Relating the Annihilation Number and the 2-Domination Number of a Tree

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Abstract

A set S of vertices in a graph G is a 2-*dominating set* if every vertex of G not in S is adjacent to at least two vertices in S. The 2-*domination number* $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set in G. The *annihilation number* $a(G)$ is the largest integer k such that the sum of the first k terms of the nondecreasing degree sequence of G is at most the number of edges in G. The conjecture-generating computer program, Graffiti.pc, conjectured that $\gamma_2(G) \le a(G) + 1$ holds for every connected graph G. It is known that this conjecture is true when the minimum degree is at least 3. The conjecture remains unresolved for minimum degree 1 or 2. In this paper, we prove that the conjecture is indeed true when G is a tree, and we characterize the trees that achieve equality in the bound. It is known that if T is a tree on n vertices with n_1 vertices of degree 1, then $\gamma_2(T) \le (n + n_1)/2$. As a consequence of our characterization, we also characterize trees T that achieve equality in this bound.

*Research supported in part by the University of Johannesburg and the South African National Research Foundation.

†Research supported in part by the University of Johannesburg.

Keywords: 2-domination, 2-domination number, annihilation number
AMS subject classification (2010): 05C69

1 Introduction

In this paper, we study upper bounds on the 2-domination numbers of trees in terms of their annihilation numbers. For $k \geq 1$, a *k-dominating set* of a graph G is a set S of vertices in G such that every vertex outside S is adjacent to at least k vertices in S. Every graph G has a k-dominating set, since $V(G)$ is such a set. The *k-domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality of a k-dominating set of G. In particular, a 1-dominating set is a dominating set, and the 1-domination number $\gamma_1(G)$ is the domination number $\gamma(G)$. A k-dominating set of G of cardinality $\gamma_k(G)$ is called a *γ_k-set* of G. The concept of a k-dominating set was first introduced by Fink and Jacobson in 1985 [6] and is now well-studied in the literature. We refer the reader to the two books on domination by Haynes, Hedetniemi, and Slater [9, 10], as well as to the excellent survey on k-domination in graphs by Chellali, Favaron, Hansberg, and Volkmann [2].

As explained in [11], the *annihilation number* of a graph was first introduced by Pepper in [13]. Originally it was defined in terms of a reduction process on the degree sequence similar to the Havel-Hakimi process (see [7, 14]). In [13], Pepper showed an equivalent way to define the annihilation number, this is the version we will use in this work. The *annihilation number* of a graph G, denoted $a(G)$, is the largest integer k such that the sum of the first k terms of the degree sequence of G arranged in nondecreasing order is at most the number of edges. That is if $d_1, \ldots, d_n$ is the degree sequence of a graph G with m edges, where $d_1 \leq \cdots \leq d_n$, then the annihilation number of G is the largest integer k such that $\sum_{i=1}^{k} d_i \leq m$ or, equivalently, the largest integer k such that $\sum_{i=1}^{k} d_i \leq \sum_{i=k+1}^{n} d_i$.

The conjecture-generating computer program, Graffiti.pc, made the following conjecture relating the 2-domination number of a graph and its annihilation number.

Conjecture 1. ([5]) *If G is a connected graph with at least 2 vertices, then $\gamma_2(G) \leq a(G)+1$.*

It is known that Conjecture 1 is true when the minimum degree is at least 3. Conjecture 1 is still unresolved when the minimum degree of G is 1 or 2. Proving the conjecture for trees may be the most interesting case. Our aim in this paper is threefold: First to prove that Conjecture 1 is indeed true for trees. Secondly to characterize the extremal trees achieving equality in the upper bound of Conjecture 1. Thirdly to characterize trees with the largest possible 2-domination number.

1.1 Notation

In this paper, the word "graph" is used to denote a "simple graph" with no loops or multiple edges. For notation and graph theory terminology not defined herein, we in general follow [9]. We write $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G. Usually, we use n for the number of vertices and m for the number of edges. We write $N_G(v)$ and $d_G(v)$ for the neighborhood and degree of a vertex $v \in V(G)$. We extend the notion of neighborhood to sets by letting $N_G(S) = \bigcup_{v\in S} N(v)$ for any $S \subseteq V(G)$. If the graph G is clear from the context, we simply write $N(v)$, $N(S)$, and $d(v)$ rather than $N_G(v)$, $N_G(S)$, and $d_G(v)$, respectively. The minimum degree among the vertices of G is denoted by $\delta(G)$. The *matching number* is the maximum size of a matching in G and is denoted by $\alpha'(G)$. A vertex of degree 1 is called a *leaf*, its neighbor is a *support vertex*, and its incident edge is a *pendant edge*. We denote the set of leaves of a tree T by $L(T)$. A *star* is a tree with at most one non-leaf vertex. The *corona* of a graph G, denoted $G \circ K_1$, is formed from G by adding for each $v \in V(G)$, a new vertex v' and the pendant edge vv'.

For a set $S \subseteq V(G)$, we let $G[S]$ denote the subgraph induced by S. The graph obtained from G by deleting the vertices in S and all edges incident with vertices in S is denoted by $G - S$. In the special case when $S = \{v\}$, we also denote $G - S$ by $G - v$ for simplicity. For a set $S \subseteq V(G)$ and $v \in V(G)$, we denote by $d_S(v)$ the number of all vertices in S that are adjacent to v. In particular, when $S = V(G)$, we note $d_S(v) = d(v)$. For a subset $S \subseteq V(G)$, we define

$$\Sigma(S, G) = \sum_{v\in S} d_G(v).$$

For a graph G with m edges, we define an a-set of G to be a (not necessarily unique) set S of vertices in G such that $|S| = a(G)$ and $\sum_{v\in S} d_G(v) \le m$. We define an $a_{\min}$-set of G to be an a-set S of G, such that $\Sigma(S, G)$ is a minimum. Hence if S is an $a_{\min}$-set of G, then S is a set of (not necessarily unique) vertices corresponding to the first $a(G)$ vertices in the nondecreasing degree sequence of G.

In order to prove Conjecture 1 for trees, we introduce a slight variation of the annihilation number of a graph. We define the *upper annihilation number* of a graph G, denoted $a^*(G)$, to be the largest integer k such that the sum of the first k terms of the degree sequence of G arranged in nondecreasing order is at most $|E(G)|+1$. That is if $d_1, \ldots, d_n$ is the degree sequence of a graph G with m edges, where $d_1 \le \cdots \le d_n$, then the upper annihilation number of G is the largest integer k such that $\sum_{i=1}^{k} d_i \le m + 1$. We define an $a^*_{\min}$-set of G to be a (not necessarily unique) set S^* of vertices in G such that $|S^*| = a^*(G)$ and S^* corresponds to the first $a^*(G)$ vertices in the nondecreasing degree sequence of G.

1.2 Known Results and Observations

In their introductory paper on k-domination, Fink and Jacobson [6] established the following lower bound on the k-domination number of a tree.

Theorem 1. ([6]) *For $k \ge 1$, if T is a tree with n vertices, then $\gamma_k(T) \ge ((k-1)n+1)/k$.*

As a special case of Theorem 1, if T is a tree with n vertices, then $\gamma_2(T) \ge (n + 1)/2$. The following upper bound on the 2-domination number of a tree was observed in several papers.

Theorem 2. ([4, 8, 12]) *If T is a tree with n vertices and n_1 leaves, then $\gamma_2(T) \le (n+n_1)/2$.*

Caro and Roditty [1] and Stracke and Volkmann [15] established the following upper bound on the k-domination number of a graph.

Theorem 3. ([1, 15]) *For every graph G with n vertices and every integer $k \ge 1$, if $\delta(G) \ge 2k-1$, then $\gamma_k(G) \le \lfloor n/2 \rfloor$.*

In the special case when $k=2$, the result of Theorem 3 states that if G is a graph with n vertices and $\delta(G) \ge 3$, then $\gamma_2(G) \le \lfloor n/2 \rfloor$. Since $\alpha'(G) \le \lfloor n/2 \rfloor$ for any graph G with n vertices, this result was improved in the following theorem.

Theorem 4. ([3]) *Let k be a positive integer. If G is any graph with $\delta(G) \ge 2k-1$, then $\gamma_k(G) \le \alpha'(G)$.*

We remark that both Theorems 2 and 3 follow from a more general result in Hansberg et al. [8].

Theorem 5. ([8]) *If G is an r-partite graph with n vertices and k is a positive integer, then*

$$\gamma_k(G) \le \frac{1}{r}\left((r-1)n + |x \in V(G)\colon d_G(x) \le k-1|\right).$$

Before we continue, a definition is in order.

Definition 1. ([12]) A set of vertices is *j-independent* if each vertex of the set has at most $j-1$ neighbors in the set. The *j-independence number* of G, denoted $\alpha_j(G)$, is the cardinality of a largest j-independent set in G (when $j=1$, this is just the standard independence number and a 1-independent set is a standard independent set).

As can be seen in [12], Theorems 2, 3 and 5 all follow as corollaries of the following result.

Theorem 6. ([12]) *Given positive integers j, k, m, and n, let G be a graph with n vertices, and let H_m be the subgraph of G induced by the vertices having degree at least m. If $m = k+j-1$, then $\gamma_k(G) \le n - \alpha_j(H_m)$.*

Pepper was the first to observe that if G is an n-vertex graph with $n \ge 2$, then $a(G) \ge \lfloor n/2 \rfloor$. This observation of Pepper's as well as Theorem 3 when $k=2$, lead to the following observation due to West.

Observation 7. ([16]) *Conjecture 1 is true if $\delta(G) \ge 3$.*

We remark that Pepper constructed an infinite family $\mathcal{F}$ of graphs G for which $\delta(G)=2$ and $\gamma_2(G) = a(G)+1$, as follows. Let $\mathcal{F}$ be the family of graphs formed from r disjoint copies of C_5 in the following manner. Add to this graph a matching with $r-1$ edges, where each edge of the matching joins two 5-cycles and on each 5-cycle the vertices incident to the matching are nonadjacent. When $r=3$, an example of a graph in the family $\mathcal{F}$ is shown in Figure 1. As observed by Pepper, if $G \in \mathcal{F}$, then G has $6r-1$ edges, with $a(G) = 3r-1$ and $\gamma_2(G) = 3r$.

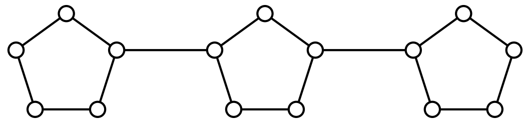


Figure 1: A graph $G \in \mathcal{F}$.

2 The Family $\mathcal{T}$

For integers r and s such that $r, s \geq 1$, a *double star* $S(r, s)$ is a tree with exactly two vertices that are not leaves, one of which is adjacent to r leaves and the other to s leaves. In order to state our main result, we construct a family $\mathcal{T}$ of trees as follows. For any integer j greater than 1, let $T_{2,j}$ denote the double star $S(1, j-1)$.

Definition 2. (The Family $\mathcal{T}$) For integers i and j, where $2 \leq i \leq j$, we construct the family $\mathcal{T}_{i,j}$ of trees defined recursively as follows. Initially we let $T_{2,j} \in \mathcal{T}_{2,j}$. For every tree $T \in \mathcal{T}_{i,j}$, do the following.

$\mathcal{O}_1$**:** If $v \in V(T)$ is a leaf in T, then add the set $\{t, s_1, s_2, \ldots, s_\ell\}$ of $\ell + 1$ new vertices to $V(T)$, where $\ell \geq i - 1$ is arbitrary, and add the edge ts_1 and the edges vs_i for all $i = 1, 2, \ldots, \ell$ to $E(T)$. Add the resulting tree to the family $\mathcal{T}_{i,\min\{j,\ell+1\}}$.

$\mathcal{O}_2$**:** If $v \in V(T)$ has $d_T(v) \leq \min\{i, j-1\}$, then add the set $\{t, s_1, s_2, \ldots, s_\ell\}$ of $\ell + 1$ new vertices to $V(T)$, where $\ell \geq \max\{d_T(v) + 1, i\} - 1$ is arbitrary, and add the edge tv and the edges $s_i t$ for all $i = 1, 2, \ldots, \ell$ to $E(T)$. Add the resulting tree to the family $\mathcal{T}_{\max\{d_T(v)+1,i\},\min\{j,\ell+1\}}$.

For an integer $i \geq 2$, let

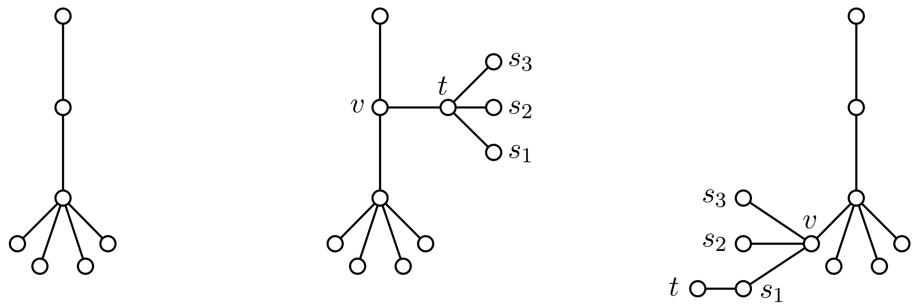
$$\mathcal{T}_i = \bigcup_{j \geq i} \mathcal{T}_{i,j} \qquad \text{and let} \qquad \mathcal{T} = \{K_2\} \cup \left(\bigcup_{i \geq 2} \mathcal{T}_i \right).$$

Figure 2 illustrates the operations $\mathcal{O}_1$ and $\mathcal{O}_2$ on the tree $T_{2,5}$.

We remark that for $2 \leq i \leq j$, the family $\mathcal{T}_{i,j}$ is defined as a set of trees. Further, since it is the union of a set of trees, $\mathcal{T}_i$ is also a set of trees, as is the family $\mathcal{T}$. In [16] it is mentioned that Pepper found an infinite family of trees for which the 2-domination number equals the upper bound of Conjecture 1. This is the family of trees formed by taking any tree, forming the corona of that tree and then forming the corona of the resulting tree. We remark that Pepper's family of trees is contained in our family $\mathcal{T}$ constructed above.

3 Main Results

Observation 7 states that Conjecture 1 is true when the minimum degree is at least 3. However, as remarked earlier Conjecture 1 remains unresolved when $\delta(G) \in \{1, 2\}$. We



(a) $T_{2,5}$ (b) Operation $\mathcal{O}_2$ (c) Operation $\mathcal{O}_1$

Figure 2: Operations $\mathcal{O}_1$ and $\mathcal{O}_2$ applied to the tree $T_{2,5}$.

prove that Conjecture 1 is true for trees and we characterize the trees achieving equality in Conjecture 1. We shall prove the following results. A proof of Theorem 8 is given in Section 5.

Theorem 8. *For a tree T, the following hold.*
(a) $\gamma_2(T) \le a^*(T)$.
(b) $\gamma_2(T) \le a(T) + 1$.
(c) $\gamma_2(T) = a(T) + 1$ *if and only if* $T \in \mathcal{T}$.

Recall that by Theorem 2, if T is a tree with n vertices and n_1 leaves, then $\gamma_2(T) \le (n + n_1)/2$. As a consequence of our main result, namely Theorem 8, we characterize the trees achieving equality in this bound. Recall that $\mathcal{T}_2 = \bigcup \mathcal{T}_{2,j}$, where the union is taken over all integers j with $j \ge 2$. A proof of Theorem 9 is given in Section 6.

Theorem 9. *If T is a tree with n vertices and n_1 leaves, then $\gamma_2(T) \le (n + n_1)/2$, with equality if and only if $T \in \mathcal{T}_2 \cup \{K_2\}$.*

4 Preliminary Results

In this section, we establish some preliminary results that we will need when proving our main results. Recall that for a subset $S \subseteq V(G)$, we define $\Sigma(S, G) = \sum_{v \in S} d_G(v)$. We begin with the following trivial observation.

Observation 10. *For an $a^*_{\min}$-set S^* in a tree T with m edges, the following hold.*
(a) *If $\Sigma(S^*, T) \le m$, then $a^*(T) = a(T)$.*
(b) *If $\Sigma(S^*, T) = m + 1$, then $a^*(T) = a(T) + 1$.*

Proof. For an $a^*_{\min}$-set S^* in a tree T with m edges, $|S^*| = a^*(T)$ and $\Sigma(S^*, T) \le m + 1$. By definition, $a^*(T) \ge a(T)$. If $\Sigma(S^*, T) \le m$, then $a(T) \ge |S^*| = a^*(T)$, implying that $a^*(T) = a(T)$, which establishes Part (a). If $\Sigma(S^*, T) = m+1$, then let $v \in S^*$ be arbitrary and note that $\Sigma(S^* \setminus \{v\}, T) \le m$, and so $a(T) \ge |S^* \setminus \{v\}| = |S^*| - 1 = a^*(T) - 1$. Since S^* is a set of vertices corresponding to the first $a^*(T)$ vertices in the nondecreasing degree sequence of T and $\Sigma(S^*, T) = m+1$, we have $a(T) \le a^*(T) - 1$. Consequently, $a^*(T) = a(T) + 1$. $\square$

As a consequence of Observation 10, if T is a tree then $a^*(T) = a(T)$ or $a^*(T) = a(T)+1$. We next establish some useful properties of trees that belong to the family $\mathcal{T} \setminus \{K_2\}$. We shall need the following notation. For a set $S \subseteq V(G)$, we define

$$\Delta_S(G) = \max\{d_G(v) \mid v \in S\}$$

and

$$\overline{\delta}_S(G) = \min\{d_G(v) \mid v \in V(G) \setminus S\}$$

Thus, $\Delta_S(G)$ is the maximum degree in G among all the vertices in S, while $\overline{\delta}_S(G)$ is the minimum degree in G among all the vertices that do not belong to S.

Lemma 11. *Let $T' \in \mathcal{T} \setminus \{K_2\}$, and so $T' \in \mathcal{T}_{i,j}$ for some integers $j \ge i \ge 2$. If S' is an $a_{\min}$-set in T' and $S^{*\prime}$ is an $a^*_{\min}$-set in T', then the following hold.*

- (a) $\overline{\delta}_{S'}(T') = i$ *and* $\overline{\delta}_{S^{*\prime}}(T') = j$.
- (b) $\Delta_{S'}(T') \le i$ *and* $\Delta_{S^{*\prime}}(T') = i$.
- (c) $\Sigma(S^{*\prime}, T') = m(T') + 1$.
- (d) $a^*(T') = a(T') + 1$.
- (e) $\gamma_2(T') = a^*(T')$.

Proof. We proceed by induction on the minimum number k of operations needed to construct a tree T' in the family $T' \in \mathcal{T} \setminus \{K_2\}$. When $k = 0$, we have $T' = T_{2,j} \in \mathcal{T}_{2,j}$ for some $j \ge 2$. Let u and v denote the two vertices of T' that are not leaves, where u has one leaf neighbor and v has $j-1$ leaf neighbors. Now $\gamma_2(T') = j+1$, $a(T') = j$ and $a^*(T') = j+1$. In particular, $a^*(T') = a(T') + 1$ and $\gamma_2(T') = a^*(T')$. The set $S' = L(T')$ of leaves of T' is the unique $a_{\min}$-set in T', implying that $d_{T'}(u) = \overline{\delta}_{S'}(T') = 2 = i$ and $\Delta_{S'}(T') = 1 < i$. Furthermore, the set $L(T') \cup \{u\}$ is the unique $a^*_{\min}$-set in T', unless $j = 2$, in which case $L(T') \cup \{v\}$ is also an $a^*_{\min}$-set in T'. Hence if $S^{*\prime}$ is an $a^*_{\min}$-set in T', then $d_{T'}(v) = \overline{\delta}_{S^{*\prime}}(T') = j$, $d_{T'}(u) = \Delta_{S^{*\prime}}(T') = 2 = i$, and $\Sigma(S^{*\prime}, T') = m(T') + 1 = j+2$. Hence the tree T' satisfies properties (a)–(e). This establishes the base case when $k = 0$.

Suppose that $k \ge 0$, and let $T \in \mathcal{T} \setminus \{K_2\}$ be constructed using k operations. Thus, $T \in \mathcal{T}_{i,j}$ for some integers $j \ge i \ge 2$. For the inductive hypothesis, assume that the tree T satisfies properties (a)–(e). Let T' be obtained from T by using Operation $\mathcal{O}_1$ or $\mathcal{O}_2$ in Definition 2, and so T' is constructed using $k+1$ operations. We show that properties (a)–(e) also hold for the tree T', which will complete the proof by induction. We consider two possibilities.

Case 1. T' is obtained from T by applying Operation $\mathcal{O}_1$. We adopt the notation used in Definition 2. Let $L_{T'} = \{s_2, \ldots, s_\ell\} \cup \{t\}$. Let S^* be a $a^*_{\min}$-set in T. Let z be a vertex in S^* of highest possible degree in T, and so $d_T(z) = \Delta_{S^*}(T)$. By Property (b), we have $d_T(z) = i \ge 2$, and by Property (c) we deduce that the set $S^* \setminus \{z\}$ is an $a_{\min}$-set in T. Since

v is a leaf in T, $z \in S^*$, and $d_T(z) > 1$, we note $v \in S^*$. Let $S^{*\prime} = (S^* \cup L_{T'} \cup \{s_1\}) \setminus \{v\}$. Applying Properties (a)–(e) for the tree T, we have

$$\begin{aligned}
\Sigma(S^{*\prime}, T') &= \Sigma(S^* \setminus \{v\}, T) + \ell + 2 \\
&= \Sigma(S^*, T) + \ell + 1 \\
&= (m(T) + 1) + \ell + 1 \\
&= m(T') + 1.
\end{aligned}$$

By Property (b), all vertices in S^* have degree at most i in T. Since the degrees of vertices in $S^* \setminus \{v\}$ remain unchanged in T' and since $i \geq 2$, all vertices in $S^{*\prime}$ have degree at most i in T'. Recall that $z \in S^*$ and $d_T(z) = i$. Since $v \neq z$, we have $z \in S^{*\prime}$ and $d_{T'}(z) = i$, implying $\Delta_{S^{*\prime}}(T') = i$.

By Property (a), $\overline{\delta}_{S^*}(T) = j$. Hence since $V(T') \setminus S^{*\prime} = (V(T) \setminus S^*) \cup \{v\}$, we have

$$\overline{\delta}_{S^{*\prime}}(T') = \min\{\overline{\delta}_{S^*}(T), d_{T'}(v)\} = \min\{j, \ell + 1\}.$$

Since $T' \in \mathcal{T}_{i,\min\{j,\ell+1\}}$, it follows that the second statement of property (a) is satisfied.

In particular, since $\ell+1 \geq i$ and $j \geq i$, every vertex in T' that does not belong to the set $S^{*\prime}$ has degree at least i in T'. As observed earlier, $\Sigma(S^{*\prime}, T') = m(T')+1$. Furthermore, every vertex in $S^{*\prime}$ has degree at most i in T', and there exists a vertex in $S^{*\prime}$ of degree exactly i in T'. Therefore, the set $S^{*\prime}$ is an $a^*_{\min}$-set in T', and so $a^*(T') = |S^{*\prime}| = |S^*|+\ell = a^*(T)+\ell$. As shown earlier, $\Delta_{S^{*\prime}}(T') = i$ and $\overline{\delta}_{S^{*\prime}}(T') = \min\{j, \ell+1\}$. Since $S^{*\prime}$ is defined as a set of vertices corresponding to the first $a^*(T')$ vertices in the nondecreasing degree sequence of T', and $\Sigma(S^{*\prime}, T') = m(T') + 1$, we have $a(T') \leq a^*(T') - 1$. Consequently, by Observation 10, $a^*(T') = a(T') + 1$. By our properties of the set $S^{*\prime}$, it follows that if S' is an $a_{\min}$-set in T', then $\overline{\delta}_{S'}(T') = i$, so $\Delta_{S'}(T') \leq i$.

It remains to show that $\gamma_2(T') = a^*(T')$. Let D be a γ_2-set of T'. Since the set D contains all leaves of T', we have $L_{T'} \subset D$. If $s_1 \in D$, then we can replace s_1 in D by v. Hence we may assume that $s_1 \notin D$. In order to 2-dominate s_1, we have $v \in D$. Therefore the set $D \setminus L_{T'}$ is a 2-dominating set in T, so $\gamma_2(T) \leq |D| - |L_{T'}| = \gamma_2(T') - \ell$. Conversely, noting that v, which is a leaf in T, belongs to every 2-dominating set of T, we have that adding the set $L_{T'}$ to an arbitrary γ_2-set of T produces a 2-dominating set of T', and so $\gamma_2(T') \leq \gamma_2(T) + \ell$. Consequently, $\gamma_2(T') = \gamma_2(T)+\ell$. Since T satisfies Property (e), we have $\gamma_2(T) = a^*(T)$. As established earlier, $a^*(T') = a^*(T) + \ell$. Therefore, $\gamma_2(T') = \gamma_2(T) + \ell = a^*(T) + \ell = a^*(T')$. Hence in Case 1, Properties (a)–(e) hold for the tree T'.

Case 2. T' is obtained from T by applying Operation $\mathcal{O}_2$. We once again adopt the notation used in Definition 2. In this case let $L_{T'} = \{s_1, s_2, \ldots, s_\ell\}$. By definition, $T' \in \mathcal{T}_{i',j'}$, where $i' = \max\{d_T(v) + 1, i\}$ and $j' = \min\{j, \ell + 1\}$. The definition of ℓ yields $\ell \geq i' - 1$.

Let S^* be a $a^*_{\min}$-set in T. By the inductive hypothesis applied to the tree T, we have $\Sigma(S^*, T) = m(T) + 1$ and $\Delta_{S^*}(T) = i$. This implies that S^* contains at least one vertex of degree i. By the choice of v, we have $d_T(v) \leq i$. If $v \notin S^*$, then $d_T(v) = i$ and we can simply replace a vertex of S^* of degree i in T with v. Hence we may choose S^* so that $v \in S^*$. Let $S^{*\prime} = S^* \cup L_{T'}$. Since $v \in S^*$, we have

$$\begin{array}{rcl}\Sigma(S^{*\prime}, T') & = & (\Sigma(S^*, T) + 1) + \ell \\ & = & (m(T) + 1) + \ell + 1 \\ & = & m(T') + 1.\end{array}$$

We next consider the degrees of the vertices of $S^{*\prime}$ in the tree T'. If $x \in L_{T'}$, then $d_{T'}(x) = 1$. If $x \in S^*$, then either $x = v$, in which case $d_{T'}(v) = d_T(v) + 1$, or $x \neq v$, in which case $d_{T'}(x) = d_T(x) \le \Delta_{S^*}(T) = i$. Hence, $\Delta_{S^{*\prime}}(T') = \max\{d_T(v) + 1, \Delta_{S^*}(T)\} = \max\{d_T(v) + 1, i\}$; that is, $\Delta_{S^{*\prime}}(T') = i'$. Thus all vertices in $S^{*\prime}$ have degree at most i' in T'. By Property (a), $\overline{\delta}_{S^*}(T) = j$. Hence since $V(T') \setminus S^{*\prime} = (V(T) \setminus S^*) \cup \{t\}$, we have

$$\overline{\delta}_{S^{*\prime}}(T') = \min\{\overline{\delta}_{S^*}(T), d_{T'}(t)\} = \min\{j, \ell + 1\} = j'.$$

We also note that by the choice of v, we have $j \ge d_T(v) + 1$. Further, $j \ge i$, and so $j \ge \max\{d_T(v) + 1, i\} = i'$. Moreover, $\ell + 1 \ge i'$. Therefore,

$$j' = \min\{j, \ell + 1\} \ge i'.$$

Hence all vertices in T' that do not belong to the set $S^{*\prime}$ have degree at least i' in T'. As observed earlier, all vertices in $S^{*\prime}$ have degree at most i' in T', and there exists a vertex in $S^{*\prime}$ of degree exactly i' in T'. Thus since $\Sigma(S^{*\prime}, T') = m(T') + 1$, the set $S^{*\prime}$ is an $a^*_{\min}$-set in T', and so $a^*(T') = |S^{*\prime}| = |S^*| + \ell = a^*(T) + \ell$.

As shown earlier, $\Delta_{S^{*\prime}}(T') = i'$ and $\overline{\delta}_{S^{*\prime}}(T') = j'$. Since $S^{*\prime}$ is defined as a set of vertices corresponding to the first $a^*(T')$ vertices in the nondecreasing degree sequence of T' and $\Sigma(S^{*\prime}, T') = m(T') + 1$, we have $a(T') \le a^*(T') - 1$. Consequently, by Observation 10, $a^*(T') = a(T') + 1$, thus proving property (d). Let S' be an $a_{\min}$-set in T'. Since $a^*(T') = a(T') + 1$ and $\Delta_{S^{*\prime}}(T') = i'$, it follows that $S^{*\prime}$ has exactly one more vertex of degree i' than S' does. Hence $\overline{\delta}_{S'}(T') = i'$, implying that $\Delta_{S'}(T') \le i'$.

It remains to show that $\gamma_2(T') = a^*(T')$. Let D be a γ_2-set in T'. Since the set D contains all leaves of T', we have $L_{T'} \subset D$. Note that t is not a leaf in T' and $N_{T'}(t) = L_{T'} \cup \{v\}$. If $t \in D$, then we can replace t in D by v. Hence we may assume $t \notin D$. Therefore the set $D \setminus L_{T'}$ is a 2-dominating set in T, so $\gamma_2(T) \le |D| - |L_{T'}| = \gamma_2(T') - \ell$. Conversely, adding the set $L_{T'}$ to an arbitrary γ_2-set of T produces a 2-dominating set of T', and so $\gamma_2(T') \le \gamma_2(T) + \ell$. Consequently, $\gamma_2(T') = \gamma_2(T) + \ell$. Since T satisfies Property (e), we have $\gamma_2(T) = a^*(T)$. As established earlier, $a^*(T') = a^*(T) + \ell$. Therefore $\gamma_2(T') = \gamma_2(T) + \ell = a^*(T) + \ell = a^*(T')$. Hence in Case 2, Properties (a)–(e) hold for the tree T'. This completes the proof of Lemma 11. $\Box$

5 Proof of Theorem 8

In this section, we present a proof of Theorem 8. Recall the statement of the theorem. Since our proof by induction involves constructing a tree T' from a tree $T \in \mathcal{T}$ we state Theorem 8 in terms of T'.

Theorem 8. *For a tree T', the following hold.*
(a) $\gamma_2(T') \le a^*(T')$.
(b) $\gamma_2(T') \le a(T') + 1$.
(c) $\gamma_2(T') = a(T') + 1$ *if and only if* $T' \in \mathcal{T}$.

Proof. We proceed by induction on n, the number of vertices of the tree T'. When $n = 1$, we have $\gamma_2(T') = 1 = a^*(T') = a(T')$ and $T' \notin \mathcal{T}$. When $n = 2$, we have $\gamma_2(T') = 2 = a^*(T') = a(T') + 1$ and $T' \in \mathcal{T}$. This establishes the base cases. For the induction hypothesis, consider $n \ge 3$ and assume that every tree T with less than n vertices satisfies Properties (a), (b) and (c) in the statement of the theorem. Let T' be a tree with n vertices.

If T' is a star $K_{1,s}$ with $s \ge 2$, then $\gamma_2(T') = s = a^*(T') = a(T')$ and $T' \notin \mathcal{T}$. Hence we may assume that T' is not a star, for otherwise the tree T' satisfies the desired properties (a), (b) and (c) and we are done. Hence no vertex dominates all other vertices in the graph. Let r be any vertex in T' and call r the *root* of T'. We now consider the tree rooted at r. (We note that r is not necessarily a leaf in T'.) Let x be a vertex in T' at maximum distance from r. Since T' is not a star, x is not a neighbor of r. Thus, $x \notin N_{T'}(r)$. By our choice of x, the vertex x is a leaf in T'. Let y be the unique neighbor of x. Since no vertex dominates all other vertices in T', we have $y \neq r$ and $d_{T'}(y) \ge 2$. Let z be the vertex adjacent to y on the unique path from r to y in T'; z is the parent of y in T' when T' is rooted at r. Since x is a vertex at maximum distance from r, every neighbor of y other than z must be a leaf. We now consider the following two cases, which exhaust all possibilities.

Case 1: $d_{T'}(y) \ge 3$. Let Q denote the set of all leaves adjacent to y, and so $Q = N(y)\backslash\{z\}$. Let $T = T' - (Q \cup \{y\})$ and note that $|E(T')| = |E(T)| + |Q| + 1$. Let S be a $a^*_{\rm min}$-set in T, and so $\Sigma(S,T) \le |E(T)| + 1$. Letting $S_1 = S \cup Q$, we have

$$\begin{array}{rcl}\Sigma(S_1,T') & \le & (\Sigma(S,T)+1)+|Q| \\ & \le & |E(T)|+1+|Q|+1 \\ & = & |E(T')|+1,\end{array}$$

so $a^*(T') \ge |S_1| = |S| + |Q| = a^*(T) + |Q|$. Every 2-dominating set in T can be extended to a 2-dominating set in T' by combining it with Q, so $\gamma_2(T') \le \gamma_2(T) + |Q|$. Since T satisfies Properties (a), (b) and (c), we now have

$$\begin{array}{rcl}\gamma_2(T') & \le & \gamma_2(T)+|Q| \\ & \le & a^*(T)+|Q| \\ & \le & (a^*(T')-|Q|)+|Q| \\ & = & a^*(T').\end{array}$$

Hence, T' satisfies Property (a). As a consequence of Observation 10, $a^*(T') \leq a(T')+1$, and so T' satisfies Property (b). It remains to show that T' satisfies Property (c). By Lemma 11, if $T' \in \mathcal{T}$ (and still $n \geq 3$), then $\gamma_2(T') = a(T')+1$, as desired. Hence it suffices to show that if $\gamma_2(T') = a(T')+1$, then $T' \in \mathcal{T}$. Suppose then that $\gamma_2(T') = a(T')+1$.

Let A be an $a_{\min}$-set in T and let $A_1 = A \cup Q$. We have

$$\begin{array}{rcl} \Sigma(A_1, T') & \leq & (\Sigma(A,T)+1)+|Q| \\ & \leq & |E(T)|+|Q|+1 \\ & = & |E(T')|, \end{array}$$

so $a(T') \geq |A_1| = |A|+|Q| = a(T)+|Q|$. If $\gamma_2(T) \leq a(T)$, then $\gamma_2(T') \leq \gamma_2(T)+|Q| \leq a(T)+|Q| \leq a(T')$, a contradiction to our assumption of $\gamma_2(T') = a(T')+1$. Therefore, $\gamma_2(T) = a(T)+1$. Since T satisfies Property (c) by the inductive hypothesis, we have $T \in \mathcal{T}$. If $T = K_2$, then $T' = T_{2,d(y)-1} \in \mathcal{T}$. Hence we may assume $T \neq K_2$, for otherwise the desired result follows. Thus, $T \in \mathcal{T}_{i,j}$ for some i and j, where $2 \leq i \leq j$.

Suppose $d_T(z) > \min\{i, j-1\}$. Recall that S is an $a^*_{\min}$-set in T. We show that we may assume $z \notin S$. Since $T \in \mathcal{T}_{i,j}$, the tree T satisfies Lemma 11. In particular, every vertex in S has degree at most i in T. Hence if $\min\{i, j-1\} = i$, then $d_T(z) > i$, implying that $z \notin S$, as desired. If $\min\{i, j-1\} < i$, then $i = j$ and $d_T(z) > \min\{i, j-1\} = i-1$, implying $d_T(z) \geq i$. In this case, if $z \in S$, then $d_T(z) = i$. By Parts (a) and (b) in Lemma 11, we have $\overline{\delta}_S(T') = i$ and $\Delta_S(T') = i$. Therefore some vertex z' of degree i in T' does not belong to the set S. Replacing z in S with z' produces a new $a^*_{\min}$-set in T that does not contain z. Hence we may assume $z \notin S$. Letting $S^* = S \cup Q$, we have

$$\begin{array}{rcl} \Sigma(S^*, T') & \leq & \Sigma(S,T)+|Q| \\ & \leq & (|E(T)|+1)+|Q| \\ & = & |E(T')|, \end{array}$$

implying $a(T') \geq |S^*| = |S|+|Q| = a^*(T)+|Q| = a(T)+1+|Q|$. Now $\gamma_2(T') \leq \gamma_2(T)+|Q| = a(T)+1+|Q| \leq a(T')$, a contradiction. Hence, $d_T(z) \leq \min\{i, j-1\}$.

Let $i' = \max\{d_T(z)+1, i\}$, and suppose $d_{T'}(y) < i'$. If $i' = d_T(z)+1$, then $d_{T'}(y) \leq d_T(z)$, while if $i' = i$, then $d_{T'}(y) < i$. Hence, $d_{T'}(y) \leq d_T(z)$ or $d_{T'}(y) < i$.

Suppose $d_{T'}(y) \leq d_T(z)$. If $z \notin S$, then since $d_T(z) \leq i$ and $\Delta_S(T) = i$, we can simply replace a vertex in S of degree i in T with z. Hence in this case we may choose S to contain z. Letting $S^* = (S \cup Q \cup \{y\}) \setminus \{z\}$, we have

$$\begin{array}{rcl} \Sigma(S^*, T') & \leq & \Sigma(S \setminus \{z\}, T)+d_{T'}(y)+|Q| \\ & = & \Sigma(S,T)-d_T(z)+d_{T'}(y)+|Q| \\ & \leq & \Sigma(S,T)+|Q| \\ & = & (|E(T)|+1)+|Q| \\ & = & |E(T')|, \end{array}$$

implying $a(T') \geq |S^*| = |S|+|Q| = a^*(T)+|Q| = a(T)+1+|Q|$. Now then $\gamma_2(T') \leq \gamma_2(T)+|Q| = a(T)+1+|Q| \leq a(T')$, a contradiction. Hence, $d_{T'}(y) > d_T(z)$, and so

$d_{T'}(y) < i$. In this case, let z' be a vertex in S of degree i and note that $d_{T'}(y) \leq d_T(z') - 1$. Therefore letting $S^* = (S \cup Q \cup \{y\}) \setminus \{z'\}$, we have

$$\begin{aligned}
\Sigma(S^*, T') &\leq \Sigma(S \setminus \{z'\}, T) + d_{T'}(y) + |Q| \\
&= (\Sigma(S,T) + 1 - d_T(z')) + d_{T'}(y) + |Q| \\
&\leq \Sigma(S,T) + |Q| \\
&= (|E(T)| + 1) + |Q| \\
&= |E(T')|,
\end{aligned}$$

implying as before that $\gamma_2(T') \leq a(T')$, a contradiction. Therefore, $d_{T'}(y) \geq i'$.

Recall that $i' = \max\{d_T(z) + 1, i\}$. We now let $j' = \min\{d_{T'}(y) - 1, j\}$. We have shown that $d_T(z) \leq \min\{i, j-1\}$ and $d_{T'}(y) \geq i'$. Thus, by Definition 2, the tree T' is obtained from the tree T by applying Operation $\mathcal{O}_2$. Further, $T' \in \mathcal{T}_{i',j'} \subseteq \mathcal{T}$, as desired. Thus the tree T' satisfies Property (c). This completes the proof of Case 1.

Case 2: $d_{T'}(y) = 2$. Let $Y = \{y_1, y_2, \ldots, y_\ell\}$ be the children of z in T' when T' is rooted at r. Renaming vertices, if necessary, we may assume that $y = y_1$. By Case 1, we may assume $1 \leq d(y_i) \leq 2$ for all $y_i \in Y$, for otherwise $T' \in \mathcal{T}$, as desired. We now consider the following two subcases.

Subcase 2a: There exists a vertex in $Y \setminus \{y_1\}$ *of degree* 2 *in* T'. Renaming vertices, if necessary, we may assume that $d_{T'}(y_2) = 2$. Let x_2 be the neighbor of y_2 different from z. In this case, we consider the tree $T = T' - \{x, y\}$. Let S be an $a^*_{\min}$-set in T, and so $\Sigma(S,T) \leq |E(T)| + 1$. Assume that $d_{T'}(z) = 2$. This implies that $r = z$, $T' = P_5$ is a path on five vertices and $\gamma_2(T') = 3 = a^*(T') = a(T')$. Thus $T' \notin \mathcal{T}$. Hence we may assume that $r \neq z$ and consequently $d_{T'}(z) \geq 3$ for otherwise the desired result follows. If $z \in S$, then let $S_2 = (S \cup \{x, y\}) \setminus \{z\}$ and if $z \notin S$, then let $S_2 = S \cup \{x\}$. In both cases, $\Sigma(S_2, T') \leq \Sigma(S,T) + 1$. Thus since $|E(T')| = |E(T)| + 2$, we have

$$\begin{aligned}
\Sigma(S_2, T') &\leq \Sigma(S,T) + 1 \\
&\leq (|E(T)| + 1) + 1 \\
&= |E(T')|,
\end{aligned}$$

implying that $a(T') \geq |S_2| = |S| + 1 = a^*(T) + 1$. Let D be a γ_2-set in T. If $y_2 \in D$, then we can simply replace y_2 with z. Hence we can choose the set D to contain z. But then $D \cup \{x\}$ is a 2-dominating set of T', and so $\gamma_2(T') \leq \gamma_2(T) + 1$. By Lemma 11(e), we have $\gamma_2(T) = a^*(T)$, implying that $\gamma_2(T') \leq a^*(T) + 1 \leq a(T') \leq a^*(T')$. Hence, T' satisfies Properties (a), (b) and (c), which completes the proof of Subcase 2a.

Subcase 2b: Every vertex in $Y \setminus \{y_1\}$ *is a leaf in* T'. In this case, we consider the tree $T = T' - (Y \cup \{x\})$. Let S be an $a^*_{\min}$-set in T, and so $\Sigma(S,T) \leq |E(T)| + 1$. Since $d_T(z) \leq 1$, we can choose S so that $z \in S$. Let $S_3 = (S \cup Y \cup \{x\}) \setminus \{z\}$. Since $|E(T')| = |E(T)| + \ell + 1$, we have

$$\begin{aligned}
\Sigma(S_3,T') &= \Sigma(S\setminus\{z\},T)+\ell+2\\
&= \Sigma(S,T)+\ell+1\\
&\le (|E(T)|+1)+\ell+1\\
&= |E(T')|+1,
\end{aligned}$$

implying that $a^*(T')\ge|S_3|=|S|+\ell=a^*(T)+\ell$. Let D be a γ_2-set in T. Since $d_T(z)\le 1$, we have $z\in D$ and therefore $(D\cup(Y\cup\{x\}))\setminus\{y\}$ is a 2-dominating set of T', implying that $\gamma_2(T')\le|D|+\ell=\gamma_2(T)+\ell$. By the inductive hypothesis, $\gamma_2(T)\le a^*(T)$. By Observation 10, $a^*(T')\le a(T')+1$. Hence,

$$\begin{aligned}
\gamma_2(T') &\le \gamma_2(T)+\ell\\
&\le a^*(T)+\ell\\
&\le a^*(T')\\
&\le a(T')+1.
\end{aligned}$$

Hence, T' satisfies Property (a) and Property (b). It remains to show that T' satisfies Property (c). By Lemma 11, if $T'\in\mathcal{T}$ (and still $n\ge 3$), then $\gamma_2(T')=a(T')+1$, as desired. Hence it suffices to show that if $\gamma_2(T')=a(T')+1$, then $T'\in\mathcal{T}$. Suppose that $\gamma_2(T')=a(T')+1$. In this case, we must have equality throughout the previous inequality chain. In particular, $\gamma_2(T')=\gamma_2(T)+\ell$, $\gamma_2(T)=a^*(T)$, $a^*(T)+\ell=a^*(T')$ and $a^*(T')=a(T')+1$.

If $T=K_1$, then $T'=T_{2,\ell}\in\mathcal{T}_{2,\ell}\subseteq\mathcal{T}$. Hence we may assume that $T\ne K_1$, for otherwise $T'\in\mathcal{T}$ as desired. Let A be an $a_{\min}$-set in T. Since $d_T(z)\le 1$, we can choose the set A so that $z\in A$. Let $S_3'=(A\cup Y\cup\{x\})\setminus\{z\}$. Since $|E(T')|=|E(T)|+\ell+1$, we have

$$\begin{aligned}
\Sigma(S_3',T') &= \Sigma(A\setminus\{z\},T)+\ell+2\\
&= \Sigma(A,T)+\ell+1\\
&\le |E(T)|+\ell+1\\
&= |E(T')|,
\end{aligned}$$

implying that $a(T')\ge|S_3'|=|A|+\ell=a(T)+\ell$. If $\gamma_2(T)\le a(T)$, then $\gamma_2(T')=\gamma_2(T)+\ell\le a(T)+\ell\le a(T')$, a contradiction to the assumption that $\gamma_2(T')=a(T')+1$. Therefore, $\gamma_2(T)=a(T)+1$. By the inductive hypothesis, $T\in\mathcal{T}$. If $T=K_2$, then $T'=T_{2,\ell+1}\in\mathcal{T}_{2,\ell+1}\subset\mathcal{T}$. Hence we may assume $T\ne K_2$, for otherwise the desired result follows. Thus, $T\in\mathcal{T}_{i,j}$ for some i and j, where $2\le i\le j$.

Suppose that $\ell\le i-2$. Since $T\in\mathcal{T}_{i,j}$, by Lemma 11 there is a vertex $w\in S$ such that $d_T(w)=i$. Since $i\ge 2$ and $d_T(z)=1$, we note $w\ne z$. If $S^*=(S\cup Y\cup\{x\})\setminus\{w\}$, then

$$\begin{aligned}
\Sigma(S^*,T') &= \Sigma(S\setminus\{w,z\},T)+d_{T'}(z)+\ell+2\\
&= (\Sigma(S,T)-d_T(w)-d_T(z))+(\ell+1)+\ell+2\\
&= (\Sigma(S,T)-i-1)+2\ell+3\\
&\le (|E(T)|+1)+\ell+(\ell+2-i)\\
&\le |E(T)|+\ell+1\\
&= |E(T')|,
\end{aligned}$$

implying that $a(T') \geq |S^*| = |S| + \ell = a^*(T) + \ell$. However as observed earlier, $a^*(T) + \ell = a^*(T')$ and $a^*(T') = a(T') + 1$, implying that $a(T') \geq a^*(T) + \ell = a^*(T') = a(T') + 1$, which is impossible. Therefore $\ell \geq i - 1$. By Definition 2, the tree T' can be obtained from the tree T by applying Operation $\mathcal{O}_1$. Further, $T' \in \mathcal{T}_{i,\min\{j,\ell+1\}} \subseteq \mathcal{T}$, as desired. □

6 Proof of Theorem 9

In this section, we present a proof of Theorem 9. Recall the statement of the theorem.

Theorem 9. *If T is a tree with n vertices and n_1 leaves, then $\gamma_2(T) \leq (n + n_1)/2$, with equality if and only if $T \in \mathcal{T}_2 \cup \{K_2\}$.*

Proof. Let T be a tree with n vertices and n_1 leaves. By Theorem 2, we have $\gamma_2(T) \leq (n+n_1)/2$. Hence it remains to prove that $\gamma_2(T) = (n+n_1)/2$ if and only if $T \in \mathcal{T}_2 \cup \{K_2\}$.

First suppose $T \in \mathcal{T}_2 \cup \{K_2\}$. If $T = K_2$, then $\gamma_2(T) = 2 = (n+n_1)/2$, as desired. Hence we may assume that $T \in \mathcal{T}_2$. Thus, $T \in \mathcal{T}_{2,j}$ for some integer $j \geq 2$. Let S^* be an $a^*_{\rm min}$-set in T. By Lemma 11, we have $\Sigma(S^*, T) = |E(T)| + 1 = n$. Furthermore, every vertex in S^* has degree 1 or degree 2 in T, and some vertex u in S^* has degree exactly 2 in T. If some leaf v is not in S^*, then letting $S = (S^* \setminus \{u\}) \cup \{v\}$ yields $|S| = |S^*|$ and $\Sigma(S,T) = \Sigma(S^*,T) - 1$, contradicting the minimality of the set S^*. Hence S^* contains all leaves of T, while every vertex in S^* that is not a leaf of T has degree 2 in T. Let x_2 be the number of vertices in S^* that are not leaves in T. We have, $x_2 \geq 1$ and $n = \Sigma(S^*,T) = n_1 + 2x_2$, implying that $a^*(T) = n_1 + x_2 = n_1 + (n-n_1)/2 = (n+n_1)/2$. By Lemma 11(e), we have $\gamma_2(T) = a^*(T)$, so $\gamma_2(T) = (n+n_1)/2$, as desired.

Conversely, suppose $\gamma_2(T) = (n+n_1)/2$. Let S^* be an $a^*_{\rm min}$-set in T and let $\Delta = \Delta(T)$. For $i = 1, 2, \ldots, \Delta$, let x_i denote the number of vertices in S^* that have degree i in T. We have,

$$\sum_{i=1}^{\Delta} ix_i = \Sigma(S^*, T) \leq |E(T)| + 1 = n, \qquad \text{and so} \qquad \sum_{i=2}^{\Delta} \frac{ix_i}{2} \leq \frac{n - x_1}{2}.$$

Therefore,

$$a^*(T) = \sum_{i=1}^{\Delta} x_i \leq x_1 + \left(\sum_{i=2}^{\Delta} \frac{ix_i}{2} \right) \leq x_1 + \left(\frac{n - x_1}{2} \right) = \frac{n + x_1}{2} \leq \frac{n + n_1}{2}. \tag{1}$$

Hence by Theorem 8 and our supposition that $\gamma_2(T) = (n+n_1)/2$, we have

$$\frac{n+n_1}{2} = \gamma_2(T) \leq a^*(T) \leq \frac{n+n_1}{2}. \tag{2}$$

Therefore we must have equality throughout (2), and hence throughout (1). Equality throughout (2) implies that $\gamma_2(T) = a^*(T)$. Since $1 < i/2$ for $i \geq 3$, equality throughout (1) implies that $x_i = 0$ for all i with $3 \leq i \leq \Delta$. Further, $x_1 = n_1$ and $\Sigma(S^*, T) = |E(T)|+1 = n$. If $a^*(T) = n_1$, then every vertex in S^* is a leaf, so $\Sigma(S^*, T) = n_1$ and hence $n = n_1$ and $T = K_2$. Hence we may assume that at least one vertex in S^* is not a leaf, for otherwise the desired result follows. Thus, $|S^*| = x_1 + x_2$ and $x_2 > 0$; that is, S^* contains at least one vertex of degree 2. Since S^* is a set of vertices corresponding to the first $a^*(T)$ vertices in the nondecreasing degree sequence of T and $\Sigma(S^*, T) = |E(T)| + 1$, we have $a(T) = a^*(T) - 1$. Hence, $\gamma_2(T) = a^*(T) = a(T) + 1$, and so, by Theorem 8, $T \in \mathcal{T}$. Since $x_2 > 0$, we have $T \neq K_2$, and so $T \in \mathcal{T}_{i,j}$ for some integers i and j with $2 \leq i \leq j$. However, since $\Delta_{S^*}(T) = 2$, Lemma 11 implies that $T \in \mathcal{T}_{2,j}$ for some integer j with $j \geq 2$, and so $T \in \mathcal{T}_2$, as desired. $\square$

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