On well-covered Cartesian products

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August 7, 2018

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Abstract

In 1970, Plummer defined a well-covered graph to be a graph G in which all maximal independent sets are in fact maximum. Later Hartnell and Rall showed that if the Cartesian product $G \square H$ is well-covered, then at least one of G or H is well-covered. In this paper, we consider the problem of classifying all well-covered Cartesian products. In particular, we show that if the Cartesian product of two nontrivial, connected graphs of girth at least 4 is well-covered, then at least one of the graphs is K_2. Moreover, we show that $K_2 \square K_2$ and $C_5 \square K_2$ are the only well-covered Cartesian products of nontrivial, connected graphs of girth at least 5.

Keywords: well-covered graph, Cartesian product, 1-well-covered, isolatable vertex

AMS subject classification (2010): 05C69, 05C65

1   Introduction

A graph G is called well-covered if all maximal independent sets of G have the same cardinality. This class of graphs was introduced by Plummer [10] in 1970. To date the study of the class of well-covered graphs seems to be primarily concentrated on finding good characterizations of various subclasses. Examples of subclasses of well-covered graphs that have been characterized include those of girth 8 or more [2], those of girth at least 5 [4], those with neither 4-cycles nor 5-cycles [5], and subcubic [1]. See the survey articles by Plummer [11] and Hartnell [7].

Topp and Volkmann [12] initiated the study of well-covered graphs in the realm of several of the standard graph products. They proved that a lexicographic product $G[H]$ is well-covered if and only if both of G and H are well-covered. Although they

did not characterize the direct products that are well-covered, they did prove that if G and H have no isolated vertices and the direct product $G \times H$ is well-covered, then both of G and H are well-covered and $\alpha(G) \cdot |V(H)| = \alpha(H) \cdot |V(G)|$. (Here α is the vertex independence number.) Cartesian products turn out to be more difficult to deal with as far as well-covered is concerned. Topp and Volkmann showed that the Cartesian product of any two complete graphs is well-covered and the Cartesian product of two cycles is well-covered if and only if at least one of the cycles is C_3. Left unanswered in [12] was the question of whether a Cartesian product $G \square H$ being well-covered implies that at least one of G or H is well-covered.

Fradkin [6] pursued this question and showed that the Cartesian product of any two triangle-free graphs, neither of which is well-covered, is also not well-covered. Hartnell and Rall [8] settled the problem of Topp and Volkmann.

Theorem 1.1 ([8])**.** *If G and H are graphs such that $G \square H$ is well-covered, then at least one of G or H is well-covered.*

This suggests an interesting problem.

Problem 1. *For a given graph H, characterize those graphs G such that $G \square H$ is well-covered.*

In Section 2 we give two constructions that suggest the general solution to this problem will likely be difficult. In Section 3 we provide a partial solution to Problem 1 by first considering the Cartesian product of two nontrivial, connected graphs of girth at least 4 and show that if this Cartesian product is well-covered, then one of the two factors is K_2. Moreover, we show that for any two, nontrivial, connected graphs G and H, both with girth at least 5, the Cartesian product $G \square H$ is well-covered if and only if $G \square H \in \{K_2 \square K_2, C_5 \square K_2\}$.

2 Notation and Preliminary Results

A subset A of the vertex set of a graph is *independent* if the vertices in A are pairwise nonadjacent. If an independent set I of $V(G)$ is maximal (with respect to being independent, then every vertex in $V(G) - I$ has at least one neighbor in I. Thus, a maximal independent set I in G is also dominating. That is, $N[I] = V(G)$. The smallest cardinality, $i(G)$, of a maximal independent set in G is the *independent domination number* of G. The *vertex independence number* of G is the largest cardinality of the maximal independent sets of G and is denoted $\alpha(G)$. A graph G is *well-covered* if all the maximal independent sets of G have the same cardinality. Thus, G is well-covered if and only if $i(G) = \alpha(G)$. If J is any independent set in a graph G, then J can be extended to a maximal independent set of G by repeatedly adding new vertices that are not dominated by the current set. Thus, in a well-covered graph G any independent set (in particular, any vertex) is contained in an independent set of cardinality $\alpha(G)$.

If G and H are two finite, simple graphs, then their *Cartesian product*, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \square H$ if one of the following holds:

- $g_1 = g_2$ and $h_1 h_2 \in E(H)$,

- $h_1 = h_2$ and $g_1 g_2 \in E(G)$.

The graphs G and H are called the *factors* of $G \square H$. For a given $g \in V(G)$, the subgraph of $G \square H$ induced by the set of vertices $\{(g, h) \colon h \in V(H)\}$ is called an *H-layer* and is denoted by ${}^g H$. In a similar way, for a fixed $h \in V(H)$ the subgraph G^h induced by $\{(g, h) \colon g \in V(G)\}$ is called a *G-layer*. By definition, every H-layer is isomorphic to H, and every G-layer is isomorphic to G. A significant portion of this paper is dedicated to the study of Cartesian products where one of the factors is K_2. We assume that $V(K_2) = [2] = \{1, 2\}$. (For a positive integer n we use $[n]$ to denote the set of positive integers no larger than n.) The Cartesian product $G \square K_2$ is called the *prism of G* and G is called the *base* of the prism. In the prism of a graph G we will simplify the notation and write g^i in place of (g, i) for $i \in [2]$ and $g \in V(G)$. Also for $i \in [2]$, if $A \subseteq V(G)$ we write A^i to denote the set of vertices $\{a^i \colon a \in A\}$.

A vertex of degree 1 in a graph is called a *leaf* and its unique neighbor is called a *support vertex*. An edge incident with a leaf is a *pendant* edge. A support vertex that has more than one leaf as a neighbor is called a *strong* support vertex. If a graph G has a strong support vertex x, then G is not well-covered. This is easily seen by letting I be a maximal independent set of $G - N[x]$. The set $I \cup \{x\}$ is a maximal independent set of G, and yet if L denotes the set of leaves adjacent to x, then $I \cup L$ is a larger independent set. The *girth* of a graph G, denoted by $g(G)$, is the length of its shortest cycle unless G is a forest, in which case we define the girth to be ∞. For $u, v \in V(G)$, the length of the shortest path in G joining u to v is denoted $d_G(u, v)$, or simply $d(u, v)$ if the graph is clear from the context.

If I is any independent set in a well-covered graph G and J_1 and J_2 are maximal independent sets in the induced subgraph $G - N[I]$, then both $I \cup J_1$ and $I \cup J_2$ are maximal independent in G. It follows immediately that $|J_1| = |J_2|$, and thus $G - N[I]$ is well-covered. This important and useful property of a well-covered graph was proved by Finbow et al. [4]. We state its contrapositive form since we will often use it to show that a graph is not well-covered.

Lemma 2.1 ([4]). *If G is a graph and I is an independent set of G such that $G - N[I]$ is not well-covered, then G is not well-covered.*

A vertex w in an arbitrary graph H is *isolatable* in H if there exists an independent set M in H such that $\{w\}$ is a (isolated) component in $H - N[M]$. Equivalently, w is isolatable in H if there exists an independent set J of H such

that $V(H) - N[J] = \{w\}$. As in [4], we say a vertex x of a well-covered graph G is *extendable* in G if $G - x$ is well-covered and $\alpha(G - x) = \alpha(G)$. A well-covered graph G is *1-well-covered* if every $v \in V(G)$ is extendable in G. In [4] it is proved that in a well-covered graph the notions of extendable and not isolatable are equivalent.

Theorem 2.2 ([4]). *Let G be a well-covered graph. A vertex x of G is an extendable vertex of G if and only if x is not isolatable in G.*

The next result by Finbow and Hartnell [3] shows that graphs of girth at least 4 having no isolatable vertices are actually well-covered. We provide a slightly expanded proof for the sake of completeness.

Theorem 2.3 ([3]). *Let G be a graph with $g(G) \geq 4$. If no vertex in G is isolatable, then G is well-covered.*

Proof. Suppose that G is a graph of girth at least 4 in which no vertex is isolatable, but that G is not well-covered. Consider the set $\mathcal{F}$ of certain pairs of maximal independent sets of G defined by

$$\mathcal{F} = \{(L, K) : \ K \text{ and } L \text{ are maximal independent in } G \text{ and } |K| < |L| = \alpha(G)\}\,.$$

Since G is not well-covered, $\mathcal{F} \neq \emptyset$. Let $n = \min\{|K - L| : (L, K) \in \mathcal{F}\}$ and let $m = \min\{|L \cap N(w)| : (L, K) \in \mathcal{F}, n = |K - L| \text{ and } w \in K - L\}$. Choose a pair (L_0, K_0) from $\mathcal{F}$ such that $n = |K_0 - L_0|$ and a vertex $v \in K_0 - L_0$ such that $m = |L_0 \cap N(v)|$. Let $u \in L_0 \cap N(v)$ and set $J = L_0 - \{u\}$. Since u is not dominated by the independent set J and since u is not isolatable (G has no isolatable vertices by assumption), there is a vertex $x \in N(u) - N(J)$. Since $g(G) \geq 4$, $xv \notin E(G)$. Let $M = J \cup \{x\}$. We see that M is a maximum independent set of G, and thus $(M, K_0) \in \mathcal{F}$. If $x \notin K_0$, then $|K_0 - M| = n$ but $|M \cap N(v)| = m - 1$, which is a contradiction to the choice of m. On the other hand, if $x \in K_0$, then $|K_0 - M| = n - 1$, which is a contradiction to the choice of n. Consequently, $\mathcal{F}$ is empty, and G is well-covered. $\square$

Finbow et al. [4] were able to give a complete description of the well-covered graphs of girth at least 5. By analyzing this collection Pinter [9] observed the following characterization of connected 1-well-covered graphs of girth at least 5.

Theorem 2.4 ([9]). *If G is a nontrivial, connected well-covered graph of girth at least 5 that has no isolatable vertex, then $G \cong K_2$ or $G \cong C_5$.*

To see that a general solution to Problem 1 is most likely very difficult, consider the following. Let H be a graph with maximum degree k, let $n > k$, and let I be any maximal independent set of $K_n \square H$. If I does not intersect some K_n-layer, say K_n^h, then the vertices of K_n^h are dominated by $\cup_{w \in N(h)}(I \cap V(K_n^w))$. Since $n > k$, the Pigeonhole Principle implies that $|I \cap V(K_n^w)| \geq 2$ for some $w \in N(h)$. This

contradicts the independence of I, and so I contains exactly one vertex from each K_n-layer. Hence, $|I| = |V(H)|$ and we see that $K_n \square H$ is well-covered.

In fact, by generalizing this idea we can, for any graph H of maximum degree k, construct many graphs G that have girth 3 such that $G \square H$ is well-covered. Here is one family of such graphs. Let r be any positive integer. Let G be any graph whose vertex set $V(G)$ can be partitioned as $V_1 \cup \cdots \cup V_r$ in such a way that

1. The subgraph $G[V_i]$ of G is a complete graph of order at least 3 for each $i \in [r]$, and

2. For each $j \in [r]$ there is a subset W_j of V_j such that $|W_j| \geq k+1$ and $N(W_j) \subseteq V_j$.

It is clear that such a graph G is well-covered and $\alpha(G) = r$. We claim that $G \square H$ is a well-covered graph with independence number $r|V(H)|$. For each $i \in [r]$ let $G_i = G[V_i]$. Any subset of $V(G)$ formed by choosing one vertex from each of $W_1, \ldots, W_r$ is independent, and is in fact maximal independent, in G. In addition, for each $h \in V(H)$ we can choose a subset M_h of $W_1 \cup \cdots \cup W_r$ that contains one vertex from W_i for each $i \in [r]$ in such a way that $M_h \cap M_{h'} = \emptyset$ for any $h' \in N(h)$. This last sentence is true since $\Delta(H) = k$ and $|W_i| \geq k+1$. It now follows that the set $\cup_{h \in V(H)}(M_h \times \{h\})$ is a maximal independent set of $G \square H$ with cardinality $r|V(H)|$. On the other hand, let J be any maximal independent set of $G \square H$. Suppose there exists a G-layer, say G^h, such that $|J \cap V(G^h)| < r$. This implies that there exists $j \in [r]$ such that $J \cap (V_j \times \{h\}) = \emptyset$. In this case no vertex of $W_j \times \{h\}$ is dominated by $J \cap V(G^h)$. Since $|W_j \times \{h\}| > k = \Delta(H)$, we arrive at a contradiction. Hence, $|J \cap V(G^x)| \geq r$, for every $x \in V(H)$. Since $V(G)$ is covered by r complete subgraphs we conclude that $|J \cap V(G^x)| = r$ for every $x \in V(H)$. Consequently, $|J| = r|V(H)|$, and this implies that $G \square H$ is well-covered.
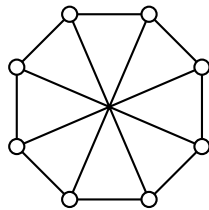

Figure 1: The graph H.

For a specific example, let H be the graph shown in Figure 1. Let G_1, G_2 and G_3 be complete graphs of order 10 with $V(G_1) = \{x_1, \ldots, x_{10}\}$, $V(G_2) = \{y_1, \ldots, y_{10}\}$, and $V(G_3) = \{z_1, \ldots, z_{10}\}$. The graph G is formed from the disjoint union of G_1, G_2, G_3 by adding any subset of edges among vertices in the set $\cup_{i=1}^{6}\{x_i, y_i, z_i\}$.

3 Factors of Girth at Least 4

If a graph G is not connected and has components $C_1, \ldots, C_m$, then for any graph H, the product $G \square H$ is the disjoint union of $C_1 \square H, \ldots, C_m \square H$. This graph, $G \square H$, is well-covered if and only if $C_i \square H$ is well-covered for every $i \in [m]$. Consequently, to determine which Cartesian products are well-covered we can restrict our attention to those in which both factors are connected. As the examples presented at the end of Section 2 show, the characterization of well-covered Cartesian products if at least one of the factors has a triangle is unlikely. So we focus on Cartesian products of connected factors that have girth larger than 3.

In this section we will prove the following characterization of well-covered Cartesian products of two connected graphs that each have girth at least 5.

Theorem 3.1. *Let G and H be nontrivial, connected graphs, both of which have girth at least 5. The Cartesian product $G \square H$ is well-covered if and only if $G \square H \cong K_2 \square K_2$ or $G \square H \cong C_5 \square K_2$.*

We first prove through a series of reductions that if G and H are both nontrivial, connected, triangle-free graphs such that $G \square H$ is well-covered, then $G \square H$ is a prism.

Lemma 3.2. *Let G and H be connected graphs, both of which have order at least 3 and girth at least 4. If either G or H has an isolatable vertex having degree at least 2, then $G \square H$ is not well-covered.*

Proof. Assume without loss of generality that G has an isolatable vertex x and let y_1 and y_2 be distinct neighbors of x. Let I be an independent set in G such that $G - N[I] = \{x\}$. Since H has order at least 3, we fix a vertex s in H with distinct neighbors t_1 and t_2. We assume without loss of generality that $\deg_H(t_1) \leq \deg_H(t_2)$. If $\deg_H(t_1) = 1 = \deg_H(t_2)$, then let $J = I \times \{t_1, t_2\}$. If $1 = \deg_H(t_1) < \deg_H(t_2)$, then let $J = (I \times \{t_1, t_2\}) \cup (\{y_1\} \times (N_H(t_2) - \{s\}))$. Finally, suppose that $1 < \deg_H(t_1)$. Let $T = N_H(\{t_1, t_2\}) - \{s\}$, let $A = T - N_H(t_2)$, let $B = T - A$, and let $J = (I \times \{t_1, t_2\}) \cup (\{y_1\} \times A) \cup (\{y_2\} \times B)$.

Since H is triangle-free and because of the definition of I, we see that all three cases above J is independent in $G \square H$. In addition, the vertex (x, s) is a strong support vertex adjacent to leaves (x, t_1) and (x, t_2) in $G \square H - N[J]$. Therefore, by Lemma 2.1 it follows that $G \square H$ is not well-covered. $\square$

The following corollary follows immediately from Lemma 3.2, Theorem 2.2, and Theorem 2.3.

Corollary 3.3. *Let G and H be connected graphs, both of which have minimum degree at least 2 and girth at least 4. If $G \square H$ is well-covered, then both G and H are 1-well-covered.*

Since a leaf in a graph of order at least 3 is isolatable, we are now able to weaken the hypothesis of Lemma 3.2 to cover the case when one of the factors has an isolatable vertex of degree 1 while making no assumption about the order of the other factor.

Lemma 3.4. *Let G and H be nontrivial, connected graphs both having girth at least 4. If G has minimum degree 1 and order at least 3, then $G \square H$ is not well-covered.*

Proof. Assume that x is a leaf in G that is adjacent to a support vertex y, and assume that $N_G(y) = \{x, z_1, \ldots, z_r\}$ where $r \geq 1$. Let $I = \{z_1, \ldots, z_r\}$. Note that since $g(G) \geq 4$, I is independent in G and x is isolated in $G - N[I]$. Fix any vertex s of H and assume that $N_H(s) = \{t_1, \ldots, t_m\}$. If $\deg_H(t_1) = 1$, then let $A = \emptyset$; otherwise, let $A = N_H(t_1) - \{s\}$. Let $J = (I \times N_H(s)) \cup (\{y\} \times A)$. Since both G and H have girth at least 4, it follows that J is independent in $G \square H$. The vertex (x, s) is a strong support vertex in $G \square H - N[J]$, adjacent to leaves (y, s) and (x, t_1). By Lemma 2.1 we conclude that $G \square H$ is not well-covered. $\square$

We now make the final reduction in the case when both nontrivial, connected factors have girth at least 4. Here we do not assume that one of the factors has an isolatable vertex.

Lemma 3.5. *If G and H are connected graphs both having order at least 3 and girth at least 4, then $G \square H$ is not well-covered.*

Proof. Assume that G and H are both of order at least 3 and girth at least 4. Suppose, in order to arrive at a contradiction, that $G \square H$ is well-covered. By Lemma 3.2 and Lemma 3.4, neither G nor H has an isolatable vertex, and hence $\delta(G) \geq 2$ and $\delta(H) \geq 2$. In addition, it follows from Theorem 2.3 that both G and H are well-covered. Fix a vertex s_2 in H, and distinguish one of its neighbors, say s_1. Let $N_H(s_2) = \{s_1, t_1, \ldots, t_m\}$ for some $m \geq 1$ and let $N_H(s_1) = \{s_2, z_1, \ldots, z_r\}$ for some $r \geq 1$. Select a vertex y in G, and let I be a maximal independent set of $G - N[y]$. Since y is not isolatable in G, the graph $G - N[I]$ contains at least one neighbor x of y. Furthermore, since $g(G) \geq 4$ and G is well-covered, it follows from Lemma 2.1 that $G - N[I]$ contains exactly the two adjacent vertices x and y. Let $w \in N_G(x) - \{y\}$ and let

$$J = (I \times \{s_1\}) \cup (N_G(y) \times \{t_1, \ldots, t_m\}) \cup (\{w\} \times \{z_1, \ldots, z_r\}).$$

We note that I was chosen to be independent; $N_G(y) \times \{t_1, \ldots, t_m\}$ is independent since both G and H are triangle-free; and $\{w\} \times \{z_1, \ldots, z_r\}$ is independent since $g(H) \geq 4$. As a result, J is independent by the definition of the edge set of $G \square H$. However, by appealing to Lemma 2.1, we now arrive at a contradiction since $G \square H - N[J]$ has a strong support vertex (y, s_1) that is adjacent to the leaves (y, s_2) and (x, s_1). Consequently, $G \square H$ is not well-covered. $\square$

Corollary 3.6. *If G and H are nontrivial, connected graphs with girth at least 4 such that $G\Box H$ is well-covered, then at least one of G or H is the graph K_2.*

Because of Corollary 3.6, for the remainder of the paper we restrict ourselves to prisms of graphs that have girth at least 4. The following corollary, concerning the prism of certain graphs, follows directly from Lemma 3.4.

Corollary 3.7. *If a connected graph G has girth at least 4, has order at least 3 and has minimum degree 1, then $G\Box K_2$ is not well-covered.*

Recall from Theorem 2.3 that a graph is well-covered if it has girth at least 4 and has no isolatable vertices. We next ask whether a graph G with an isolatable vertex can have a well-covered prism. The next result shows the answer is no if G has girth at least 5.

Theorem 3.8. *If G has girth at least 5 and has an isolatable vertex, then $G\Box K_2$ is not well-covered.*

Proof. By Corollary 3.7 we may assume G has minimum degree at least 2. Let x be an isolatable vertex of G, let $N(x) = \{y_1, \ldots, y_k\}$, and let A_i represent the vertices other than x that are adjacent to y_i for $i \in [k]$. Since G has girth at least 5, $N(x)$ is independent and the sets $A_1, \ldots, A_k$ are independent and pairwise disjoint.

Suppose first that one of the sets $A_1, \ldots, A_k$, say A_1, has cardinality 1. Let $A_1 = \{a\}$. Note by assumption that there exists an independent set J that isolates x in $G - N[J]$ and necessarily it must contain a. This implies that y_1^1 and x^2 are leaves adjacent to the same support vertex in $G\Box K_2 - N[J^2]$. We conclude that in this case $G\Box K_2$ is not well-covered.

So we may assume that $|A_i| \geq 2$ for all $i \in [k]$. Let I be a maximal independent set of $G - N[\{y_1, \ldots, y_k\}]$. It is possible that $V(G) = \{x, y_1, \ldots, y_k\} \cup \left(\cup_{i\in[k]} A_i\right)$, in which case $I = \emptyset$. This is handled in (b) below.

(a) Suppose first that I dominates at least one of the sets $A_1, \ldots, A_k$, say A_1. Let J be an independent set that isolates x in $G - N[J]$ and let $S = I^1 \cup \left(\cup_{i\in[k]}(J \cap A_i)\right)^2$. It follows that S is independent in $G\Box K_2$, and y_1^1 and x^2 are leaves in $G\Box K_2 - N[S]$ both adjacent to x^1. Again this implies that $G\Box K_2$ is not well-covered.

(b) Now suppose that there exists $i \in [k]$ such that I does not dominate at least two vertices of A_i. Since $|A_j| \geq 2$ for all $j \in [k]$, we note that this is necessarily the case in the event that $I = \emptyset$. Without loss of generality we may assume $i = 1$ and a_1 and b_1 are vertices in A_1 not dominated by I. Suppose first that $\deg_G(a_1) > 2$ and let $w \in N(b_1) - \{y_1\}$. Thus, w can be adjacent to only one vertex of $N(a_1) - \{y_1\}$ for otherwise G contains a 4-cycle. In this case, choose $z \in N(a_1) - \{y_1\}$ that is not adjacent to w and choose

$S = I^1 \cup \{y_2^1, \ldots, y_k^1, w^2, z^2\}$ so that a_1^1 and b_1^1 are leaves in $G \square K_2 - N[S]$. A similar argument can be used when $\deg_G(b_1) > 2$.

Next, suppose that $\deg_G(a_1) = 2$ and $\deg_G(b_1) = 2$ where $N(a_1) = \{y_1, z\}$ and $N(b_1) = \{y_1, w\}$. If $wz \notin E(G)$, then again choose $S = I^1 \cup \{y_2^1, \ldots, y_k^1, w^2, z^2\}$ so that a_1^1 and b_1^1 are leaves in $G \square K_2 - N[S]$. So we may assume that $wz \in E(G)$.

Suppose first that $z \notin \cup_{i=2}^k A_i$. Let J be any independent set in G that isolates x in $G - N[J]$. Let $M = (J - \{z\}) - (J \cap A_1)$ and choose $S = M^1 \cup \{z^2, y_2^2\}$. One can easily verify that S is indeed an independent set in $G \square K_2$. We claim that a_1^1 and x^1 are leaves in $G \square K_2 - N[S]$. Note that $N(a_1^1) = \{a_1^2, y_1^1, z^1\}$ and $N(x^1) = \{x^2, y_1^1, \ldots, y_k^1\}$. The vertices z^1 and a_1^2 are dominated by z^2 so a_1^1 is a leaf, M^1 dominates $y_2^1, \ldots, y_k^1$, and y_2^2 dominates x^2. Thus, x^1 is also a leaf. A similar argument works if $w \notin \cup_{i=2}^k A_i$.

So we may assume that $z \in A_i$ and $w \in A_j$ for some $i, j \in \{2, \ldots, k\}$. It follows that $i \neq j$ since A_i is an independent set. If every independent set that isolates x contains z, then none of these independent sets contain w as $wz \in E(G)$. In this case, let J be such an independent set that doesn't contain w, and let $M = J - (J \cap A_1)$. Let $S = M^1 \cup \{w^2, y_i^2\}$. Note that S is independent since $w \notin J \cup A_i$. We claim that b_1^1 and x^1 are leaves in $G \square K_2 - N[S]$. Note that $N(b_1^1) = \{b_1^2, w^1, y_1^1\}$ and $N(x^1) = \{x^2, y_1^1, \ldots, y_k^1\}$. The vertex w^2 dominates b_1^2 and w^1 so b_1^1 is a leaf. Since M^1 dominates $\{y_2^1, \ldots, y_k^1\}$ and y_i^2 dominates x^2, it follows that x^1 is a leaf. On the other hand, if there exists an independent set that isolates x and does not contain z, we may choose a set S so that a_1^1 and x^1 are leaves in $G \square K_2 - N[S]$. In each of these cases the removal of the closed neighborhood of an independent set from $G \square K_2$ created a strong support, and hence $G \square K_2$ is not well-covered by Lemma 2.1.

(c) Finally, suppose that I does not dominate exactly one vertex from each A_i for $i \in [k]$, say a_i. Note that if $\deg_G(x) \geq 3$, then there exist i and j such that $a_i a_j \notin E(G)$ for otherwise G would contain a triangle. Without loss of generality, we may assume $i = 1$ and $j = 2$. In this case, choose $S = I^1 \cup \{a_1^2, a_2^2\}$ and note that y_1^1 and y_2^1 are leaves, both adjacent to the same support vertex in $G \square K_2 - N[S]$. If $\deg_G(x) = 2$ and $a_1 a_2 \notin E(G)$, then the above set S still works. So we may assume that $a_1 a_2 \in E(G)$. Choose $S = I^1 \cup \{a_2^1, a_1^2, t^2\}$ where $t \in A_2 - \{a_2\}$. One can verify that S is an independent set. We claim that y_1^1 and x^2 are leaves, both adjacent to x^1 in $G \square K_2 - N[S]$. Note that $N(y_1^1) = \{x^1, y_1^2\} \cup A_1^1$ and $N(x^2) = \{x^1, y_1^2, y_2^2\}$. The set $I^1 \cup \{a_2^1\}$ dominates all of A_1^1 and $\{a_1^2, t^2\}$ dominates y_1^2 and y_2^2. Thus, y_1^1 and x^2 are indeed leaves. Hence, by Lemma 2.1 $G \square K_2$ is not well-covered.

Having considered all cases, we may conclude that $G \square K_2$ is not well-covered. $\square$

It is straightforward to verify that both of the prisms $K_2 \square K_2$ and $C_5 \square K_2$ are

well-covered. This together with Theorems 2.3, 2.4, 3.8 proves Theorem 3.1, which we restate here for completeness.

Theorem 3.1. *Let G and H be nontrivial, connected graphs, both of which have girth at least 5. The Cartesian product $G \square H$ is well-covered if and only if $G \square H \cong K_2 \square K_2$ or $G \square H \cong C_5 \square K_2$.*

We now show that, regardless of girth, if the base graph has no isolatable vertices, then its prism is well-covered. Of course by the characterization theorems of Finbow et al. every well-covered graph of girth larger than 5 has an isolatable vertex.

Theorem 3.9. *If G is a well-covered graph with no isolatable vertices, then $G \square K_2$ is well-covered.*

Proof. Let $m = \alpha(G)$ and let I be any maximal independent set of $G \square K_2$. Suppose that $|I \cap V(G^1)| < m$. If necessary, enlarge $I \cap V(G^1)$ to an independent set A of G^1 such that $|A| = m - 1$. The subgraph $G^1 - N[A]$ is a clique of order at least 2, for otherwise G^1 would contain an independent set of cardinality at least $m + 1$, which is a contradiction. This implies that $G^1 - N[I \cap V(G^1)]$ contains a clique C with vertices $x_1^1, \ldots, x_k^1$ for some $k \geq 2$. The vertices in C are dominated by the set $I \cap V(G^2)$, which implies that $x_1^2, x_2^2 \in I$. Since C is a clique, we have that $x_1 x_2 \in E(G)$, which contradicts the assumption that I is an independent set. Consequently, $|I \cap V(G^1)| = m$. Similarly, $|I \cap V(G^2)| = m$. We conclude that $|I| = 2m$, and hence $G \square K_2$ is well-covered. $\qquad \square$

In particular, if $g(G) = 4$ and G has no isolatable vertex, then by Theorem 2.3 it follows that G is well-covered. This proves the following corollary to Theorem 3.9.

Corollary 3.10. *If G has girth 4 and has no isolatable vertex, then the prism of G is well-covered.*

There are many examples of 1-well-covered graphs of girth 4 as in the hypothesis of Corollary 3.10. One such graph, commonly known as WL_8, is shown in Figure 2.
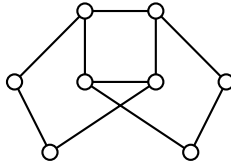

Figure 2: The graph WL_8.

4   Summary

We have shown that if a Cartesian product of two nontrivial, connected, triangle-free graphs is well-covered, then this Cartesian product is a prism, say $G \square K_2$. In

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addition, if G has girth at least 5, then G is either K_2 or C_5, and indeed the prisms $K_2 \square K_2$ and $C_5 \square K_2$ are both well-covered. If the girth of G is 4 and G has no isolatable vertex, then the prism $G \square K_2$ is well-covered. We suspect that being well-covered and having no isolatable vertex is also a necessary condition for the prism of a graph of girth 4 to be well-covered. We end by stating this as a conjecture.

Conjecture 4.1. *Let G be a connected, triangle-free graph that contains a cycle of order 4. If $G \square K_2$ is well-covered, then G has no isolatable vertex.*

Acknowledgements

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