On well-dominated direct, Cartesian and strong product graphs

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Abstract

If each minimal dominating set in a graph is a minimum dominating set, then the graph is called well-dominated. Since the seminal paper on well-dominated graphs appeared in 1988, the structure of well-dominated graphs from several restricted classes have been studied. In this paper we give a complete characterization of nontrivial direct products that are well-dominated. We prove that if a strong product is well-dominated, then both of its factors are well-dominated. When one of the factors of a strong product is a complete graph, the other factor being well-dominated is also a sufficient condition for the product to be well-dominated. Our main result gives a complete characterization of well-dominated Cartesian products in which at least one of the factors is a complete graph. In addition, we conjecture that this result is actually a complete characterization of the class of nontrivial, well-dominated Cartesian products.

Keywords: well-dominated, Cartesian product, direct product, strong product
AMS subject classification: 05C69, 05C76

1 Introduction

A dominating set of a finite graph $G$ is a set $D$ of vertices such that each vertex of $G$ is within distance 1 of at least one vertex in $D$. Finding a (set inclusion) minimal dominating set is straightforward and can be accomplished in linear time by simply ordering the vertices and then discarding them one at a time if the remaining set still
dominates the graph. Depending on the graph, the resulting minimal dominating set may be much larger than the domination number of $G$, which is the minimum cardinality among all its dominating sets. On the other hand, for a given graph $G$ and positive integer $k$, it is well known that determining whether $G$ has a dominating set of cardinality at most $k$ is an $NP$-complete problem. In most applications, finding the size of a smallest dominating set is the typical goal. Well-dominated graphs are those for which the above algorithm always finds a dominating set of minimum cardinality. The seminal paper on well-dominated graphs was by Finbow, Hartnell and Nowakowski [3]. They characterized the well-dominated graphs of girth at least 5 and showed that the only well-dominated bipartite graphs are those with domination number one-half their order. Several other groups of authors have studied the concept of well-dominated graphs within restricted graph classes. See, for example, [12] (block and unicyclic graphs), [11] (simplicial and chordal graphs), [5] (4-connected, 4-regular claw-free), and [4] (planar triangulations). In [6] Gőzüpek, Hujdurovic and Milanic characterized the well-dominated graphs that are nontrivial lexicographic products.

Our focus in this paper is the class of well-dominated graphs that have a nontrivial factorization as a Cartesian, direct or strong product. These three graph products are referred to as the “fundamental products” in the book [7] by Hammack, Imrich and Klavzar. Along with the lexicographic product, they are the most studied graph products in the literature.

Anderson, Kuenzel and Rall [1] characterized the direct products that are well-dominated under the assumption that at least one of the factors has no isolatable vertices. (A vertex $x$ in a graph $X$ is isolatable if there is an independent set $A$ in $X$ such that $\{x\}$ is a component in $X - N[A]$.)

**Theorem 1.** [1, Theorem 3] Let $G$ and $H$ be nontrivial connected graphs such that at least one of $G$ or $H$ has no isolatable vertices. The direct product $G \times H$ is well-dominated if and only if $G = H = K_3$ or at least one of the factors is $K_2$ and the other factor is a 4-cycle or the corona of a connected graph.

In this paper we complete the characterization of well-dominated direct products by removing the requirement on isolatable vertices. In fact, we prove that if both factors of a direct product have order at least 3 and one of them has an isolatable vertex, then the direct product is not well-dominated. The main result on well-dominated direct products is then the following theorem.

**Theorem 2.** Let $G$ and $H$ be connected graphs. The direct product $G \times H$ is well-dominated if and only if $G \times H = K_3 \times K_3$, $G \times H = K_2 \times C_4$, or $G \times H = K_2 \times (F \odot K_1)$ for a connected graph $F$.

In the same paper Anderson et al. proved that if a Cartesian product $G \Box H$ is well-dominated, then at least one of $G$ or $H$ is well-dominated. In addition, they
provided a characterization of the well-dominated Cartesian products of triangle-free graphs. Namely, they proved that the Cartesian product of two connected, triangle-free graphs of order at least 2 is well-dominated if and only if both factors are complete graphs of order 2. Here we explore the more general case of well-dominated Cartesian products in which (at least) one of the factors has girth 3. In particular, we prove the following characterization of well-dominated graphs of the form $K^m \Box H$, and we conjecture that every well-dominated Cartesian product of nontrivial connected graphs is isomorphic to one of these.

**Theorem 3.** Let $m$ be a positive integer with $m \geq 2$ and let $H$ be a nontrivial, connected graph. The Cartesian product $K^m \Box H$ is well-dominated if and only if either $m \neq 3$ and $H = K^m$ or $m = 3$ and $H \in \{K_3, P_3\}$.

For the strong product we prove that both factors of a well-dominated strong product are well-dominated. If one of the factors of a strong product is a complete graph, then the other factor being well-dominated is also a sufficient condition for the product to be well-dominated.

**Theorem 4.** Let $n$ be a positive integer. For any graph $H$ the strong product $K_n \boxtimes H$ is well-dominated if and only if $H$ is well-dominated.

The remainder of the paper is organized in the following way. In the next section we provide the necessary definitions for the remainder of the paper. In Section 3 we settle the relatively straightforward result for strong products. Theorem 2, the complete characterization of well-dominated direct products, is verified in Section 4. Proving Theorem 3 is the main task of Section 5. We also derive a number of necessary conditions on two connected graphs whose Cartesian product is well-dominated, which leads us to conjecture that the characterization in Theorem 3 captures all connected well-dominated Cartesian products.

## 2 Definitions

All graphs in this paper are finite, undirected, simple and have order at least 2. For a positive integer $n$, we let $[n] = \{1, \ldots, n\}$. This set will be the vertex set of the complete graph of order $n$. In general, we follow the terminology and notation of Hammack, Imrich, and Klavžar [7]. The order of a graph $G$ is the number of vertices in $G$ and is denoted by $n(G)$; $G$ is nontrivial if $n(G) \geq 2$. For a vertex $v$ in a graph $G$, the open neighborhood $N(v)$ and the closed neighborhood $N[v]$ are defined by $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. For $A \subseteq V(G)$ we let $N(A) = \bigcup_{v \in A} N(v)$ and $N[A] = N(A) \cup A$. The subgraph of $G$ induced by $A$ is denoted by $\langle A \rangle$. Any vertex subset $D$ such that $N[D] = V(G)$ is a dominating set of
$G$, and $D$ is then a minimal dominating set if no proper subset of $D$ is a dominating set. The domination number of $G$ is denoted by $\gamma(G)$ and is the minimum cardinality among the dominating sets of $G$. The upper domination number, denoted by $\Gamma(G)$, is the largest cardinality of a minimal dominating set of $G$. A set $M \subseteq V(G)$ is an independent set if its vertices are pairwise non-adjacent. An independent set is maximal if it is not a proper subset of an independent set. The cardinalities of a smallest and a largest maximal independent set in $G$ are denoted by $i(G)$ and $\alpha(G)$, respectively. Note that a maximal independent set is a dominating set, which gives

$$\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G).$$

(1)

A graph $G$ is called well-covered if $i(G) = \alpha(G)$ and is well-dominated if $\gamma(G) = \Gamma(G)$. It is clear from (1) that the class of well-dominated graphs is a subclass of the class of well-covered graphs.

Let $G$ be a graph and let $u \in A \subseteq V(G)$. The private neighborhood of $u$ with respect to $A$ is the set $\text{pn}[u, A]$ defined by $\text{pn}[u, A] = \{x \in V(G) : N[x] \cap A = \{u\}\}$. Equivalently, $\text{pn}[u, A] = N[u] - N[A - \{u\}]$. The vertices in $\text{pn}[u, A]$ are called private neighbors of $u$ with respect to $A$. The subset $A \subseteq V(G)$ is irredundant if $\text{pn}[u, A] \neq \emptyset$ for every $u \in A$. It follows from the definitions that a dominating set $D$ of $G$ is a minimal dominating set if and only if every vertex in $D$ has a private neighbor with respect to $D$; that is, $D$ is irredundant. In this paper we will also need a more restricted type of private neighbor. The external private neighborhood of $u$ with respect to $A$ is the set $\text{epn}[u, A]$ defined by $\text{epn}[u, A] = \text{pn}[u, A] - \{u\} = N(u) - N[A - \{u\}]$. If $v \in \text{epn}[u, A]$, then $v$ is called an external private neighbor of $u$ with respect to $A$. Such a vertex $v$, if it exists, belongs to $V(G) - A$, which is the reason to use the word external. A property related to irredundance, and one that is important for this paper, is that of being open irredundant. The set $A$ is open irredundant if every vertex of $A$ has an external private neighbor with respect to $A$.

A vertex $x$ in a graph $G$ is an isolatable vertex in $G$ if there exists an independent set $I$ of vertices in $G$ such that $x$ is isolated in the induced subgraph $G - N[I]$ of $G$. Concerning the closed neighborhood of an independent set we have the following useful fact about well-dominated graphs first observed by Finbow, Hartnell and Nowakowski. The proof is straightforward and follows from the fact that if $M$ is an independent set in $G$ and $D$ is a minimal dominating set of $G - N[M]$, then $D \cup M$ is a minimal dominating set of $G$.

**Observation 5.** [3] If $G$ is a well-dominated graph and $M$ is any independent set of vertices in $G$, then $G - N[M]$ is well-dominated.

Let $G$ and $H$ be finite, undirected graphs. The Cartesian product of $G$ and $H$, denoted by $G \Box H$, has as its vertex set the Cartesian (set) product $V(G) \times V(H)$. The direct product, denoted $G \times H$, and the strong product, $G \boxtimes H$, also have $V(G) \times V(H)$ as their set of vertices. Distinct vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent in
• $G \square H$ if either $(g_1 = g_2$ and $h_1 h_2 \in E(H))$ or $(h_1 = h_2$ and $g_1 g_2 \in E(G))$;

• $G \times H$ if $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$;

• $G \boxtimes H$ if they are adjacent in $G \square H$ or they are adjacent in $G \times H$.

All three of these graph products are associative and commutative. A product graph is called nontrivial if both of its factors are nontrivial. See [7] for specific information on these and other graph products. The corona of a graph $G$, denoted by $G \odot K_1$, is the graph of order $2n(G)$ obtained by adding, for each vertex $u$ of $G$ a new vertex $u'$ together with a new edge $uu'$.

3 Well-dominated strong products

Recall that two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent in the strong product $G \boxtimes H$ if one of the following holds.

• $g_1 = g_2$ and $h_1 h_2 \in E(H)$

• $h_1 = h_2$ and $g_1 g_2 \in E(G)$

• $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$

Nowakowski and Rall [10] established the following relationships between ordinary domination invariants on strong products.

**Proposition 6.** [10, Corollary 2.2] If $G$ and $H$ are finite graphs, then

$$\gamma(G \boxtimes H) \leq \gamma(G)\gamma(H) \text{ and } \Gamma(G \boxtimes H) \geq \Gamma(G)\Gamma(H).$$

The following corollary follows immediately from Proposition 6.

**Corollary 7.** Let $G$ and $H$ be finite graphs. If $G \boxtimes H$ is well-dominated, then $G$ and $H$ are well-dominated.

The converse of Corollary 7 is not true in general. For example, the 5-cycle is well-dominated, but

$$\gamma(C_5 \boxtimes C_5) = 4 < 6 = \Gamma(C_5 \boxtimes C_5).$$

However, we are able to show in this paper that if at least one of the factors is a complete graph, then the strong product of this complete graph and a well-dominated graph is well-dominated.
Let $D$ be a dominating set of $G \boxtimes H$ and let $A = \{ g \in V(G) : (g, h) \in D \text{ for some } h \in V(H) \}$. If $u \in V(G) - A$ and $v \in V(H)$, then from the definition of the edge structure of the strong product it follows that $(u, v)$ is dominated by $D$ only if $u$ is dominated by $A$. Thus, $A$ dominates $G$ and therefore $\gamma(G) \leq |A| \leq |D|$. Interchanging the roles of $G$ and $H$ proves the following lemma.

**Lemma 8.** For all pairs of graphs $G$ and $H$, we have $\gamma(G \boxtimes H) \geq \max\{\gamma(G), \gamma(H)\}$.

We now proceed to prove Theorem 4, which is restated here.

**Theorem 4** Let $n$ be a positive integer. For any graph $H$ the strong product $K_n \boxtimes H$ is well-dominated if and only if $H$ is well-dominated.

**Proof.** Suppose that $H$ is a well-dominated graph. Let $D$ be any minimal dominating set of $K_n \boxtimes H$ and let $S = \{ h \in V(H) : (i, h) \in D \text{ for some } i \in [n] \}$. Note that for any $u \in V(H)$ and for $1 \leq i < j \leq n$, we have $N[(i, u)] = N[(j, u)]$. Since $D$ is an irredundant set, we infer that $|S| = |D|$. We claim that $S$ is a minimal dominating set of $H$. As in the paragraph preceding Lemma 8 we see that $S$ dominates $H$. Let $h$ be any vertex of $S$ and let $i \in [n]$ such that $(i, h) \in D$. If $(i, h)$ is isolated in the subgraph of $K_n \boxtimes H$ induced by $D$, then $h$ is isolated in the subgraph of $H$ induced by $S$ and $h \in \text{pn}[h, S]$. On the other hand, if $(i, h) \notin \text{pn}[(i, h), D]$, then there exists $(k, x) \in \text{pn}[(i, h), D]$ such that $x \notin S$. Again by definition of the edge structure of the strong product it follows that $x \in \text{pn}[h, S]$. We conclude that $S$ is irredundant in $H$, and hence $S$ is a minimal dominating set of $H$. Therefore, $|D| = |S| = \gamma(H)$. That is, all minimal dominating sets of $K_n \boxtimes H$ have the same cardinality, which implies that $K_n \boxtimes H$ is well-dominated.

The converse follows from Corollary 7. \hfill \Box

Still unanswered is the following natural question.

**Question 1.** What properties on the well-dominated graphs $G$ and $H$ are necessary and sufficient for $G \boxtimes H$ to be well-dominated?

### 4 Well-dominated direct products

In this section we complete the characterization of well-dominated direct products. Throughout we assume that the factors are connected and have order at least 2. We prove Theorem 2, which we now restate for convenience of the reader.

**Theorem 2** Let $G$ and $H$ be connected graphs. The direct product $G \times H$ is well-dominated if and only if $G \times H = K_3 \times K_3$, $G \times H = K_2 \times C_4$, or $G \times H = K_2 \times (F \circ K_1)$ for a connected graph $F$. 

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It is easy to see that if $M$ is any maximal independent set of $G$ then $M \times V(H)$ is a maximal independent set, and thus also a minimal dominating set, of $G \times H$. If, in addition, $G \times H$ is well-dominated, then it follows that $M \times V(H)$ is a minimum dominating set of $G \times H$ and so $\gamma(G \times H) = \alpha(G)n(H)$.

**Lemma 9.** Let $G$ and $H$ be connected graphs of order at least 3. If $G \times H$ is well-dominated, then $\gamma(G \times H) = \gamma(G)n(H) = \gamma(H)n(G)$. Furthermore, $\gamma(G) = \alpha(G)$ and $\gamma(H) = \alpha(H)$.

**Proof.** Let $D$ be a minimum dominating set of $G$. It is easy to see that $D \times V(H)$ dominates $G \times H$. We get

$$\gamma(G)n(H) = |D \times V(H)| \geq \gamma(G \times H) = \alpha(G)n(H) \geq \gamma(G)n(H),$$

and we have equality throughout. Therefore, $\gamma(G \times H) = \gamma(G)n(H)$, and $\gamma(G) = \alpha(G)$. By reversing the roles of $G$ and $H$ we also have $\gamma(G \times H) = \gamma(H)n(G)$ and $\gamma(H) = \alpha(H)$. \hfill \Box

As we have observed several times, $D \times V(H)$ dominates the direct product $G \times H$ if $D$ dominates $G$. The next lemma shows that if both factors have order at least 3, then the set $D \times V(H)$ is not a minimal dominating set unless $D$ is independent in $G$.

**Lemma 10.** Let $G$ and $H$ be connected graphs of order at least 3, and let $D$ be a minimal dominating set of $G$. The set $D \times V(H)$ is a minimal dominating set of $G \times H$ if and only if $D$ is independent in $G$.

**Proof.** If $D$ is an independent dominating set of $G$, then $D \times V(H)$ is a maximal independent set, and hence a minimal dominating set, of $G \times H$. For the converse, suppose that $a$ is a vertex that is not isolated in the subgraph of $G$ induced by $D$. Let $x$ be a vertex in $H$ that is not a support vertex. We will show that $\text{pn}([(a, x), D \times V(H)]) = \emptyset$. First, since $N(a) \cap D \neq \emptyset$, we see that $(a, x)$ has a neighbor in $D \times V(H)$, which implies that $(a, x) \notin \text{pn}([(a, x), D \times V(H)])$. Next, let $(u, y) \in N((a, x)) - (D \times V(H))$. Since $x$ is not a support vertex of $H$, we see that there exists a path, say $x, y, z$, in $H$. However, this means that $(a, z)$ is adjacent to $(u, y)$, and hence $(u, y) \notin \text{pn}([(a, x), D \times V(H)])$. It follows that $\text{pn}([(a, x), D \times V(H)]) = \emptyset$, which implies that $D \times V(H)$ is not a minimal dominating set. \hfill \Box

As a result of Lemma 10 we can now state a very restrictive condition that must be satisfied by both factors of a well-dominated direct product if both have order at least 3.

**Corollary 11.** Let $G$ and $H$ be connected graphs of order at least 3. If $G \times H$ is well-dominated, then all minimum dominating sets of $G$ and all minimum dominating sets of $H$ are independent.
Proof. Suppose \( G \) and \( H \) are connected of order at least 3 such that \( G \times H \) is well-dominated. By Lemma 9, we have \( \gamma(G \times H) = \gamma(G)n(H) = \gamma(H)n(G) \). Let \( D \) be any minimum dominating set of \( G \). Since \( D \times V(H) \) dominates \( G \times H \) and \( \gamma(G \times H) = |D \times V(H)| \), it follows from Lemma 10 that \( D \) is independent. Similarly, every minimum dominating set of \( H \) is independent.

Using these results we now proceed to the proof of Theorem 2.

It was verified in [1] that \( G \times H \) is well-dominated if \( G \times H = K_3 \times K_3 \), \( G \times H = K_2 \times C_4 \), or \( G \times H = K_2 \times (F \odot K_1) \) for a connected graph \( F \). For the converse we assume that \( G \times H \) is well-dominated. Suppose for the sake of contradiction that \( G \times H \neq K_3 \times K_3 \), \( G \times H \neq K_2 \times C_4 \), and \( G \times H \neq K_2 \times (F \odot K_1) \) for a connected graph \( F \). By Theorem 1, it follows that both \( G \) and \( H \) possess an isolatable vertex, and hence \( n(G) \geq 3 \) and \( n(H) \geq 3 \). Let \( M \) be an independent set in \( G \) such that \( G - N[M] = \{x\} \). Since \( G \) is connected, there exists a vertex \( x' \in N(M) \) that is adjacent to \( x \). By Lemma 9 and the inequality chain (1), \( \gamma(G) = i(G) = \alpha(G) \). Since \( M \cup \{x\} \) is an independent dominating set of \( G \), it follows that \( \gamma(G) = |M| + 1 \). However, \( M \cup \{x'\} \) dominates \( G \) and is therefore a minimum dominating set of \( G \) that is not independent, which is a contradiction by Corollary 11.

In the first paper on well-dominated graphs, Finbow et al. proved the following characterization of the class of well-dominated bipartite graphs.

**Theorem 12.** [3, Theorem 3] Let \( G \) be a connected, bipartite graph. Then \( G \) is well-dominated if and only if \( G = C_4 \) or \( G \) is the corona of a bipartite graph.

Let \( F \) be a connected graph. It is straightforward to verify that \( (F \odot K_1) \times K_2 = (F \times K_2) \odot K_1 \). Thus the well-dominated graphs of the form \( K_2 \times (F \odot K_1) \) are a subclass of the easily recognizable well-dominated, bipartite coronas of Theorem 12.

5 Well-dominated Cartesian products

In this section we prove the following characterization of well-dominated Cartesian products in which at least one of the factors is a complete graph. In particular, we prove our main theorem in this study.

**Theorem 3** Let \( m \) be a positive integer with \( m \geq 2 \) and let \( H \) be a nontrivial, connected graph. The Cartesian product \( K_m \Box H \) is well-dominated if and only if either \( m \neq 3 \) and \( H = K_m \) or \( m = 3 \) and \( H \in \{K_3, P_3\} \).

We begin by deriving some preliminary results that will prove to be useful in its proof. After that we prove Proposition 21, which is a special case of Theorem 3 that assumes \( n(H) \leq m \).
As mentioned in the introduction, the first step in this process was obtained by Anderson, Kuenzel and Rall [1].

**Theorem 13.** [1, Theorem 1] Let $G$ and $H$ be connected graphs. If $G \Box H$ is well-dominated, then $G$ or $H$ is well-dominated.

If a graph $G$ admits a dominating set that is open irredundant, then the following result provides a method for constructing a minimal dominating set in any Cartesian product that has $G$ as one of its factors.

**Lemma 14.** If $D$ is an open irredundant dominating set of a graph $G$, then $D \times V(H)$ is an open irredundant, minimal dominating set of $G \Box H$, for any graph $H$.

**Proof.** If $(x, y) \in V(G \Box H) - (D \times V(H))$, then there exists a vertex $x' \in N(x) \cap D$ since $D$ is a dominating set of $G$. This implies that $(x, y)$ is adjacent to $(x', y)$, and thus $D \times V(H)$ is a dominating set of $G \Box H$. To see that $D \times V(H)$ is a minimal dominating set, let $(d, h)$ be an arbitrary vertex in $D \times V(H)$. Since $D$ is an open irredundant set of $G$, there exists a vertex $d' \in \text{epn}(d, D)$. By definition, $d' \in N(d) - D$, and $d'$ is not adjacent to any vertex of $D - \{d\}$. Consequently, $(d', h) \in \text{epn}[(d, h), D \times V(H)]$. Therefore, $D \times V(H)$ is an open irredundant, minimal dominating set of $G \Box H$. \qed

Bollobás and Cockayne proved that every graph with minimum degree at least 1 has an open irredundant, minimum dominating set $D$.

**Proposition 15.** [2, Proposition 6] If a graph $G$ has no isolated vertices, then $G$ has a minimum dominating set that is open irredundant.

By using Proposition 15, we now establish a relationship that must hold between two graphs if their Cartesian product is well-dominated.

**Proposition 16.** Let $G$ and $H$ be nontrivial connected graphs. If $G \Box H$ is well-dominated, then $\gamma(G \Box H) = \gamma(G) \cdot n(H) = \gamma(H) \cdot n(G)$.

**Proof.** Let $D_1$ be a minimum dominating set of $G$ that is open irredundant and let $D_2$ be a minimum dominating set of $H$ that is open irredundant. These minimum dominating sets of $G$ and $H$ exist by Proposition 15. By Lemma 14, it follows that both of $D_1 \times V(H)$ and $V(G) \times D_2$ are minimal dominating sets of $G \Box H$. Since $G \Box H$ is well-dominated, it follows that $\gamma(G \Box H) = |D_1 \times V(H)| = |V(G) \times D_2|$, and therefore

$$\gamma(G \Box H) = \gamma(G) \cdot n(H) = \gamma(H) \cdot n(G).$$

\qed

If $D$ is a minimal dominating set of a graph $G$, then, in general, some vertices of $D$ will have external private neighbors with respect to $D$ and some vertices will not.
The following notation will be useful in what follows. If $D$ is any dominating set of $G$, we define $c(D)$ as follows:

$$c(D) = \{ x \in D : \text{pn}[x, D] = \{x\} \}.$$ 

That is, a vertex $x$ in $D$ belongs to $c(D)$ if and only if $x$ is isolated in the subgraph induced by $D$, and $N(x) - D \subseteq N(D - \{x\})$. In particular, $c(D)$ is a (possibly empty) independent set of $G$. Note also that if $c(D) \neq \emptyset$, then $|D| \geq 2$ since we are assuming that $G$ is of order at least 2.

**Lemma 17.** Let $S$ be a minimal dominating set of a graph $H$. If $D$ is a minimal dominating set of a graph $G$ such that $c(D) \neq \emptyset$, then $(\{u\} \times S) \cup ((D - \{u\}) \times V(H))$ is a dominating set of $G \Box H$ for every $u \in c(D)$. If, in addition, $c(D) = \{u\}$, then $(\{u\} \times S) \cup ((D - \{u\}) \times V(H))$ is a minimal dominating set of $G \Box H$.

**Proof.** Let $u \in c(D)$. For simplification, let $A = (\{u\} \times S) \cup ((D - \{u\}) \times V(H))$. Suppose $(x, y) \in V(G \Box H) - A$. If $x \notin D$, then there exists $d \in D - \{u\}$ such that $dx \in E(G)$. This follows since $D$ dominates $G$ and $x \notin \text{pn}[u, D] = \{u\}$. Hence, $(x, y)$ has a neighbor in $(D - \{u\}) \times V(H)$. On the other hand, if $x \in D$, then $x = u$ and $y \notin S$. Since $S$ dominates $H$, it follows that $(x, y)$ has a neighbor in $\{u\} \times S$. Therefore, $A$ is a dominating set of $G \Box H$.

Now, suppose that $c(D) = \{u\}$. To show that $A$ is a minimal dominating set let $(a, b) \in A$. If $a \neq u$, then $a$ has an external private neighbor, say $a'$, with respect to $D$, and $(a', b)$ is a private neighbor of $(a, b)$ with respect to $A$. On the other hand, suppose $a = u$. This implies that $b \in S$. Since $S$ is a minimal dominating set of $H$, there exists $b' \in \text{pn}[b, S]$. It now follows that $(u, b')$ is a private neighbor of $(a, b)$ with respect to $A$ since $u$ is an isolated vertex in $G[D]$. We have shown that $\text{pn}[(a, b), A] \neq \emptyset$, and it follows that $A$ is a minimal dominating set of $G \Box H$. 

If $G$ is any finite graph, then for large enough $m$, namely for $m > \Delta(G)$, the Cartesian product $G \Box K_m$ is well-covered. (See page 1262 of [8].) This is not true in the well-dominated class as we now prove.

**Proposition 18.** If $G$ is a nontrivial connected graph and has a minimum dominating set $D$ such that $c(D) \neq \emptyset$, then $G \Box H$ is not well-dominated for every nontrivial connected graph $H$.

**Proof.** Let $G$ be a nontrivial, connected graph and suppose that $D$ is a minimum dominating set of $G$ with a vertex $u \in c(D)$. Let $H$ be a nontrivial connected graph and suppose that $S$ is any minimum dominating set of $H$. By Lemma 17, $(\{u\} \times S) \cup ((D - \{u\}) \times V(H))$ is a dominating set of $G \Box H$. While $(\{u\} \times S) \cup ((D - \{u\}) \times V(H))$ may not be a minimal dominating set, it contains one. Since $|((\{u\} \times S) \cup ((D - \{u\}) \times V(H))| \leq \gamma(H) + (\gamma(G) - 1) \cdot n(H)$

$$= \gamma(G) \cdot n(H) + (\gamma(H) - n(H)) < \gamma(G) \cdot n(H),$$
we conclude by Proposition 16 that $G \Box H$ is not well-dominated. \hfill \Box

The following corollary of Proposition 18 further limits which graphs can be a factor of a well-dominated Cartesian product.

**Corollary 19.** If $G$ is a nontrivial connected graph that is well-dominated and has an isolatable vertex, then $G \Box H$ is not well-dominated for any nontrivial connected graph $H$.

**Proof.** Suppose $G$ is a nontrivial connected, well-dominated graph and suppose $x$ is an isolatable vertex of $G$. Let $H$ be any connected graph of order at least 2. Since $G$ is well-dominated, we have $\gamma(G) = i(G) = \alpha(G) = \Gamma(G)$. Let $I$ be an independent set in $G$ such that $G - N[I] = \{x\}$. The set $J = I \cup \{x\}$ is an independent dominating set, and is therefore also a minimum dominating set of $G$. Since $G - N[I] = \{x\}$, we see that $x \in c(J)$. It follows by Proposition 18 that $G \Box H$ is not well-dominated.

The following result follows immediately from Corollary 19.

**Corollary 20.** If $G$ is a connected, well-dominated graph of order at least 3 and $G \Box H$ is well-dominated for some nontrivial connected graph $H$, then $\delta(G) \geq 2$.

Proceeding with the proof of Theorem 3 we first deal with the case where the order of the complete factor in the Cartesian product is at least as large as the order of the other factor.

**Proposition 21.** Let $m$ be a positive integer such that $m \geq 2$ and let $H$ be a nontrivial connected graph such that $n(H) \leq m$. The Cartesian product $K_m \Box H$ is well-dominated if and only if one of the following holds.

1. $m = 2$ and $H = K_2$.
2. $m = 3$ and $H \in \{K_3, P_3\}$.
3. $m \geq 4$ and $H = K_m$.

**Proof.** If $n(H) < m$, then $K_m \Box H$ is not well-dominated follows immediately from Proposition 16. Hence, we now assume that $H$ has order $m$. It is straightforward to show that the result is correct for $m = 2$ or $m = 3$. Now, let $m \geq 4$. It is easy to show that $K_m \Box K_m$ is well-dominated. For the converse, suppose that $H$ is a connected graph of order $m$ such that $K_m \Box H$ is well-dominated but such that $H \neq K_m$. Throughout the proof we let $\{h_1, h_2, \ldots, h_m\}$ denote the vertex set of $H$.

By Proposition 16, we infer that $\gamma(H) = 1$. Without loss of generality we assume that $\{h_1\}$ dominates $H$. Let $A$ be a maximum independent set of $H$. Since $H$ is not a complete graph, $|A| \geq 2$ and $A$ does not contain $h_1$. If $|A| = m - 1$, then $H = K_{1,m-1}$. In this case let $S = ([m - 1] \times \{h_m\}) \cup (\{m\} \times \{h_2, \ldots, h_{m-1}\})$. Note
that \(|S| = 2m - 3 > m\). We claim that \(S\) is a minimal dominating set of \(K_m \square H\). It is clear that \(S\) dominates \(K_m \square H\). Furthermore, \(pn[(i, h_m), S] = \{(i, h_1)\}\) for \(i \in [m - 1]\), and \((m, h_j) \in pn[(m, h_j), S]\) for \(2 \leq j \leq m - 1\). This proves that \(S\) is a minimal dominating set of \(K_m \square H\), which is a contradiction. Thus, we may assume that \(|A| < m - 1\), and we may also assume that \(A = V(H) - \{h_1, \ldots, h_k\}\) for some \(k\) such that \(2 \leq k \leq m - 2\). Let \(R = ([m - 1] \times \{h_1\}) \cup (\{m\} \times A)\). It is easy to show that \(R\) dominates \(K_m \square H\). For \(i \in [m - 1]\), we see that \((i, h_2) \in pn[(i, h_1), R]\), and \((m, a) \in pn[(m, a), R]\), for \(a \in A\). This implies that \(R\) is a minimal dominating set.

Therefore, if \(m \geq 4\) and \(H\) is a connected graph of order \(m\), then \(K_m \square H\) is well-dominated if and only if \(H = K_m\). \(\Box\)

In the remainder of the proof of Theorem 3 we consider \(K_m \square H\) where \(n(H) > m\).

The proofs for \(m = 2\) and \(m = 3\) are straightforward, while the proof of the more general case for \(m \geq 4\) occupies the rest of this section.

**Proposition 22.** If \(H\) is a connected graph of order at least 3, then \(K_2 \square H\) is not well-dominated.

**Proof.** Suppose there exists a connected graph \(H\) of order at least 3, such that \(K_2 \square H\) is well-dominated. By Proposition 16, we have \(\gamma(K_2 \square H) = \gamma(K_2)n(H) = n(H)\). Since \(H\) has order at least 3 and is connected, there exists a vertex \(h \in V(H)\) such that \(\deg(h) \geq 2\). Now, if \(S\) is the set defined by \(S = ([2] \times \{h\}) \cup (\{1\} \times (V(H) - N_H[h]))\) we arrive at a contradiction since \(S\) dominates \(K_2 \square H\) and \(|S| \leq n(H) - 1\). \(\Box\)

We have a similar result for Cartesian products with \(K_3\).

**Proposition 23.** If \(H\) is a connected graph of order at least 4, then \(K_3 \square H\) is not well-dominated.

**Proof.** Suppose there exists a connected graph \(H\) of order more than 3, such that \(K_3 \square H\) is well-dominated. By Proposition 16, we have \(\gamma(K_3 \square H) = \Gamma(K_3 \square H) = n(H)\). Suppose first that \(\Delta(H) \geq 3\); let \(h\) be a vertex of \(H\) such that \(\deg(h) = r \geq 3\). This implies that \(S = ([3] \times \{h\}) \cup (\{1\} \times (V(H) - N_H[h]))\) dominates \(K_3 \square H\). This is a contradiction since \(|S| = 3 + (n(H) - (r + 1)) < n(H)|\). Consequently, \(\Delta(H) = 2\), and thus \(H\) is either a cycle or a path of order at least 4. It is easy to verify that \(\gamma(K_3 \square P_4) = 4 < 6 = \Gamma(K_3 \square P_4)\) and \(\gamma(K_3 \square C_4) = 3 < 6 = \Gamma(K_3 \square C_4)\). Thus, we may assume that \(H\) contains a path of order 5, say \(h_1h_2h_3h_4h_5\). Let \(A = ([2, 3] \times \{h_3\}) \cup (\{1\} \times (V(H) - \{h_2, h_3, h_4\}))\). This set \(A\) dominates \(K_3 \square H\) and yet \(|A| = n(H) - 1\). This final contradiction establishes the proposition. \(\Box\)

**Proposition 24.** Let \(m\) be a positive integer such that \(m \geq 4\). If \(H\) is a connected graph of order more than \(m\), then \(K_m \square H\) is not well-dominated.
Proof. We proceed by induction on \( m \). For the base case we suppose for the sake of contradiction that there exists a connected graph \( H \) of order more than 4 such that \( K_4 \Box H \) is well-dominated. Let \( k = \gamma(H) \). By Proposition 16, \( \gamma(K_4 \Box H) = n(H) = 4k \), and hence every minimal dominating set of the well-dominated graph \( K_4 \Box H \) has cardinality 4\( k \). Let \( D = \{h_1, \ldots, h_k\} \) be a minimum dominating set of \( H \). Suppose first that \( H \) has a vertex \( x \) of degree at least 4. If \( A = V(H) - N[x] \), then the set \( S \) defined by \( S = ([4] \times \{x\}) \cup (\{1\} \times A) \) is a dominating set of \( K_4 \Box H \). However, \( |S| = 4 + |A| = 4 + (n(H) - |N[x]|) < 4k \), which is a contradiction. Therefore, \( \Delta(H) \leq 3 \). Since \( V(H) = \cup_{i=1}^{k} N[h_i] \) and \( n(H) = 4k \), it follows that \( \deg(h_i) = 3 \), for every \( i \in [k] \), and we see that \( N_H[h_1], \ldots, N_H[h_k] \) is a partition of \( V(H) \). Let \( M = \{4\} \times D \). The set \( M \) is independent in the well-dominated graph \( K_4 \Box H \) since \( N_H[h_1], \ldots, N_H[h_k] \) is a partition of \( V(H) \). By Observation 5, it follows that the graph \( G \) defined by \( G = K_4 \Box H - N[M] \) is well-dominated. Note that \( G = K_3 \Box F \), where \( F \) is the subgraph of \( H \) induced by \( V(H) - D \). This implies that each component of \( G \) is well-dominated. Using Proposition 23 we infer that each component of \( F \) has order at most 3. Furthermore, \( \Delta(F) \leq 2 \) since \( \Delta(H) = 3 \).

Note that \( K_3 \Box K_2 \) is not well-dominated, which then implies that each component of \( F \) has order 1 or 3. At least one of the components of \( F \) has order 3, for otherwise \( H \) is not connected. For each \( i \in [k] \), let \( X_i = N(h_i) = \{x_{i1}, x_{i2}, x_{i3}\} \). Each component of \( F \) that has order 3 intersects either one, two or three of the sets \( X_1, \ldots, X_k \). Suppose first that there exists \( i \in [k] \), say \( i = 1 \), such that \( \langle X_i \rangle \) is a component of \( F \). This implies that the subgraph of \( H \) induced by \( X_1 \cup \{h_1\} \) is a component of \( H \) of order 4, which contradicts the assumption that \( H \) is connected and has order at least 5. We thus assume that each component of \( F \) that has order 3 has a nonempty intersection with either two or three of the sets \( X_1, \ldots, X_k \). Any such component is clearly either a path of order 3 or a complete graph of order 3.

Suppose there exists \( 1 \leq i < j \leq k \) such that \( \langle X_i \cup X_j \rangle \) contains a \( P_3 \) or \( K_3 \) involving at least one vertex from each of \( X_i \) and \( X_j \). Without loss of generality we assume that \( i = 1 \) and \( j = 2 \) and that \( x_{13} \) has degree at least 2 in \( \langle X_1 \cup X_2 \rangle \). We assume without loss of generality that \( x_{13}x_{21} \in E(H) \). If \( x_{13} \) is adjacent to another vertex in \( X_2 \), say \( x_{13}x_{22} \in E(H) \), then let \( B = (\{1\} \times \{h_1, x_{11}, x_{12}, x_{23}\}) \cup (\{2, x_{21}\}, (3, x_{21}), (4, x_{22}) \}. \) On the other hand, if \( x_{13} \) is adjacent to another vertex in \( X_1 \), say \( x_{13}x_{12} \in E(H) \), then let \( B = (\{1\} \times \{h_1, x_{11}, x_{12}\}) \cup (\{2, x_{12}\}, (3, x_{21}), (4, x_{21}) \}. \) In both cases we see that \( B \cup ([4] \times \{h_3, \ldots, h_k\}) \) dominates \( K_4 \Box H \) and has cardinality \( 4k - 1 \), which is a contradiction.

Hence, every component of \( F \) that has order 3 contains one vertex from three distinct members of the partition \( X_1, \ldots, X_k \) of \( V(F) \). We assume without loss of generality that \( x_{11}, x_{21}, x_{31} \) is a path in \( F \). Let \( B = (\{1\} \times \{h_1, h_2, x_{12}, x_{13}, x_{22}, x_{23}, x_{32}\}) \cup (\{2, x_{11}\}, (2, x_{33}), (3, x_{31}), (4, x_{31}) \}. \) It now follows that \( B \cup ([4] \times \{h_4, \ldots, h_k\}) \) is a dominating set of \( K_4 \Box H \) and has cardinality \( 4k - 1 \), which is a contradiction. Therefore, if \( H \) is a connected graph of order more than 4, then \( K_4 \Box H \) is not well-dominated.
Now let $m \geq 5$. Our inductive hypothesis is that if $G$ is any connected graph of order at least $m$, then $K_{m-1} \Box G$ is not well-dominated. Again, for the sake of arriving at a contradiction, suppose there exists a connected graph $H$ of order more than $m$ such that $K_m \Box H$ is well-dominated. For consistency and ease of understanding we use the same notation as in the case $m = 4$. Let $k = \gamma(H)$ and let $D = \{h_1, \ldots, h_k\}$ be a minimum dominating set of $H$. By Proposition 16, $\gamma(K_m \Box H) = n(H) = mk$, and hence every minimal dominating set of the well-dominated graph $K_m \Box H$ has cardinality $mk$. Suppose first that $H$ has a vertex $x$ of degree at least $m$. If $A = V(H) - N[x]$, then the set $S$ defined by $S = ([m] \times \{x\}) \cup (\{1\} \times A)$ is a dominating set of $K_m \Box H$. However, $|S| = m + |A| = m + (n(H) - |N[x]|) < mk$, which is a contradiction. Therefore, $\Delta(H) \leq m - 1$.

Since $V(H) = \bigcup_{i=1}^k N[h_i]$, it follows that $\deg(h_i) = m - 1$, for every $i \in [k]$ and $N_H[h_1], \ldots, N_H[h_k]$ is a partition of $V(H)$. Similar to the case above (for $m = 4$) we let $M$ be the independent set defined by $M = \{m\} \times D$, and we note by Observation 5 that $G = K_m \Box H - N[M]$ is well-dominated, and hence every component of $G$ is well-dominated. Since $G$ is isomorphic to $K_{m-1} \Box F$, where $F$ is the subgraph of $H$ induced by $V(H) - D$, it follows from the inductive hypothesis and Proposition 21 that every nontrivial component of $F$ is isomorphic to $K_{m-1}$. Also, since $K_{m-1} \Box F$ is well-dominated, it follows by Proposition 16 that $\gamma(K_{m-1} \Box F) = (m - 1)\gamma(F) = n(F) = k(m - 1)$. Thus, $\gamma(F) = k$, which implies that $F$ has no components of order 1. That is, $F$ is the disjoint union of $k$ complete graphs of order $m - 1$. Suppose that for some $i \in [k]$ there exists a vertex $x \in N_H(h_i)$ such that $N_F(x) \subseteq N_H(h_i)$. Since the component, say $C_x$, of $F$ that contains $x$ is a complete graph of order $m - 1$, we infer that $C_x = N_H(h_i)$, which implies that $H$ is not connected. This contradiction means there is no such $i \in [k]$. In particular, for each $u \in N_H(h_1)$ we have $N(u) \cap (N_H(h_2) \cup \cdots \cup N_H(h_k)) \neq \emptyset$. Let

$$S = (\{1\} \times \bigcup_{i=1}^k N_H[h_i]) \cup (\{2, \ldots, m\} \times \{h_1\}).$$

This set $S$ dominates $K_m \Box H$ and $|S| = mk - 1$, which is a contradiction. This establishes the proposition. \[\square\]

By combining the results of Propositions 21, 22, 23, and 24, the proof of Theorem 3 is complete.

We close this section on well-dominated Cartesian products with the following conjecture.

**Conjecture 1.** Let $G$ and $H$ be nontrivial connected graphs. If $G \Box H$ is well-dominated, then at least one of $G$ or $H$ is a complete graph.

If Conjecture 1 is true, then we would have a complete characterization of the well-dominated Cartesian products. That is, if Conjecture 1 is true, then by Theorem 3 it follows that the Cartesian product $G \Box H$ of two nontrivial, connected graphs is
well-dominated if and only if $G \square H = K_m \square K_m$ for some positive integer $m \geq 2$ or $G \square H = K_3 \square P_3$.

References


