does not footprint itself. Thus $\gamma_{gr}^{'}(T_i) \geq k_i$. Altogether we get $k = \sum_{i=1}^{r} k_i + 1 = \sum_{i=1}^{l} \gamma_{gr}^{''}(T_i) + \sum_{i=l+1}^{r} \gamma_{gr}^{'}(T_i) + 1 \leq \sum_{i=1}^{r} \gamma_{gr}^{'}(T_i) + 1 = A$.

Case 3: the vertex v is in S, but it does not footprint itself. Note that in this case $r > 1$. Thus at least one of the children of v precedes v in S, and at least one of its children is not dominated before v is added. Therefore the case of sequence that realizes $\gamma_{gr}^{'}(T)$ with

v being in S applies. Hence we may use the same argument as the one in the proof of Lemma 5.10 to conclude that $k \le B \le \max\{A, B\}$. $\square$

From Corollary 5.11 and Lemma 5.13 we get the following result.

Corollary 5.14 *If T is a rooted tree, then*

$$\gamma^+_{gr}(T) = \begin{cases} \max\{1 + \sum_{1\le i\le r} \gamma^{'}_{gr}(T_i), \gamma^{'}_{gr}(T)\} & \text{if } r > 1, \\ 1 + \sum_{1\le i\le r} \gamma^{'}_{gr}(T_i) & \text{if } r \le 1. \end{cases}$$

Corollary 5.15 *If T is a rooted tree, then*

$$\gamma^+_{gr}(T) \ge \gamma^-_{gr}(T).$$

Proof. If $r = 0$ then the result is clear. If $r = 1$ then $\gamma^+_{gr}(T) = 1 + \gamma^{'}_{gr}(T_1) \ge 1 + \gamma^{''}_{gr}(T_1) = 1 + \gamma^-_{gr}(T_1) = \gamma^-_{gr}(T)$. Finally if $r > 1$, then $\gamma^+_{gr}(T) \ge \gamma^{'}_{gr}(T) \ge \gamma^{''}_{gr}(T) = \gamma^-_{gr}(T)$. $\square$

Corollary 5.16 *If T is a rooted tree, then*

$$\gamma_{gr}(T) = \gamma^+_{gr}(T).$$

Corollaries 5.8, 5.9, 5.11, 5.14 and 5.16 imply the following relations.

Corollary 5.17 *If T is a rooted tree, then*

- $\gamma^{''}_{gr}(T) = \begin{cases} \max_{1\le i\le r}\{\gamma^{''}_{gr}(T_i) + \sum_{j\ne i} \gamma_{gr}(T_j) + 1\} & \text{if } r \ge 1, \\ 0 & \text{if } r = 0. \end{cases}$
- $\gamma^{'}_{gr}(T) = \begin{cases} \max_{i\ne j}\{(\gamma^{''}_{gr}(T_i) + 1) + \sum_{p\ne i,j} \gamma_{gr}(T_p) + 1 + \gamma^{''}_{gr}(T_j)\} & \text{if } r > 1, \\ \gamma^{''}_{gr}(T) & \text{if } r \le 1. \end{cases}$
- $\gamma_{gr}(T) = \begin{cases} \max\{1 + \sum_{1\le i\le r} \gamma^{'}_{gr}(T_i), \gamma^{'}_{gr}(T)\} & \text{if } r > 1, \\ 1 + \sum_{1\le i\le r} \gamma^{'}_{gr}(T_i) & \text{if } r \le 1. \end{cases}$

Before describing a linear time algorithm for computing the Grundy domination number of a tree, let us first simplify the above corollary. Observe that:

$$\max_{1\le i\le r}\left\{\gamma^{''}_{gr}(T_i) + \sum_{j\ne i} \gamma_{gr}(T_j) + 1\right\} \quad = \quad \sum_{j=1}^{r} \gamma_{gr}(T_j) + \max_{1\le i\le r}\left\{\gamma^{''}_{gr}(T_i) - \gamma_{gr}(T_i)\right\} + 1$$

and

$$\begin{aligned} &\max_{i\ne j}\left\{(\gamma^{''}_{gr}(T_i) + 1) + \sum_{p\ne i,j} \gamma_{gr}(T_p) + 1 + \gamma^{''}_{gr}(T_j)\right\} \\ = \quad &\sum_{p=1}^{r} \gamma_{gr}(T_p) + \max_{i\ne j}\left\{\left(\gamma^{''}_{gr}(T_i) - \gamma_{gr}(T_i)\right) + \left(\gamma^{''}_{gr}(T_j) - \gamma_{gr}(T_j)\right)\right\} + 2\,. \end{aligned}$$

Hence, denoting $\delta\gamma_{gr}(T) = \gamma^{''}_{gr}(T) - \gamma_{gr}(T)$, we can rephrase the above corollary in the following equivalent way:

Corollary 5.18 *If T is a rooted tree, then*

- $\gamma''_{gr}(T) = \begin{cases} \sum_{j=1}^{r} \gamma_{gr}(T_j) + \max_{1\le i\le r}\{\delta\gamma_{gr}(T_i)\} + 1 & \textit{if } r \ge 1, \\ 0 & \textit{if } r = 0. \end{cases}$

- $\gamma'_{gr}(T) = \begin{cases} \sum_{p=1}^{r} \gamma_{gr}(T_p) + \max_{i\ne j}\Big\{\delta\gamma_{gr}(T_i) + \delta\gamma_{gr}(T_j)\Big\} + 2 & \textit{if } r > 1, \\ \gamma''_{gr}(T) & \textit{if } r \le 1. \end{cases}$

- $\gamma_{gr}(T) = \begin{cases} \max\{\gamma'_{gr}(T), \sum_{1\le i\le r} \gamma'_{gr}(T_i) + 1\} & \textit{if } r > 1, \\ \sum_{1\le i\le r} \gamma'_{gr}(T_i) + 1 & \textit{if } r \le 1. \end{cases}$

Consequently, Algorithm 2 below correctly computes the Grundy domination number of a tree T in time $O(n)$. Let us root T at an arbitrary vertex r. For $w \in V(T)$, we denote by T_w the subtree of T rooted at w.

Algorithm 2: Grundy domination number of a tree

Input: A tree T on n vertices.
Output: $\gamma_{gr}(T)$.

Fix a root $r \in V(T)$, and let $v_1, v_2, \ldots, v_{n-1}, v_n = r$ be the vertices of T listed in reverse order with respect to the time they are visited by a breadth-first traversal of T from r

for $i = 1, \ldots, n$ **do**
  Let $C(v_i)$ be the set of children of v_i
  if $C(v_i) = \emptyset$ **then**
    $\gamma''_{gr}(T_{v_i}) \leftarrow 0$, $\gamma'_{gr}(T_{v_i}) \leftarrow 0$, $\gamma_{gr}(T_{v_i}) \leftarrow 1$, $\delta\gamma_{gr}(T_{v_i}) \leftarrow -1$
  else
    $S \leftarrow \sum_{u\in C(v_i)} \gamma_{gr}(T_u)$
    $M \leftarrow \max\big\{\delta\gamma_{gr}(T_u) \,:\, u \in C(v_i)\big\}$
    $\gamma''_{gr}(T_{v_i}) \leftarrow S + M + 1$
    if $|C(v_i)| = 1$ **then**
      $\gamma'_{gr}(T_{v_i}) \leftarrow \gamma''_{gr}(T_{v_i})$
      $\gamma_{gr}(T_{v_i}) \leftarrow 1 + \gamma'_{gr}(T_u)$, where u is the unique child of v_i
    else
      $M' \leftarrow$ second largest element in the multiset $\big\{\delta\gamma_{gr}(T_u) \,:\, u \in C(v_i)\big\}$
      $\gamma'_{gr}(T_{v_i}) \leftarrow S + M + M' + 2$
      $\gamma_{gr}(T_{v_i}) \leftarrow \max\left\{\gamma'_{gr}(T_{v_i}), \displaystyle\sum_{u\in C(v_i)} \gamma'_{gr}(T_u) + 1\right\}$
    $\delta\gamma_{gr}(T_{v_i}) \leftarrow \gamma''_{gr}(T_{v_i}) - \gamma_{gr}(T_{v_i})$

return $\gamma_{gr}(T_r)$

Let us now justify the algorithm's linear time complexity. The vertex ordering $v_1, \ldots, , v_n$ can be computed in $O(n)$ time using a breadth-first traversal from r. At each leaf (vertex v_i with $C(v_i) = \emptyset$), a constant number of operations is performed. At each internal vertex v_i, $O(|C(v_i)|)$ operations are performed (assuming, as usual, that adding and comparing two numbers can be done in $O(1)$ time). Denoting by $L(T)$ the set of leaves of T and by $I(T)$ the set of internal vertices of T, we can bound the time complexity of the algorithm as

$$O(|L(T)|) + \sum_{v \in I(T)} O(|C(v)|) = O(|L(T)|) + O(|E(T)|) = O(n)\,.$$

We have thus proved:

Theorem 5.19 *There exists a linear time algorithm for computing the Grundy domination number of a tree.*

Remark 5.20 *The results of this section and their proofs imply that Algorithm 2 can be modified so that it also computes a Grundy dominating sequence of the input tree T. At every vertex v_i, an optimal dominating sequences for $\gamma_{gr}(T_{v_i}), \gamma_{gr}^{'}(T_{v_i})$ and $\gamma_{gr}^{''}(T_{v_i})$ has to be computed. The time complexity of the so modified algorithm becomes $O(n^2)$.*

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