On well-dominated direct, Cartesian and strong product graphs

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Abstract

If each minimal dominating set in a graph is a minimum dominating set, then the graph is called *well-dominated*. Since the seminal paper on well-dominated graphs appeared in 1988, the structure of well-dominated graphs from several restricted classes have been studied. In this paper we give a complete characterization of nontrivial direct products that are well-dominated. We prove that if a strong product is well-dominated, then both of its factors are well-dominated. When one of the factors of a strong product is a complete graph, the other factor being well-dominated is also a sufficient condition for the product to be well-dominated. Our main result gives a complete characterization of well-dominated Cartesian products in which at least one of the factors is a complete graph. In addition, we conjecture that this result is actually a complete characterization of the class of nontrivial, well-dominated Cartesian products.

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1 Introduction

A dominating set of a finite graph G is a set D of vertices such that each vertex of G is within distance 1 of at least one vertex in D. Finding a (set inclusion) minimal dominating set is straightforward and can be accomplished in linear time by simply ordering the vertices and then discarding them one at a time if the remaining set still dominates the graph. Depending on the graph, the resulting minimal dominating set may be much larger than the domination number of G, which is the minimum cardinality among all its dominating sets. On the other hand, for a given graph G and positive integer k, it is well known that determining whether G has a dominating set of cardinality at most k is an NP-complete problem. In most applications, finding the size of a smallest dominating set is the typical goal. *Well-dominated* graphs are those for which the above algorithm always finds a dominating set of minimum cardinality. The seminal paper on well-dominated graphs was by Finbow, Hartnell and Nowakowski [3]. They characterized the well-dominated graphs of girth at least 5 and showed that the only well-dominated bipartite graphs are those with domination number one-half their order. Several other groups of authors have studied the concept of well-dominated graphs within restricted graph classes. See, for example, [12] (block and unicyclic graphs), [11] (simplicial and chordal graphs), [5] (4-connected, 4-regular claw-free), and [4] (planar triangulations). In [6] Gözüpek, Hujdurović and Milanič characterized the well-dominated graphs that are nontrivial lexicographic products.

Our focus in this paper is the class of well-dominated graphs that have a nontrivial factorization as a Cartesian, direct or strong product. These three graph products are referred to as the "fundamental products" in the book [7] by Hammack, Imrich and Klavžar. Along with the lexicographic product, they are the most studied graph products in the literature.

Anderson, Kuenzel and Rall [1] characterized the direct products that are well-dominated under the assumption that at least one of the factors has no isolatable vertices. (A vertex x in a graph X is *isolatable* if there is an independent set A in X such that $\{x\}$ is a component in $X - N[A]$.)

Theorem 1. [1, Theorem 3] *Let G and H be nontrivial connected graphs such that at least one of G or H has no isolatable vertices. The direct product $G \times H$ is well-dominated if and only if $G = H = K_3$ or at least one of the factors is K_2 and the other factor is a 4-cycle or the corona of a connected graph.*

In this paper we complete the characterization of well-dominated direct products by removing the requirement on isolatable vertices. In fact, we prove that if both factors of a direct product have order at least 3 and one of them has an isolatable vertex, then the direct product is not well-dominated. The main result on well-dominated direct products is then the following theorem.

Theorem 2. *Let G and H be connected graphs. The direct product $G \times H$ is well-dominated if and only if $G \times H = K_3 \times K_3$, $G \times H = K_2 \times C_4$, or $G \times H = K_2 \times (F \odot K_1)$ for a connected graph F.*

In the same paper Anderson et al. proved that if a Cartesian product $G \,\Box\, H$ is well-dominated, then at least one of G or H is well-dominated. In addition, they provided a characterization of the well-dominated Cartesian products of triangle-free graphs. Namely, they proved that the Cartesian product of two connected, triangle-free graphs of order at least 2 is well-dominated if and only if both factors are complete graphs of order 2. Here we explore the more general case of well-dominated Cartesian products in which (at least) one of the factors has girth 3. In particular, we prove the following characterization of well-dominated graphs of the form $K_m \,\Box\, H$, and we conjecture that every well-dominated Cartesian product of nontrivial connected graphs is isomorphic to one of these.

Theorem 3. *Let m be a positive integer with $m \geq 2$ and let H be a nontrivial, connected graph. The Cartesian product $K_m \,\Box\, H$ is well-dominated if and only if either $m \neq 3$ and $H = K_m$ or $m = 3$ and $H \in \{K_3, P_3\}$.*

For the strong product we prove that both factors of a well-dominated strong product are well-dominated. If one of the factors of a strong product is a complete graph, then the other factor being well-dominated is also a sufficient condition for the product to be well-dominated.

Theorem 4. *Let n be a positive integer. For any graph H the strong product $K_n \boxtimes H$ is well-dominated if and only if H is well-dominated.*

The remainder of the paper is organized in the following way. In the next section we provide the necessary definitions for the remainder of the paper. In Section 3 we settle the relatively straightforward result for strong products. Theorem 2, the complete characterization of well-dominated direct products, is verified in Section 4. Proving Theorem 3 is the main task of Section 5. We also derive a number of necessary conditions on two connected graphs whose Cartesian product is well-dominated, which leads us to conjecture that the characterization in Theorem 3 captures all connected well-dominated Cartesian products.

2 Definitions

All graphs in this paper are finite, undirected, simple and have order at least 2. For a positive integer n, we let $[n] = \{1, \ldots, n\}$. This set will be the vertex set of the complete graph of order n. In general, we follow the terminology and notation of Hammack, Imrich, and Klavžar [7]. The order of a graph G is the number of vertices in G and is denoted by $n(G)$; G is *nontrivial* if $n(G) \geq 2$. For a vertex v in a graph G, the *open neighborhood* $N(v)$ and the *closed neighborhood* $N[v]$ are defined by $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. For $A \subseteq V(G)$ we let $N(A) = \cup_{v \in A} N(v)$ and $N[A] = N(A) \cup A$. The subgraph of G induced by A is denoted by $\langle A \rangle$. Any vertex subset D such that $N[D] = V(G)$ is a *dominating set* of

G, and D is then a *minimal* dominating set if no proper subset of D is a dominating set. The *domination number* of G is denoted by $\gamma(G)$ and is the minimum cardinality among the dominating sets of G. The *upper domination number*, denoted by $\Gamma(G)$, is the largest cardinality of a minimal dominating set of G. A set $M \subseteq V(G)$ is an *independent* set if its vertices are pairwise non-adjacent. An independent set is *maximal* if it is not a proper subset of an independent set. The cardinalities of a smallest and a largest maximal independent set in G are denoted by $i(G)$ and $\alpha(G)$, respectively. Note that a maximal independent set is a dominating set, which gives

$$\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G)\,. \tag{1}$$

A graph G is called *well-covered* if $i(G) = \alpha(G)$ and is *well-dominated* if $\gamma(G) = \Gamma(G)$. It is clear from (1) that the class of well-dominated graphs is a subclass of the class of well-covered graphs.

Let G be a graph and let $u \in A \subseteq V(G)$. The *private neighborhood of u with respect to A* is the set $\text{pn}[u, A]$ defined by $\text{pn}[u, A] = \{x \in V(G) : N[x] \cap A = \{u\}\}$. Equivalently, $\text{pn}[u, A] = N[u] - N[A - \{u\}]$. The vertices in $\text{pn}[u, A]$ are called *private neighbors* of u with respect to A. The subset $A \subseteq V(G)$ is *irredundant* if $\text{pn}[u, A] \neq \emptyset$ for every $u \in A$. It follows from the definitions that a dominating set D of G is a minimal dominating set if and only if every vertex in D has a private neighbor with respect to D; that is, D is irredundant. In this paper we will also need a more restricted type of private neighbor. The *external private neighborhood* of u with respect to A is the set $\text{epn}[u, A]$ defined by $\text{epn}[u, A] = \text{pn}[u, A] - \{u\} = N(u) - N[A - \{u\}]$. If $v \in \text{epn}[u, A]$, then v is called an *external* private neighbor of u with respect to A. Such a vertex v, if it exists, belongs to $V(G) - A$, which is the reason to use the word external. A property related to irredundance, and one that is important for this paper, is that of being open irredundant. The set A is *open irredundant* if every vertex of A has an external private neighbor with respect to A.

A vertex x in a graph G is an *isolatable vertex* in G if there exists an independent set I of vertices in G such that x is isolated in the induced subgraph $G - N[I]$ of G. Concerning the closed neighborhood of an independent set we have the following useful fact about well-dominated graphs first observed by Finbow, Hartnell and Nowakowski. The proof is straightforward and follows from the fact that if M is an independent set in G and D is a minimal dominating set of $G - N[M]$, then $D \cup M$ is a minimal dominating set of G.

Observation 5. [3] *If G is a well-dominated graph and M is any independent set of vertices in G, then $G - N[M]$ is well-dominated.*

Let G and H be finite, undirected graphs. The *Cartesian product* of G and H, denoted by $G \,\square\, H$, has as its vertex set the Cartesian (set) product $V(G) \times V(H)$. The *direct product*, denoted $G \times H$, and the *strong product*, $G \boxtimes H$, also have $V(G) \times V(H)$ as their set of vertices. Distinct vertices (g_1, h_1) and (g_2, h_2) are adjacent in

- $G \square H$ if either ($g_1 = g_2$ and $h_1h_2 \in E(H)$) or ($h_1 = h_2$ and $g_1g_2 \in E(G)$);
- $G \times H$ if $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$;
- $G \boxtimes H$ if they are adjacent in $G \square H$ or they are adjacent in $G \times H$.

All three of these graph products are associative and commutative. A product graph is called *nontrivial* if both of its factors are nontrivial. See [7] for specific information on these and other graph products. The *corona* of a graph G, denoted by $G \odot K_1$, is the graph of order $2n(G)$ obtained by adding, for each vertex u of G a new vertex u' together with a new edge uu'.

3 Well-dominated strong products

Recall that two vertices (g_1, h_1) and (g_2, h_2) are adjacent in the strong product $G \boxtimes H$ if one of the following holds.

- $g_1 = g_2$ and $h_1h_2 \in E(H)$
- $h_1 = h_2$ and $g_1g_2 \in E(G)$
- $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$

Nowakowski and Rall [10] established the following relationships between ordinary domination invariants on strong products.

Proposition 6. [10, Corollary 2.2] *If G and H are finite graphs, then*

$$\gamma(G \boxtimes H) \leq \gamma(G)\gamma(H) \text{ and } \Gamma(G \boxtimes H) \geq \Gamma(G)\Gamma(H)\,.$$

The following corollary follows immediately from Proposition 6.

Corollary 7. *Let G and H be finite graphs. If $G \boxtimes H$ is well-dominated, then G and H are well-dominated.*

The converse of Corollary 7 is not true in general. For example, the 5-cycle is well-dominated, but

$$\gamma(C_5 \boxtimes C_5) = 4 < 6 = \Gamma(C_5 \boxtimes C_5)\,.$$

However, we are able to show in this paper that if at least one of the factors is a complete graph, then the strong product of this complete graph and a well-dominated graph is well-dominated.

Let D be a dominating set of $G \boxtimes H$ and let $A = \{g \in V(G) : (g,h) \in D \text{ for some } h \in V(H)\}$. If $u \in V(G) - A$ and $v \in V(H)$, then from the definition of the edge structure of the strong product it follows that (u,v) is dominated by D only if u is dominated by A. Thus, A dominates G and therefore $\gamma(G) \le |A| \le |D|$. Interchanging the roles of G and H proves the following lemma.

Lemma 8. *For all pairs of graphs G and H, we have $\gamma(G \boxtimes H) \ge \max\{\gamma(G), \gamma(H)\}$.*

We now proceed to prove Theorem 4, which is restated here.

Theorem 4 *Let n be a positive integer. For any graph H the strong product $K_n \boxtimes H$ is well-dominated if and only if H is well-dominated.*

Proof. Suppose that H is a well-dominated graph. Let D be any minimal dominating set of $K_n \boxtimes H$ and let $S = \{h \in V(H) : (i,h) \in D \text{ for some } i \in [n]\}$. Note that for any $u \in V(H)$ and for $1 \le i < j \le n$, we have $N[(i,u)] = N[(j,u)]$. Since D is an irredundant set, we infer that $|S| = |D|$. We claim that S is a minimal dominating set of H. As in the paragraph preceding Lemma 8 we see that S dominates H. Let h be any vertex of S and let $i \in [n]$ such that $(i,h) \in D$. If (i,h) is isolated in the subgraph of $K_n \boxtimes H$ induced by D, then h is isolated in the subgraph of H induced by S and $h \in \text{pn}[h,S]$. On the other hand, if $(i,h) \notin \text{pn}[(i,h),D]$, then there exists $(k,x) \in \text{pn}[(i,h),D]$ such that $x \notin S$. Again by definition of the edge structure of the strong product it follows that $x \in \text{pn}[h,S]$. We conclude that S is irredundant in H, and hence S is a minimal dominating set of H. Therefore, $|D| = |S| = \gamma(H)$. That is, all minimal dominating sets of $K_n \boxtimes H$ have the same cardinality, which implies that $K_n \boxtimes H$ is well-dominated.

The converse follows from Corollary 7. □

Still unanswered is the following natural question.

Question 1. *What properties on the well-dominated graphs G and H are necessary and sufficient for $G \boxtimes H$ to be well-dominated?*

4 Well-dominated direct products

In this section we complete the characterization of well-dominated direct products. Throughout we assume that the factors are connected and have order at least 2. We prove Theorem 2, which we now restate for convenience of the reader.

Theorem 2 *Let G and H be connected graphs. The direct product $G \times H$ is well-dominated if and only if $G \times H = K_3 \times K_3$, $G \times H = K_2 \times C_4$, or $G \times H = K_2 \times (F \odot K_1)$ for a connected graph F.*

It is easy to see that if M is any maximal independent set of G then $M \times V(H)$ is a maximal independent set, and thus also a minimal dominating set, of $G \times H$. If, in addition, $G \times H$ is well-dominated, then it follows that $M \times V(H)$ is a minimum dominating set of $G \times H$ and so $\gamma(G \times H) = \alpha(G)n(H)$.

Lemma 9. *Let G and H be connected graphs of order at least 3. If $G \times H$ is well-dominated, then $\gamma(G \times H) = \gamma(G)n(H) = \gamma(H)n(G)$. Furthermore, $\gamma(G) = \alpha(G)$ and $\gamma(H) = \alpha(H)$.*

Proof. Let D be a minimum dominating set of G. It is easy to see that $D \times V(H)$ dominates $G \times H$. We get

$$\gamma(G)n(H) = |D \times V(H)| \geq \gamma(G \times H) = \alpha(G)n(H) \geq \gamma(G)n(H)\,,$$

and we have equality throughout. Therefore, $\gamma(G \times H) = \gamma(G)n(H)$, and $\gamma(G) = \alpha(G)$. By reversing the roles of G and H we also have $\gamma(G \times H) = \gamma(H)n(G)$ and $\gamma(H) = \alpha(H)$. □

As we have observed several times, $D \times V(H)$ dominates the direct product $G \times H$ if D dominates G. The next lemma shows that if both factors have order at least 3, then the set $D \times V(H)$ is not a *minimal* dominating set unless D is independent in G.

Lemma 10. *Let G and H be connected graphs of order at least 3, and let D be a minimal dominating set of G. The set $D \times V(H)$ is a minimal dominating set of $G \times H$ if and only if D is independent in G.*

Proof. If D is an independent dominating set of G, then $D \times V(H)$ is a maximal independent set, and hence a minimal dominating set, of $G \times H$. For the converse, suppose that a is a vertex that is not isolated in the subgraph of G induced by D. Let x be a vertex in H that is not a support vertex. We will show that $\mathrm{pn}[(a,x), D \times V(H)] = \emptyset$. First, since $N(a) \cap D \neq \emptyset$, we see that (a,x) has a neighbor in $D \times V(H)$, which implies that $(a,x) \notin \mathrm{pn}[(a,x), D \times V(H)]$. Next, let $(u,y) \in N((a,x)) - (D \times V(H))$. Since x is not a support vertex of H, we see that there exists a path, say x, y, z, in H. However, this means that (a,z) is adjacent to (u,y), and hence $(u,y) \notin \mathrm{pn}[(a,x), D \times V(H)]$. It follows that $\mathrm{pn}[(a,x), D \times V(H)] = \emptyset$, which implies that $D \times V(H)$ is not a minimal dominating set. □

As a result of Lemma 10 we can now state a very restrictive condition that must be satisfied by both factors of a well-dominated direct product if both have order at least 3.

Corollary 11. *Let G and H be connected graphs of order at least 3. If $G \times H$ is well-dominated, then all minimum dominating sets of G and all minimum dominating sets of H are independent.*

Proof. Suppose G and H are connected of order at least 3 such that $G \times H$ is well-dominated. By Lemma 9, we have $\gamma(G \times H) = \gamma(G)n(H) = \gamma(H)n(G)$. Let D be any minimum dominating set of G. Since $D \times V(H)$ dominates $G \times H$ and $\gamma(G \times H) = |D \times V(H)|$, it follows from Lemma 10 that D is independent. Similarly, every minimum dominating set of H is independent. □

Using these results we now proceed to the proof of Theorem 2.

It was verified in [1] that $G \times H$ is well-dominated if $G \times H = K_3 \times K_3$, $G \times H = K_2 \times C_4$, or $G \times H = K_2 \times (F \odot K_1)$ for a connected graph F. For the converse we assume that $G \times H$ is well-dominated. Suppose for the sake of contradiction that $G \times H \neq K_3 \times K_3$, $G \times H \neq K_2 \times C_4$, and $G \times H \neq K_2 \times (F \odot K_1)$ for a connected graph F. By Theorem 1, it follows that both G and H possess an isolatable vertex, and hence $n(G) \geq 3$ and $n(H) \geq 3$. Let M be an independent set in G such that $G - N[M] = \{x\}$. Since G is connected, there exists a vertex $x' \in N(M)$ that is adjacent to x. By Lemma 9 and the inequality chain (1), $\gamma(G) = i(G) = \alpha(G)$. Since $M \cup \{x\}$ is an independent dominating set of G, it follows that $\gamma(G) = |M| + 1$. However, $M \cup \{x'\}$ dominates G and is therefore a minimum dominating set of G that is not independent, which is a contradiction by Corollary 11. □

In the first paper on well-dominated graphs, Finbow et al. proved the following characterization of the class of well-dominated bipartite graphs.

Theorem 12. [3, Theorem 3] *Let G be a connected, bipartite graph. Then G is well-dominated if and only if $G = C_4$ or G is the corona of a bipartite graph.*

Let F be a connected graph. It is straightforward to verify that $(F \odot K_1) \times K_2 = (F \times K_2) \odot K_1$. Thus the well-dominated graphs of the form $K_2 \times (F \odot K_1)$ are a subclass of the easily recognizable well-dominated, bipartite coronas of Theorem 12.

5 Well-dominated Cartesian products

In this section we prove the following characterization of well-dominated Cartesian products in which at least one of the factors is a complete graph. In particular, we prove our main theorem in this study.

Theorem 3 *Let m be a positive integer with $m \geq 2$ and let H be a nontrivial, connected graph. The Cartesian product $K_m \,\square\, H$ is well-dominated if and only if either $m \neq 3$ and $H = K_m$ or $m = 3$ and $H \in \{K_3, P_3\}$.*

We begin by deriving some preliminary results that will prove to be useful in its proof. After that we prove Proposition 21, which is a special case of Theorem 3 that assumes $n(H) \leq m$.

As mentioned in the introduction, the first step in this process was obtained by Anderson, Kuenzel and Rall [1].

Theorem 13. [1, Theorem 1] *Let G and H be connected graphs. If $G \,\Box\, H$ is well-dominated, then G or H is well-dominated.*

If a graph G admits a dominating set that is open irredundant, then the following result provides a method for constructing a minimal dominating set in any Cartesian product that has G as one of its factors.

Lemma 14. *If D is an open irredundant dominating set of a graph G, then $D \times V(H)$ is an open irredundant, minimal dominating set of $G \,\Box\, H$, for any graph H.*

Proof. If $(x, y) \in V(G \,\Box\, H) - (D \times V(H))$, then there exists a vertex $x' \in N(x) \cap D$ since D is a dominating set of G. This implies that (x, y) is adjacent to (x', y), and thus $D \times V(H)$ is a dominating set of $G \,\Box\, H$. To see that $D \times V(H)$ is a minimal dominating set, let (d, h) be an arbitrary vertex in $D \times V(H)$. Since D is an open irredundant set of G, there exists a vertex d' in $\text{epn}[d, D]$. By definition, $d' \in N(d) - D$, and d' is not adjacent to any vertex of $D - \{d\}$. Consequently, $(d', h) \in \text{epn}[(d, h), D \times V(H)]$. Therefore, $D \times V(H)$ is an open irredundant, minimal dominating set of $G \,\Box\, H$. □

Bollobás and Cockayne proved that every graph with minimum degree at least 1 has an open irredundant, minimum dominating set D.

Proposition 15. [2, Proposition 6] *If a graph G has no isolated vertices, then G has a minimum dominating set that is open irredundant.*

By using Proposition 15, we now establish a relationship that must hold between two graphs if their Cartesian product is well-dominated.

Proposition 16. *Let G and H be nontrivial connected graphs. If $G \,\Box\, H$ is well-dominated, then $\gamma(G \,\Box\, H) = \gamma(G) \cdot n(H) = \gamma(H) \cdot n(G)$.*

Proof. Let D_1 be a minimum dominating set of G that is open irredundant and let D_2 be a minimum dominating set of H that is open irredundant. These minimum dominating sets of G and H exist by Proposition 15. By Lemma 14, it follows that both of $D_1 \times V(H)$ and $V(G) \times D_2$ are minimal dominating sets of $G \,\Box\, H$. Since $G \,\Box\, H$ is well-dominated, it follows that $\gamma(G \,\Box\, H) = |D_1 \times V(H)| = |V(G) \times D_2|$, and therefore

$$\gamma(G \,\Box\, H) = \gamma(G) \cdot n(H) = \gamma(H) \cdot n(G) .$$

□

If D is a minimal dominating set of a graph G, then, in general, some vertices of D will have external private neighbors with respect to D and some vertices will not.

The following notation will be useful in what follows. If D is any dominating set of G, we define $c(D)$ as follows:

$$c(D) = \{x \in D : \mathrm{pn}[x, D] = \{x\}\}.$$

That is, a vertex x in D belongs to $c(D)$ if and only if x is isolated in the subgraph induced by D, and $N(x) - D \subseteq N(D - \{x\})$. In particular, $c(D)$ is a (possibly empty) independent set of G. Note also that if $c(D) \neq \emptyset$, then $|D| \geq 2$ since we are assuming that G is of order at least 2.

Lemma 17. *Let S be a minimal dominating set of a graph H. If D is a minimal dominating set of a graph G such that $c(D) \neq \emptyset$, then $(\{u\} \times S) \cup ((D - \{u\}) \times V(H))$ is a dominating set of $G \,\square\, H$ for every $u \in c(D)$. If, in addition, $c(D) = \{u\}$, then $(\{u\} \times S) \cup ((D - \{u\}) \times V(H))$ is a minimal dominating set of $G \,\square\, H$.*

Proof. Let $u \in c(D)$. For simplification, let $A = (\{u\} \times S) \cup ((D - \{u\}) \times V(H))$. Suppose $(x, y) \in V(G \,\square\, H) - A$. If $x \notin D$, then there exists $d \in D - \{u\}$ such that $dx \in E(G)$. This follows since D dominates G and $x \notin \mathrm{pn}[u, D] = \{u\}$. Hence, (x, y) has a neighbor in $(D - \{u\}) \times V(H)$. On the other hand, if $x \in D$, then $x = u$ and $y \notin S$. Since S dominates H, it follows that (x, y) has a neighbor in $\{u\} \times S$. Therefore, A is a dominating set of $G \,\square\, H$.

Now, suppose that $c(D) = \{u\}$. To show that A is a minimal dominating set let $(a, b) \in A$. If $a \neq u$, then a has an external private neighbor, say a', with respect to D, and (a', b) is a private neighbor of (a, b) with respect to A. On the other hand, suppose $a = u$. This implies that $b \in S$. Since S is a minimal dominating set of H, there exists $b' \in \mathrm{pn}[b, S]$. It now follows that (u, b') is a private neighbor of (u, b) with respect to A since u is an isolated vertex in $G[D]$. We have shown that $\mathrm{pn}[(a, b), A] \neq \emptyset$, and it follows that A is a minimal dominating set of $G \,\square\, H$. □

If G is any finite graph, then for large enough m, namely for $m > \Delta(G)$, the Cartesian product $G \,\square\, K_m$ is well-covered. (See page 1262 of [8].) This is not true in the well-dominated class as we now prove.

Proposition 18. *If G is a nontrivial connected graph and has a minimum dominating set D such that $c(D) \neq \emptyset$, then $G \,\square\, H$ is not well-dominated for every nontrivial connected graph H.*

Proof. Let G be a nontrivial, connected graph and suppose that D is a minimum dominating set of G with a vertex $u \in c(D)$. Let H be a nontrivial connected graph and suppose that S is any minimum dominating set of H. By Lemma 17, $(\{u\} \times S) \cup ((D - \{u\}) \times V(H))$ is a dominating set of $G \,\square\, H$. While $(\{u\} \times S) \cup ((D - \{u\}) \times V(H))$ may not be a minimal dominating set, it contains one. Since

$$\begin{aligned}|(\{u\} \times S) \cup ((D - \{u\}) \times V(H))\,| &= \gamma(H) + (\gamma(G) - 1) \cdot n(H) \\ &= \gamma(G) \cdot n(H) + (\gamma(H) - n(H)) < \gamma(G) \cdot n(H),\end{aligned}$$

we conclude by Proposition 16 that $G \,\Box\, H$ is not well-dominated. □

The following corollary of Proposition 18 further limits which graphs can be a factor of a well-dominated Cartesian product.

Corollary 19. *If G is a nontrivial connected graph that is well-dominated and has an isolatable vertex, then $G \,\Box\, H$ is not well-dominated for any nontrivial connected graph H.*

Proof. Suppose G is a nontrivial connected, well-dominated graph and suppose x is an isolatable vertex of G. Let H be any connected graph of order at least 2. Since G is well-dominated, we have $\gamma(G) = i(G) = \alpha(G) = \Gamma(G)$. Let I be an independent set in G such that $G - N[I] = \{x\}$. The set $J = I \cup \{x\}$ is an independent dominating set, and is therefore also a minimum dominating set of G. Since $G - N[I] = \{x\}$, we see that $x \in c(J)$. It follows by Proposition 18 that $G \,\Box\, H$ is not well-dominated. □

The following result follows immediately from Corollary 19.

Corollary 20. *If G is a connected, well-dominated graph of order at least 3 and $G \,\Box\, H$ is well-dominated for some nontrivial connected graph H, then $\delta(G) \geq 2$.*

Proceeding with the proof of Theorem 3 we first deal with the case where the order of the complete factor in the Cartesian product is at least as large as the order of the other factor.

Proposition 21. *Let m be a positive integer such that $m \geq 2$ and let H be a nontrivial connected graph such that $n(H) \leq m$. The Cartesian product $K_m \,\Box\, H$ is well-dominated if and only if one of the following holds.*

1. *$m = 2$ and $H = K_2$.*
2. *$m = 3$ and $H \in \{K_3, P_3\}$.*
3. *$m \geq 4$ and $H = K_m$.*

Proof. If $n(H) < m$, then $K_m \,\Box\, H$ is not well-dominated follows immediately from Proposition 16. Hence, we now assume that H has order m. It is straightforward to show that the result is correct for $m = 2$ or $m = 3$. Now, let $m \geq 4$. It is easy to show that $K_m \,\Box\, K_m$ is well-dominated. For the converse, suppose that H is a connected graph of order m such that $K_m \,\Box\, H$ is well-dominated but such that $H \neq K_m$. Throughout the proof we let $\{h_1, h_2, \ldots, h_m\}$ denote the vertex set of H.

By Proposition 16, we infer that $\gamma(H) = 1$. Without loss of generality we assume that $\{h_1\}$ dominates H. Let A be a maximum independent set of H. Since H is not a complete graph, $|A| \geq 2$ and A does not contain h_1. If $|A| = m - 1$, then $H = K_{1,m-1}$. In this case let $S = ([m-1] \times \{h_m\}) \cup (\{m\} \times \{h_2, \ldots, h_{m-1}\})$. Note that $|S| = 2m-3 > m$. We claim that S is a minimal dominating set of $K_m \,\Box\, H$. It is clear that S dominates $K_m \,\Box\, H$. Furthermore, $\text{pn}[(i,h_m),S] = \{(i,h_1)\}$ for $i \in [m-1]$, and $(m,h_j) \in \text{pn}[(m,h_j),S]$ for $2 \le j \le m-1$. This proves that S is a minimal dominating set of $K_m \,\Box\, H$, which is a contradiction. Thus, we may assume that $|A| < m-1$, and we may also assume that $A = V(H) - \{h_1,\ldots,h_k\}$ for some k such that $2 \le k \le m-2$. Let $R = ([m-1]\times\{h_1\}) \cup (\{m\}\times A)$. It is easy to show that R dominates $K_m \,\Box\, H$. For $i \in [m-1]$, we see that $(i,h_2) \in \text{pn}[(i,h_1),R]$, and $(m,a) \in \text{pn}[(m,a),R]$, for $a \in A$. This implies that R is a minimal dominating set. However, $|R| = m-1+|A| \ge m-1+2 > m$, which is a contradiction and implies that $K_m \,\Box\, H$ is not well-dominated.

Therefore, if $m \ge 4$ and H is a connected graph of order m, then $K_m \,\Box\, H$ is well-dominated if and only if $H = K_m$. □

In the remainder of the proof of Theorem 3 we consider $K_m \,\Box\, H$ where $n(H) > m$. The proofs for $m = 2$ and $m = 3$ are straightforward, while the proof of the more general case for $m \ge 4$ occupies the rest of this section.

Proposition 22. *If H is a connected graph of order at least 3, then $K_2 \,\Box\, H$ is not well-dominated.*

Proof. Suppose there exists a connected graph H of order at least 3, such that $K_2 \,\Box\, H$ is well-dominated. By Proposition 16, we have $\gamma(K_2 \,\Box\, H) = \gamma(K_2)n(H) = n(H)$. Since H has order at least 3 and is connected, there exists a vertex $h \in V(H)$ such that $\deg(h) \ge 2$. Now, if S is the set defined by $S = ([2]\times\{h\}) \cup (\{1\}\times(V(H)-N_H[h]))$ we arrive at a contradiction since S dominates $K_2 \,\Box\, H$ and $|S| \le n(H)-1$. □

We have a similar result for Cartesian products with K_3.

Proposition 23. *If H is a connected graph of order at least 4, then $K_3 \,\Box\, H$ is not well-dominated.*

Proof. Suppose there exists a connected graph H of order more than 3, such that $K_3 \,\Box\, H$ is well-dominated. By Proposition 16, we have $\gamma(K_3 \,\Box\, H) = \Gamma(K_3 \,\Box\, H) = n(H)$. Suppose first that $\Delta(H) \ge 3$; let h be a vertex of H such that $\deg(h) = r \ge 3$. This implies that $S = ([3]\times\{h\}) \cup (\{1\}\times(V(H)-N_H[h]))$ dominates $K_3 \,\Box\, H$. This is a contradiction since $|S| = 3 + (n(H)-(r+1)) < n(H)$. Consequently, $\Delta(H) = 2$, and thus H is either a cycle or a path of order at least 4. It is easy to verify that $\gamma(K_3 \,\Box\, P_4) = 4 < 6 = \Gamma(K_3 \,\Box\, P_4)$ and $\gamma(K_3 \,\Box\, C_4) = 3 < 6 = \Gamma(K_3 \,\Box\, C_4)$. Thus, we may assume that H contains a path of order 5, say $h_1h_2h_3h_4h_5$. Let $A = (\{2,3\}\times\{h_3\}) \cup (\{1\}\times(V(H)-\{h_2,h_3,h_4\}))$. This set A dominates $K_3 \,\Box\, H$ and yet $|A| = n(H)-1$. This final contradiction establishes the proposition. □

Proposition 24. *Let m be a positive integer such that $m \ge 4$. If H is a connected graph of order more than m, then $K_m \,\Box\, H$ is not well-dominated.*

Proof. We proceed by induction on m. For the base case we suppose for the sake of contradiction that there exists a connected graph H of order more than 4 such that $K_4 \,\Box\, H$ is well-dominated. Let $k = \gamma(H)$. By Proposition 16, $\gamma(K_4 \,\Box\, H) = n(H) = 4k$, and hence every minimal dominating set of the well-dominated graph $K_4 \,\Box\, H$ has cardinality $4k$. Let $D = \{h_1, \dots, h_k\}$ be a minimum dominating set of H. Suppose first that H has a vertex x of degree at least 4. If $A = V(H) - N[x]$, then the set S defined by $S = ([4] \times \{x\}) \cup (\{1\} \times A)$ is a dominating set of $K_4 \,\Box\, H$. However, $|S| = 4 + |A| = 4 + (n(H) - |N[x]|) < 4k$, which is a contradiction. Therefore, $\Delta(H) \leq 3$. Since $V(H) = \cup_{i=1}^{k} N[h_i]$ and $n(H) = 4k$, it follows that $\deg(h_i) = 3$, for every $i \in [k]$, and we see that $N_H[h_1], \dots, N_H[h_k]$ is a partition of $V(H)$. Let $M = \{4\} \times D$. The set M is independent in the well-dominated graph $K_4 \,\Box\, H$ since $N_H[h_1], \dots, N_H[h_k]$ is a partition of $V(H)$. By Observation 5, it follows that the graph G defined by $G = K_4 \,\Box\, H - N[M]$ is well-dominated. Note that $G = K_3 \,\Box\, F$, where F is the subgraph of H induced by $V(H) - D$. This implies that each component of G is well-dominated. Using Proposition 23 we infer that each component of F has order at most 3. Furthermore, $\Delta(F) \leq 2$ since $\Delta(H) = 3$.

Note that $K_3 \,\Box\, K_2$ is not well-dominated, which then implies that each component of F has order 1 or 3. At least one of the components of F has order 3, for otherwise H is not connected. For each $i \in [k]$, let $X_i = N(h_i) = \{x_{i1}, x_{i2}, x_{i3}\}$. Each component of F that has order 3 intersects either one, two or three of the sets $X_1, \dots, X_k$. Suppose first that there exists $i \in [k]$, say $i = 1$, such that $\langle X_i \rangle$ is a component of F. This implies that the subgraph of H induced by $X_1 \cup \{h_1\}$ is a component of H of order 4, which contradicts the assumption that H is connected and has order at least 5. We thus assume that each component of F that has order 3 has a nonempty intersection with either two or three of the sets $X_1, \dots, X_k$. Any such component is clearly either a path of order 3 or a complete graph of order 3.

Suppose there exists $1 \leq i < j \leq k$ such that $\langle X_i \cup X_j \rangle$ contains a P_3 or K_3 involving at least one vertex from each of X_i and X_j. Without loss of generality we assume that $i = 1$ and $j = 2$ and that x_{13} has degree at least 2 in $\langle X_1 \cup X_2 \rangle$. We assume without loss of generality that $x_{13}x_{21} \in E(H)$. If x_{13} is adjacent to another vertex in X_2, say $x_{13}x_{22} \in E(H)$, then let $B = (\{1\} \times \{h_1, x_{11}, x_{12}, x_{23}\}) \cup \{(2, x_{21}), (3, x_{21}), (4, x_{22})\}$. On the other hand, if x_{13} is adjacent to another vertex in X_1, say $x_{13}x_{12} \in E(H)$, then let $B = (\{1\} \times \{h_1, x_{11}, x_{22}\}) \cup \{(2, x_{12}), (2, x_{23}), (3, x_{21}), (4, x_{21})\}$. In both cases we see that $B \cup ([4] \times \{h_3, \dots, h_k\})$ dominates $K_4 \,\Box\, H$ and has cardinality $4k - 1$, which is a contradiction.

Hence, every component of F that has order 3 contains one vertex from three distinct members of the partition $X_1, \dots, X_k$ of $V(F)$. We assume without loss of generality that x_{11}, x_{21}, x_{31} is a path in F. Let $B = (\{1\} \times \{h_1, h_2, x_{12}, x_{13}, x_{22}, x_{23}, x_{32}\}) \cup \{(2, x_{11}), (2, x_{33}), (3, x_{31}), (4, x_{31})\}$. It now follows that $B \cup ([4] \times \{h_4, \dots, h_k\})$ is a dominating set of $K_4 \,\Box\, H$ and has cardinality $4k - 1$, which is a contradiction. Therefore, if H is a connected graph of order more than 4, then $K_4 \,\Box\, H$ is not well-dominated.

Now let $m \geq 5$. Our inductive hypothesis is that if G is any connected graph of order at least m, then $K_{m-1} \,\square\, G$ is not well-dominated. Again, for the sake of arriving at a contradiction, suppose there exists a connected graph H of order more than m such that $K_m \,\square\, H$ is well-dominated. For consistency and ease of understanding we use the same notation as in the case $m = 4$. Let $k = \gamma(H)$ and let $D = \{h_1, \ldots, h_k\}$ be a minimum dominating set of H. By Proposition 16, $\gamma(K_m \,\square\, H) = n(H) = mk$, and hence every minimal dominating set of the well-dominated graph $K_m \,\square\, H$ has cardinality mk. Suppose first that H has a vertex x of degree at least m. If $A = V(H) - N[x]$, then the set S defined by $S = ([m] \times \{x\}) \cup (\{1\} \times A)$ is a dominating set of $K_m \,\square\, H$. However, $|S| = m + |A| = m + (n(H) - |N[x]|) < mk$, which is a contradiction. Therefore, $\Delta(H) \leq m - 1$.

Since $V(H) = \cup_{i=1}^{k} N[h_i]$, it follows that $\deg(h_i) = m - 1$, for every $i \in [k]$ and $N_H[h_1], \ldots, N_H[h_k]$ is a partition of $V(H)$. Similar to the case above (for $m = 4$) we let M be the independent set defined by $M = \{m\} \times D$, and we note by Observation 5 that $G = K_m \,\square\, H - N[M]$ is well-dominated, and hence every component of G is well-dominated. Since G is isomorphic to $K_{m-1} \,\square\, F$, where F is the subgraph of H induced by $V(H) - D$, it follows from the inductive hypothesis and Proposition 21 that every nontrivial component of F is isomorphic to K_{m-1}. Also, since $K_{m-1} \,\square\, F$ is well-dominated, it follows by Proposition 16 that $\gamma(K_{m-1} \,\square\, F) = (m-1)\gamma(F) = n(F) = k(m-1)$. Thus, $\gamma(F) = k$, which implies that F has no components of order 1. That is, F is the disjoint union of k complete graphs of order $m - 1$. Suppose that for some $i \in [k]$ there exists a vertex $x \in N_H(h_i)$ such that $N_F(x) \subseteq N_H(h_i)$. Since the component, say C_x, of F that contains x is a complete graph of order $m - 1$, we infer that $C_x = N_H(h_i)$, which implies that H is not connected. This contradiction means there is no such $i \in [k]$. In particular, for each $u \in N_H(h_1)$ we have $N(u) \cap (N_H(h_2) \cup \cdots \cup N_H(h_k)) \neq \emptyset$. Let

$$S = \left(\{1\} \times \cup_{i=2}^{k} N_H[h_i]\right) \cup (\{2, \ldots, m\} \times \{h_1\}).$$

This set S dominates $K_m \,\square\, H$ and $|S| = m(k-1) + (m-1) = mk - 1$, which is a contradiction. This establishes the proposition. $\square$

By combining the results of Propositions 21, 22, 23, and 24, the proof of Theorem 3 is complete.

We close this section on well-dominated Cartesian products with the following conjecture.

Conjecture 1. *Let G and H be nontrivial connected graphs. If $G \,\square\, H$ is well-dominated, then at least one of G or H is a complete graph.*

If Conjecture 1 is true, then we would have a complete characterization of the well-dominated Cartesian products. That is, if Conjecture 1 is true, then by Theorem 3 it follows that the Cartesian product $G \,\square\, H$ of two nontrivial, connected graphs is well-dominated if and only if $G \,\square\, H = K_m \,\square\, K_m$ for some positive integer $m \geq 2$ or $G \,\square\, H = K_3 \,\square\, P_3$.

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