On well-edge-dominated graphs

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Abstract

A graph is said to be well-edge-dominated if all its minimal edge dominating sets are minimum. It is known that every well-edge-dominated graph G is also equimatchable, meaning that every maximal matching in G is maximum. In this paper, we show that if G is a connected, triangle-free, nonbipartite, well-edge-dominated graph, then G is one of three graphs. We also characterize the well-edge-dominated split graphs and Cartesian products. In particular, we show that a connected Cartesian product $G\Box H$ is well-edge-dominated, where G and H have order at least 2, if and only if $G\Box H = K_2\Box K_2$.

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1 Introduction

A set F of edges in a graph G is an *edge dominating set* if every edge of G that is not in F is adjacent to at least one edge in F. Mitchell and Hedetniemi [14] initiated the study of edge domination by presenting a linear algorithm that finds a smallest edge dominating set in a tree. Yannakakis and Gavril [18] showed that it is NP hard to find an edge dominating set of minimum size even when restricted to planar graphs or subcubic bipartite graphs. See [3, 8, 9] for additional results on the complexity of finding a minimum edge dominating set. Frendrup, Hartnell and Vestergaard [7] first initiated the study of *well-edge-dominated* graphs which have the property that all of its minimal edge dominating sets have the same cardinality, although this term was not used in the paper. In fact, the focus of [7] was the study of *equimatchable* graphs which have the property that all of its maximal matchings have the same cardinality. Frendrup et al. pointed out that since a maximal matching is also a minimal edge dominating set, the class of equimatchable graphs contains the subclass of well-edge-dominated graphs. Furthermore, they state that every equimatchable graph of girth 5 or more is also well-edge-dominated. However, the collection of well-edge-dominated graphs is a proper subcollection of the equimatchable graphs as $K_{3,2}$ is equimatchable yet is not well-edge-dominated. Therefore, the study of well-edge-dominated graphs is only different from the study of equimatchable graphs if one focuses on graphs of girth at most 4.

Equimatchable graphs were first studied independently by Lewin [12] and Meng [13] in 1974. Lesk, Plummer and Pulleyblank [11] gave a characterization of equimatchable graphs that gave rise to a polynomial time algorithm for recognizing membership in this class of graphs. Since then the structure of several subclasses of equimatchable graphs have been investigated. Frendrup, Hartnell and Vestergaard [7] proved that a connected equimatchable graph with no cycles of length less than 5 is either a 5-cycle, a 7-cycle or belongs to the family $\mathcal{C}$ that contains K_2 and all the bipartite graphs one of whose partite sets consists of all its support vertices. Büyükçolak, Gözüpek and S. Özkan [4] provided a complete structural characterization of the connected, triangle-free equimatchable graphs that are not bipartite. Yildiz [19] provided a linear time algorithm for recognizing an equimatchable split graph.

In this paper, we completely characterize three classes of connected well-edge-dominated graphs. Our main result on triangle-free, nonbipartite well-edge-dominated graphs is the following result, which is proved in Section 4. We use the characterization, mentioned above, by Büyükçolak, et al. [4], of the equimatchable graphs satisfying the hypothesis of Theorem 1 and determine which of these belong to the smaller class of well-edge-dominated graphs. In what follows, the graph C_7^* is the graph obtained from C_7 by adding a chord between two vertices of C_7 that are distance 3 apart.

Theorem 1. *If G is a connected, nonbipartite, well-edge-dominated graph of girth at least* 4*, then* $G \in \{C_5, C_7, C_7^*\}$.

A graph is a *split graph* if its vertex set admits a partition into two sets, one of which is independent and the other which induces a complete graph. We show that a connected split graph is well-edge-dominated if and only if it is a star, a complete graph of order at most 4, a graph obtained from C_5 by adding two adjacent chords, or belongs to one of two families of graphs constructed from K_4. These are defined in Section 5.

In Section 6 we finish by showing that C_4 is the only nontrivial, connected, well-edge-dominated Cartesian product. Furthermore, we prove that the Cartesian product of two connected, nontrivial graphs is well-edge-dominated if and only if it is equimatchable.

Theorem 2. *Let G and H be two connected, nontrivial graphs. The following statements are equivalent.*

(a) $G \Box H$ *is equimatchable.*

(b) $G \Box H$ *is well-edge-dominated.*

(c) $G = H = K_2$.

2 Preliminaries

All the graphs considered in this paper are simple and have finite order. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $n(G) = |V(G)|$. If $n(G) \geq 2$, then G is *nontrivial.* For a positive integer k the set of positive integers no larger than k is denoted $[k]$. Although edges are 2-element subsets of vertices, for simplicity we will shorten the notation of an edge $\{u, v\}$ to uv. If $X \subseteq E(G)$, then $G - X$ is the graph with vertex set $V(G)$ and edge set $E(G) - X$. For graphs G and H, the Cartesian product $G \Box H$ has vertex set $\{(g, h) : g \in V(G), h \in V(H)\}$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \Box H$ if either $g_1 = g_2$ and $h_1h_2 \in E(H)$ or $h_1 = h_2$ and $g_1g_2 \in E(G)$. For $g \in V(G)$ the *H-fiber* ${}^gH$ is the subgraph of $G \Box H$ induced by the set $\{(g, h) : h \in V(H)\}$. Similarly, the *G-fiber* G^h for a given vertex $h \in V(H)$ denotes the subgraph induced by $\{(g, h) : g \in V(G)\}$. Note that ${}^gH$ is isomorphic to H and G^h is isomorphic to G.

Two distinct edges e and f in a graph G are *adjacent* if $e \cap f \neq \emptyset$ and are *independent* if $e \cap f = \emptyset$. A vertex x of G is *incident* to an edge e if $x \in e$. If

$X \subseteq E(G)$, then the set of vertices *covered* by X is denoted by $S(X)$ and is defined by $S(X) = \{u \in V(G) : u$ is incident to an edge in $X\}$. Let $f \in E(G)$ and let $F \subseteq E(G)$. The *closed edge neighborhood* of f is the set $N_e[f]$ consisting of f together with all edges in G that are adjacent to f. The *closed edge neighborhood* of F is the set $N_e[F]$ defined by $N_e[F] = \cup_{f \in F} N_e[f]$. Let $f \in F$. The edge f is said to *dominate* the set $N_e[f]$. An edge g is called a *private edge neighbor* of f with respect to F if $g \in N_e[f] - N_e[F - \{f\}]$. If $N_e[F] = E(G)$, then F is called an *edge dominating set* of G. The *edge domination number* of G, denoted by $\gamma'(G)$, is the smallest cardinality of an edge dominating set in G, and the *upper edge domination number* of G is the largest cardinality, $\Gamma'(G)$, of a minimal edge dominating set. A *matching* in G is a set of independent edges. The *matching number* of G, denoted $\alpha'(G)$, is the number of edges in a matching of largest cardinality in G, while the *lower matching number* is the number of edges, denoted by $i'(G)$, in a smallest maximal matching. Any maximal matching M in G is clearly a minimal edge dominating set of G, which gives

$$\gamma'(G) \leq i'(G) \leq \alpha'(G) \leq \Gamma'(G)\,.$$

A graph G is called *equimatchable* if $i'(G) = \alpha'(G)$ and is called *well-edge-dominated* if $\gamma'(G) = \Gamma'(G)$. Using the inequality above it is clear that the class of well-edge-dominated graphs is a subclass of the equimatchable graphs.

It is clear that a graph is well-edge-dominated (respectively, equimatchable) if and only if each of its components is well-edge-dominated (respectively, equimatchable). We use this fact throughout the paper together with the following lemmas.

A very useful tool in our study of well-edge-dominated graphs is the following lemma, which is the "edge version" of a fact used by Finbow, Hartnell and Nowakowski in [6]. The first statement follows from the fact that $M \cup D_1$ and $M \cup D_2$ are both minimal edge dominating sets of G for any matching M and any pair D_1 and D_2 of minimal edge dominating sets of the graph $G - N_e[M]$. The second statement follows similarly since for a matching M of G and any pair M_1 and M_2 of maximal matchings of $G - N_e[M]$, the two matchings $M \cup M_1$ and $M \cup M_2$ are both maximal matchings of G.

Lemma 1. *Let M be any matching in a graph G. If G is well-edge-dominated, then $G - N_e[M]$ is well-edge-dominated. If G is equimatchable, then $G - N_e[M]$ is equimatchable.*

The next two results show that several common graph families contain only a small number of well-edge-dominated graphs.

Lemma 2. *A complete graph of order n is well-edge-dominated if and only if $n \leq 4$.*

Proof. Using the definition we see that the complete graphs of order at most 4 are well-edge-dominated. For the converse suppose $n \geq 5$. Label the vertices of K_n as $1, \ldots, n$ and consider the set $D = \{12, 13, \ldots, 1(n-1)\}$. We claim that D is a minimal edge dominating set. Indeed, $D - \{1j\}$ is not an edge dominating set since jn is not adjacent to any edge in $D - \{1j\}$. Therefore, D is in fact a minimal edge dominating set of cardinality $n-2$ where $n \geq 5$. On the other hand, we can choose a matching of K_n of cardinality $\lfloor \frac{n}{2} \rfloor$. Note that $n - 2 > \frac{n}{2}$ when $n \geq 5$. Thus, K_n is not well-edge-dominated. ☐

Any star is well-edge-dominated and we show in Theorem 4 that $K_{n,n}$ is well-edge-dominated for any $n \geq 1$. No other complete bipartite graph is well-edge-dominated as the following lemma shows.

Lemma 3. *If* $2 \leq r < s$*, then* $K_{r,s}$ *is not well-edge-dominated.*

Proof. Assume $2 \leq r < s$ and write the partite sets of $K_{r,s}$ as $\{x_1, \ldots, x_r\}$ and $\{y_1, \ldots, y_s\}$. Note that $\{x_1y_1, \ldots, x_1y_s\}$ and $\{x_1y_1, x_2y_2, \ldots, x_ry_r\}$ are two minimal edge dominating sets of different cardinalities. Therefore, $K_{r,s}$ is not well-edge-dominated. ☐

3 Randomly matchable graphs

A graph is said to be *randomly matchable* if every maximal matching is a perfect matching. That is, a randomly matchable graph is an equimatchable graph whose matching number is half its order. Sumner [16] determined all the randomly matchable graphs.

Theorem 3. ([16]) *A connected graph is randomly matchable if and only if it is isomorphic to* K_{2n} *or* $K_{n,n}$ *for* $n \geq 1$.

Using Theorem 3 we can now show which randomly matchable graphs are well-edge-dominated.

Theorem 4. *A connected graph* G *containing a perfect matching is well-edge-dominated if and only if* $G = K_4$ *or* $G = K_{n,n}$ *for* $n \geq 1$.

Proof. Suppose first that G contains a perfect matching and is well-edge-dominated. It follows that G is equimatchable and every maximal matching is of size $n(G)/2$. Therefore, G is randomly matchable and by Theorem 3, $G = K_{2n}$ or $G = K_{n,n}$ for $n \geq 1$. By Lemma 2, K_{2n} for $n \geq 3$ is not well-edge-dominated. It follows that $G = K_4$ or $G = K_{n,n}$ for $n \geq 1$.

In the other direction, suppose $G = K_4$ or $G = K_{n,n}$ for $n \geq 1$. One can easily verify that K_4 is well-edge-dominated. Therefore, we shall assume $G = K_{n,n}$ and let A and B be the partite sets of G. We show that G is well-edge-dominated. Let D be an edge dominating set of G. Suppose D does not cover $a \in A$ and $b \in B$. Then ab is not dominated by D, which is a contradiction. Thus, we may assume D covers A which implies $|D| \geq n$. Suppose that $|D| > n$. It follows that some vertex of A is incident to two edges in D, say e and f. Note that $D - \{e\}$ is an edge dominating set of G since $D - \{e\}$ covers A and every edge of G is incident to exactly one vertex of A. Thus, $|D| = n$ and G is well-edge-dominated.

□

4 Triangle-free nonbipartite graphs

In this section we prove there are only three nonbipartite, triangle-free, connected, well-edge-dominated graphs. These three graphs are the 5-cycle, the 7-cycle and C_7^*, which is depicted in Figure 1.

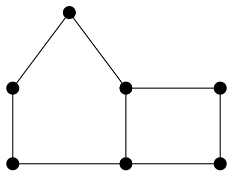

Figure 1: The graph C_7^*

We will use the structural characterization of the class of triangle-free, equimatchable graphs in the recent paper of Büyükçolak, Gözüpek and Özkan [4]. To describe their characterization, they defined several graph families using the following notation. Let H be a graph on k vertices $v_1, v_2, \ldots, v_k$ and let $m_1, m_2, \ldots, m_k$ be nonnegative integers. Then $H(m_1, m_2, \ldots, m_k)$ denotes the graph obtained from H by repeatedly replacing each vertex v_i with an independent set of m_i vertices, each of which has the same neighborhood as v_i. For example, using the graph G^* in Figure 2, we see that $G^*(1,1,1,0,1,1,1,1,0,0,0) = C_7$ and $G^*(2,0,0,0,3,0,0,0,2,3,0) = K_{4,6}$.

The following definition was made in [4].

Definition 1. ([4]) *Let G^* be the graph in Figure 2 and let $\mathcal{F}$ be the union of the following six graph families.*

1. $\mathcal{F}_{11} = \{G^*(1,1,1,1,1,n,n,0,0,0,0) : n \geq 1\}$
2. $\mathcal{F}_{12} = \{G^*(1,1,1,0,1,n+1,n+1,1,0,0,0) : n \geq 1\}$

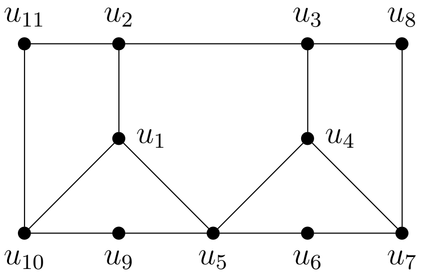

Figure 2: The graph G^*

3. $\mathcal{F}_{21} = \{G^*(1,1,1,n-r-s+1,1,r,n,s,0,0,0) : n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$

4. $\mathcal{F}_{22} = \{G^*(1,1,1,n-r-s,1,r+1,n+1,s+1,0,0,0) : n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$

5. $\mathcal{F}_3 = \{G^*(1,1,r+1,s+1,1,0,n-s,n-r,0,0,0) : n-1 \geq r \geq 1, n-1 \geq s \geq 1\}$

6. $\mathcal{F}_4 = \{G^*(r+1,n+1,s+1,1,1,0,0,0,0,0,n-r-s) : n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$

By analyzing each of the six families of equimatchable graphs listed above, we determine all the well-edge-dominated graphs in $\mathcal{F}$.

Proposition 1. *If $G \in \mathcal{F}$ is well-edge-dominated, then $G = C_7^*$.*

Proof. Throughout this proof when considering a graph from one of these six families of graphs we will always assume the variables (that is, whichever of n, r and s are used) satisfy the conditions in Definition 1 for that particular family.

First, let $G = G^*(1,1,1,1,1,n,n,0,0,0,0) \in \mathcal{F}_{11}$. Note first that if $n = 1$, then $G = C_7^*$ depicted in Figure 1. It is straightforward to show that C_7^* is well-edge-dominated. Suppose $n \geq 3$ and let $\{x_1, \ldots, x_n\}$ be the set of vertices that replace u_6 and let $\{y_1, \ldots, y_n\}$ be the set of vertices that replace u_7. Note that $K_{n-1,n}$ is a component of $G - N_e[\{x_1u_5, u_3u_4\}]$. By Lemma 3, we infer that G is not well-edge-dominated. Therefore, we shall assume $n = 2$. Now, $\{u_1u_2, u_3u_4\}$ is a matching, and $K_{2,3}$ is a component of $G - N_e[\{u_1u_2, u_3u_4\}]$. By Lemma 1 and Lemma 3, it follows that G is not well-edge-dominated.

Next, let $G = G^*(1,1,1,0,1,n+1,n+1,1,0,0,0) \in \mathcal{F}_{12}$. Let $\{x_1, \ldots, x_{n+1}\}$ be the set of vertices that replace u_6 and let $\{y_1, \ldots, y_{n+1}\}$ be the set of vertices that replace u_7. Suppose first that $n \geq 2$. Note that $K_{n,n+1}$ is a component of $G - N_e[\{x_1u_5, u_3u_8\}]$. Since $K_{n,n+1}$ is not well-edge-dominated by Lemma 3, it follows from Lemma 1 that G is not well-edge-dominated. Therefore, we shall assume $n = 1$. In this case, both $\{x_1y_1, x_2y_2, u_1u_5, u_3u_8\}$ and $\{x_1y_1, x_2y_1, u_8y_1, u_1u_5, u_2u_3\}$ are both minimal edge dominating sets, and hence G is not well-edge-dominated.

Next, let $G \in \mathcal{F}_{21} \cup \mathcal{F}_{22} \cup \mathcal{F}_4$. Note that $n \geq 2$ for every such G. Suppose $G = G^*(1,1,1,n-r-s+1,1,r,n,s,0,0,0) \in \mathcal{F}_{21}$. Note that $G - N_e[\{u_2u_3, u_1u_5\}] = K_{n,n+1}$. If $G = G^*(1,1,1,n-r-s,1,r+1,n+1,s+1,0,0,0) \in \mathcal{F}_{22}$, then $G - N_e[\{u_1u_5, u_2u_3\}] = K_{n+1,n+2}$. If $G = G^*(r+1,n+1,s+1,1,1,0,0,0,0,0,n-r-s) \in \mathcal{F}_4$, then $G - N_e[\{u_4u_5\}] = K_{n+1,n+2}$. Therefore, for every $G \in \mathcal{F}_{21} \cup \mathcal{F}_{22} \cup F_4$, we see by Lemmas 1 and 3 that G is not well-edge-dominated.

Finally, assume $G \in \mathcal{F}_3$ and $G = G^*(1,1,r+1,s+1,1,0,n-s,n-r,0,0,0)$. Let $\{x_1, \ldots, x_{s+1}\}$ be the set of vertices that replace u_4. The complete bipartite graph $K_{n-s+r+1,n-r+s}$ is a component of $G - N_e[\{u_1u_2, u_5x_1\}]$. Observe that $n-s+r+1 \neq n-r+s$ for otherwise $2r+1 = 2s$, which is not possible. Furthermore, using the conditions $n-1 \geq r \geq 1$ and $n-1 \geq s \geq 1$ we see that $n-s+r+1 \geq 3$ and $n-r+s \geq 2$. It follows by Lemma 3 that G is not well-edge-dominated. $\square$

Definition 2. ([4]) *Let G^* be the graph in Figure 2 and let $\mathcal{G}$ be the union of the following seven graph families.*

1. $\mathcal{G}_{11} = \{G^*(m+1,m+1,1,0,1,1,n+1,n+1,0,0,0) : n \geq 1, m \geq 1\}$
2. $\mathcal{G}_{12} = \{G^*(m+1,m+1,1,n-r-s,1,r+1,n+1,s+1,0,0,0) : m \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$
3. $\mathcal{G}_{21} = \{G^*(1,1,1,n-r-s+1,1,r,n,s,0,m,m) : m \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$
4. $\mathcal{G}_{22} = \{G^*(1,1,r+1,s+1,1,0,n-s,n-r,0,m,m) : m \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1\}$
5. $\mathcal{G}_{23} = \{G^*(r+1,n+1,s+1,1,1,m,m,0,0,0,n-r-s) : m \geq 1, n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s\}$
6. $\mathcal{G}_{31} = \{G^*(m-k-\ell+1,1,1,n-r-s+1,1,r,n,s,\ell,m,k) : n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s, m-1 \geq \ell \geq 1, m-1 \geq k \geq 1, m \geq k+\ell\}$
7. $\mathcal{G}_{32} = \{G^*(k+1,\ell+1,1,1,n-r-s+1,1,r,n,s,0,m-\ell,m-k) : n-1 \geq r \geq 1, n-1 \geq s \geq 1, n \geq r+s, m-1 \geq \ell \geq 1, m-1 \geq k \geq 1, m \geq k+\ell\}$

As we did in Proposition 1, an analysis of all the graphs in $\mathcal{G}$ will show that no such graph is well-edge-dominated.

Proposition 2. *If $G \in \mathcal{G}$, then G is not well-edge-dominated.*

Proof. Throughout this proof when considering a graph from one of these seven families of graphs we will always assume the variables (that is, whichever of n, m, r, s, k and ℓ are used) satisfy the conditions in Definition 2 for that particular family.

First, suppose $G \in \mathcal{G}_{11} \cup \mathcal{G}_{12}$. Let $\{x_1, \dots, x_{m+1}\}$ be the set of vertices that replace u_2 and let $\{y_1, \dots, y_{m+1}\}$ be the set of vertices that replace u_1. If $G = G^*(m+1, m+1, 1, 0, 1, 1, n+1, n+1, 0, 0, 0) \in \mathcal{G}_{11}$, then $K_{n+1,n+2}$ is a component of $G - N_e[\{x_1u_3, y_1u_5\}]$. If $G = G^*(m+1, m+1, 1, n-r-s, 1, r+1, n+1, s+1, 0, 0, 0) \in \mathcal{G}_{12}$, then $K_{n+1,n+2}$ is a component of $G - N_e[\{x_1u_3, y_1u_5\}]$. Since $n+1 \geq 2$, it follows from Lemmas 1 and 3 in both cases that G is not well-edge-dominated.

Next, suppose $G = G^*(1, 1, 1, n-r-s+1, 1, r, n, s, 0, m, m) \in \mathcal{G}_{21}$. Note that this implies $n \geq 2$ and $G - N_e[\{u_1u_5, u_2u_3\}]$ contains the component $K_{n,n+1}$. By Lemmas 1 and 3 we infer that G is not well-edge-dominated.

Next, suppose $G = G^*(1, 1, r+1, s+1, 1, 0, n-s, n-r, 0, m, m) \in \mathcal{G}_{22}$. Let $\{x_1, \dots, x_m\}$ be the set of vertices that replace u_{11} and let $\{y_1, \dots, y_{r+1}\}$ be the set of vertices that replace u_3. The complete bipartite graph $K_{n-r+s+1,n-s+r+1}$ is a component of $G - N_e[\{x_1u_2, u_1u_5\}]$. Note that $n - r + s + 1 \geq s + 2 \geq 3$ and $n-s+r+1 \geq r+2 \geq 3$. If $n-r+s+1 \neq n-s+r+1$, then $K_{n-r+s+1,n-s+r+1}$ is not well-edge-dominated by Lemma 3. On the other hand, if $n-r+s+1 = n-s+r+1$, then $G - N_e[\{u_2y_1, u_1u_5\}]$ has a component isomorphic to $K_{n-r+s+1,n-s+r}$, which is not well-edge-dominated. Again by Lemmas 1 and 3 we conclude that G is not well-edge-dominated.

Next, suppose $G = G^*(r+1, n+1, s+1, 1, 1, m, m, 0, 0, 0, n-r-s) \in \mathcal{G}_{23}$. The graph $K_{n+1,n+2}$ is a component of $G - N_e[u_4u_5]$. Using Lemmas 1 and 3 we infer that G is not well-edge-dominated.

Next, suppose $G = G^*(m-k-\ell+1, 1, 1, n-r-s+1, 1, r, n, s, \ell, m, k) \in \mathcal{G}_{31}$. Let $\{x_1, \dots, x_{m-k-\ell+1}\}$ be the set of vertices that replace u_1. Note that $n \geq 2$ and that $K_{n,n+1}$ is a component of $G - N_e[\{x_1u_5, u_2u_3\}]$. By Lemmas 1 and 3, this implies that G is not well-edge-dominated.

Finally, suppose $G = G^*(k+1, \ell+1, 1, n-r-s+1, 1, r, n, s, 0, m-\ell, m-k) \in \mathcal{G}_{32}$. Note that $n \geq 2$. Let $\{x_1, \dots, x_{\ell+1}\}$ be the set of vertices that replace u_2 and let $\{y_1, \dots, y_{k+1}\}$ be the set of vertices that replace u_1. Since $K_{n,n+1}$ is a component of $G - N_e[\{x_1u_3, y_1u_5\}]$, we conclude by Lemmas 1 and 3 that G is not well-edge-dominated.

□

Theorem 1 *If G is a connected, nonbipartite, well-edge-dominated graph of girth at least 4, then $G \in \{C_5, C_7, C_7^*\}$.*

Proof. It is straightforward to check that every graph in $\mathcal{F} \cup \mathcal{G}$ is connected, has girth 4 but is not bipartite. If we consider only nonbipartite graphs, then the main result of Büyükçolak, et. al [4, Theorem 36] states that a graph G is a connected, nonbipartite, triangle-free equimatchable graph if and only if $G \in \mathcal{F} \cup \mathcal{G} \cup \{C_5, C_7\}$. Applying Proposition 1 and Proposition 2 completes the proof. $\square$

5 Split graphs

Recall that a graph is a split graph if its vertex set can be partitioned into an independent set and a set that induces a complete graph. In this section we prove a complete characterization of the family of split graphs that are well-edge-dominated. We will use the following definitions. Let $\mathcal{H}_1$ be the family of graphs obtained by appending any finite number of leaves to a single vertex of K_4 and let $\mathcal{H}_2$ be the family of graphs obtained from K_4 by removing any edge uv and appending at least one leaf to u. Let H_3 be the graph of order 5 obtained from $K_4 - e$ by adding a new vertex adjacent to one of the vertices of degree 2 and one of the vertices of degree 3.

Lemma 4. *If $G \in \{K_2, K_3, K_4, H_3\} \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \{K_{1,n} : n \in \mathbb{N}\}$, then G is well-edge-dominated.*

Proof. By Lemma 2, K_2, K_3, and K_4 are well-edge-dominated. It is easy to see that every minimal edge dominating set of a nontrivial star $K_{1,n}$ consists of exactly one edge. Therefore, $K_{1,n}$ is well-edge-dominated. It is straightforward to check that all minimal edge dominating sets of H_3 have cardinality 2.

Next, assume $G \in \mathcal{H}_1$. Suppose the vertices v_1, v_2, v_3 and v_4 of G induce a complete graph and v_1 is the support vertex. Let D be a minimal edge dominating set of G. First assume that D contains an edge, say v_1w, where w is a leaf. Note that D cannot contain more than one edge incident with v_1 since D is minimal. The only edges not dominated by v_1w are v_2v_3, v_2v_4 and v_3v_4. Exactly one of those edges must be in D in order for it to be a minimal edge dominating set. Thus, $|D| = 2$. Next, assume D does not contain an edge incident to a leaf. Then $D \cap \{v_1v_2, v_1v_3, v_1v_4\} \neq \emptyset$. Without loss of generality, assume $v_1v_2 \in D$. The only edge of G not dominated by v_1v_2 is v_3v_4, so by minimality $|D| = 2$ and G is well-edge-dominated.

Now, assume $G \in \mathcal{H}_2$. Label the vertices of the K_4 as v_1, v_2, v_3 and v_4, remove the edge v_1v_3, and append leaves to vertex v_1. Let D be a minimal edge dominating set of G. Using a similar argument to the one above we conclude that G is well-edge-dominated.

$\square$

To show that we have identified all well-edge-dominated split graphs, we use the following result provided by Yildiz in [19].

Theorem 5 ([19]). *Let G be a simple undirected graph on $n \geq 4$ vertices with no isolated vertices. Let r and p be the number of vertices of degree 1 and $n-1$, respectively. G is an equimatchable split graph if and only if one of the following holds:*

(i) $p = n$.

(ii) $r = n-1$ *and* $p = 1$.

(iii) $p = 1$, $r \geq 2$, $n-r$ *is even, and all vertices have degree* 1, $n-r-1$, *or* $n-1$.

(iv) $p = 0$, $r \geq 2$, $n-r$ *is even, there are two vertices x and y with $xy \notin E(G)$ such that* $\deg(x) = n-2$, $\deg(y) = n-r-2$, *and all vertices in* $V(G) - \{x, y\}$ *have degree* 1 *or* $n-r-1$.

(v) There are two vertices x and y such that n is odd, $\deg(x) + \deg(y) = p + n - 2$ *and all vertices in* $V(G) - \{x, y\}$ *have degree* $n-1$ *or* $n-2$.

Theorem 6. *A nontrivial, connected split graph G is well-edge-dominated if and only if* $G \in \{K_2, K_3, K_4, H_3\} \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \{K_{1,n} : n \in \mathbb{N}\}$.

Proof. By Lemma 4, each graph in $\{K_2, K_3, K_4, H_3\} \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \{K_{1,n} : n \in \mathbb{N}\}$ is well-edge-dominated and is a split graph by definition.

For the converse let G be a connected, well-edge-dominated split graph. We let $V(G) = K \cup I$ where I is an independent set, $K = \{x_1, \ldots, x_k\}$, and $G[K]$ is a clique. Since G is equimatchable, G must be in one of the five classes provided in the statement of Theorem 5. As in Theorem 5, we shall assume the order of G is n, and G contains r vertices of degree 1 and p vertices of degree $n-1$. If G is in class (i), then $G = K_n$ and by Lemma 2, $G \in \{K_2, K_3, K_4\}$. If G is in class (ii), then $G = K_{1,n-1}$.

Therefore, we shall assume first that G is in class (iii). Let $L = \{a_1, \ldots, a_r\}$ be the set of vertices of degree 1, all of which are adjacent to x_1. By the given conditions in class (iii), $G - L$ is a clique of even order $2s$ for some $s \geq 1$. Since no vertex in this clique has degree 1, we get $2s \geq 4$. By Lemma 1, $G - N_e[\{x_1a_1\}]$ is a well-edge-dominated clique of order $2s-1$. It follows from Lemma 2 that $G \in \mathcal{H}_1$.

Next, assume G is in class (iv). Thus, $p = 0$, $r \geq 2$, $n-r$ is even, there are two vertices x and y with $xy \notin E(G)$ such that $\deg(x) = n-2$, $\deg(y) = n-r-2$, and all vertices in $V(G) - \{x, y\}$ have degree 1 or $n-r-1$. Since $r \geq 2$, $x \in K$ which implies that $y \in I$. We shall assume $x = x_1$. It follows that y is adjacent to all vertices of $K - \{x_1\}$ as $\deg(y) = n-r-2$. Furthermore, since all vertices in $I - \{y\}$ have degree 1 or $n-r-1$, $I - \{y\}$ only contains vertices of degree 1. Note that $n-r$ is even and therefore $|K|$ is odd. If $|K| = 1$, then y is isolated which contradicts our assumption that G is connected. If $|K| = 3$, then $G \in \mathcal{H}_2$. Therefore, we shall assume $|K| \geq 5$. Let ℓ be some vertex of degree 1 and notice $G - N_e[\{\ell x_1\}] = K_{|K|}$ is not well-edge-dominated. Hence, this case cannot occur.

Lastly, assume G is in class (v). There exist two vertices x and y such that n is odd, $\deg(x)+\deg(y) = p+n-2$ and all vertices in $V(G)-\{x,y\}$ have degree $n-1$ or $n-2$. If $|I| \geq 3$, then there exists a vertex in $I-\{x,y\}$ of degree neither $n-1$ or $n-2$. Thus, we may assume $|I| \leq 2$. Suppose first that $|I| = 1$. We shall write $I = \{u\}$. If u is adjacent to every vertex in K, then G is a clique and by Lemma 2 we see that $G = K_3$ since n is odd. So we shall assume $N(u) = \{x_1, \ldots, x_s\}$ where $s < k$. Must not use p here since p has a global meaning in the proof. I've changed it to s. Since n is odd, k is even. Let $M = \{x_1x_2, x_3x_4, \ldots, x_{k-1}x_k\}$ and let $M' = \{x_1x_2, x_1x_3, \ldots, x_1x_{k-1}\}$. We see that M is a maximal matching and therefore M is a minimal edge dominating set. In addition, note that M' is also a minimal edge dominating set since $M' - \{x_1x_j\}$ does not dominate x_jx_k, for $2 \leq j \leq k-1$. It follows that $k-2 = |M'| = |M| = k/2$, which gives $k = 4$.

If $s = 1$, then $G \in \mathcal{H}_1$. If $s = 2$, then G is not well-edge-dominated since $\{ux_1, x_1x_3, x_1x_4\}$ and $\{ux_1, x_2x_4\}$ are minimal edge dominating sets. If $s = 3$, then $\{x_1x_2, x_3x_4\}$ and $\{ux_1, ux_2, ux_3\}$ are minimal edge dominating sets so G is not well-edge-dominated. Thus, we assume for the remainder of the proof that $|I| = 2$. In particular, we assume that G does not have a clique of order $n-1$. I think we need this assumption to be able to assume that $\deg(u) < n-2$ for each $u \in I$ since the "clique-independent set partition" is not always unique in a split graph.

It follows that $I = \{x, y\}$ as $\deg(u) < n-2$ for each $u \in I$. For the time being we shall assume $k \geq 4$. Suppose first that we can find $1 \leq i < j \leq k$ such that x_i is adjacent to x and x_j is adjacent to y. Reindexing if necessary, we may assume x_1 is adjacent to x and x_2 is adjacent to y. Let $M = \{x_3x_4, x_5x_6, \ldots, x_{k-2}x_{k-1}\}$. Thus, $M \cup \{xx_1, yx_2\}$ and $M \cup \{x_1x_2\}$ are two different maximal matchings in G, which is a contradiction. Therefore, we may assume that x and y are adjacent to exactly one vertex in K, say x_1. This implies $|K| = 1$ for otherwise K contains vertices of degree at most $n-3$, which is a contradiction. Hence, we shall assume $k \in \{1, 2, 3\}$. Furthermore, n is assumed to be odd so $k \in \{1, 3\}$. If $k = 1$, $G = K_{1,2}$. So we shall assume $k = 3$. Moreover, we may assume x_1 is adjacent to x. Let us assume first that G contains a vertex of degree $n-1$, say x_1. If $\{xx_2, xx_3, yx_2, yx_3\} \cap E(G) = \emptyset$, then G is not equimatchable. Thus, assume $xx_2 \in E(G)$ without loss of generality. If $yx_3 \notin E(G)$ and $xx_3 \notin E(G)$, then again G is not equimatchable. Therefore, we will assume first that $yx_3 \in E(G)$. If G contains no other edges, then $G = H_3$. On the other hand, one can easily verify that adding any additional edges to G will result in a graph which is not well-edge-dominated. Next, we will assume $yx_3 \notin E(G)$ and $xx_3 \in E(G)$. If G

contains no other edges, then $G \in \mathcal{H}_1$. Thus, we shall assume $yx_2 \in E(G)$. One can easily verify that G is not well-edge-dominated. Finally, suppose G does not contain a vertex of degree $n-1$. Without loss of generality, we may assume xx_1 and yx_2 are edges in G and xx_2 and yx_1 are not edges in G. If G contains no other edges, then G is not equimatchable. Therefore, we may assume $yx_3 \in E(G)$. If G contains no other edges, then $G \in \mathcal{H}_2$. If $xx_3 \in E(G)$, then $G = H_3$.

□

6 Cartesian products

This section is devoted to proving our characterization of well-edge-dominated Cartesian products. In the process we show that among connected graphs that are the Cartesian product of nontrivial factors, the concepts of equimatchable and well-edge-dominated coincide.

Lemma 5. *Let G and H be nontrivial, connected graphs such that at least one of G or H has order at least 3. If G has a perfect matching, then $G \Box H$ is not well-edge-dominated.*

Proof. Suppose G admits a perfect matching M and suppose for the sake of contradiction that $G \Box H$ is well-edge-dominated. By “copying” M to each G-fiber we see that $G \Box H$ also has a perfect matching. Suppose $G \Box H$ has order $2n$. Since $3n \geq 6$, it follows by Theorem 4 that $G \Box H = K_{n,n}$. This is a contradiction since no complete bipartite graph of order at least 6 is the Cartesian product of nontrivial factors. □

We now prove Theorem 2, which is restated here for ease of reference.

Theorem 2 *Let G and H be two connected, nontrivial graphs. The following statements are equivalent.*

(a) $G \Box H$ is equimatchable.

(b) $G \Box H$ is well-edge-dominated.

(c) $G = H = K_2$.

Proof. The Cartesian product $K_2 \Box K_2$ is clearly well-edge-dominated, so statement (c) implies (b). As noted in Section 2, the well-edge-dominated graphs are a subclass of the class of equimatchable graphs. Thus (b) implies (a). To prove the final implication suppose G and H are connected and nontrivial such that $G \Box H$ is equimatchable. Suppose first that at least one of the graphs, say G, contains a path of order 4. Let $V(G) = \{g_1, g_2, \ldots, g_p\}$ for some $p \geq 4$ such that $\{g_1g_2, g_2g_3, g_3g_4\} \subseteq E(G)$. Let $V(H) = \{h_1, \ldots, h_q\}$ such that $h_1h_2 \in E(H)$ and let M be the matching of $G \Box H$ defined by

$$M = \left(\bigcup_{i=4}^{p} \{(g_i, h_1)(g_i, h_2)\} \right) \cup \left(\bigcup_{j=3}^{q} \{(g_1, h_j)(g_2, h_j)\} \right) \cup \left(\bigcup_{j=3}^{q} \{(g_3, h_j)(g_4, h_j)\} \right).$$

One of the components of $G \Box H - N_e[M]$ is isomorphic to either $P_3 \Box P_2$ or $K_3 \Box P_2$. Both of these have maximal matchings of size 2 and 3 and are therefore not well-edge-dominated. This contradicts Lemma 1, which allows us to assume that neither G nor H contains a path of order 4. We infer that $\{G, H\} \subseteq \{K_3, K_2\} \cup \{K_{1,n} : n \geq 2\}$. We now show that among all such Cartesian products the only one that is equimatchable is $K_2 \Box K_2$.

None of $K_3 \Box K_2$, $K_3 \Box K_3$, or $K_3 \Box K_{1,2}$ is equimatchable. Indeed, it is easy to see that $K_3 \Box K_2$ has maximal matchings of sizes 2 and 3, while both $K_3 \Box K_3$ and $K_3 \Box K_{1,2}$ admit maximal matchings of sizes 3 and 4. Now let $n \geq 3$, let $V(K_{1,n}) = \{x, x_1, \ldots, x_n\}$, let $E(K_{1,n}) = \{xx_i : i \in [n]\}$, and let $V(K_3) = \{a, b, c\}$. The sets of edges $M_1 = \{(a,x)(c,x), (b,x)(b,x_1)\} \cup \{(a,x_i)(c,x_i) : i \in [n]\}$ and $M_2 = \{(a,x)(a,x_1), (b,x)(b,x_2), (c,x)(c,x_3), (b,x_1)(c,x_1), (a,x_2)(c,x_2), (a,x_3)(b,x_3)\} \cup \{(a,x_j)(c,x_j) : 4 \leq j \leq n\}$ are maximal matchings of size $n+2$ and $n+3$, respectively.

If $n \geq 2$, then $K_2 \Box K_{1,n}$ has a perfect matching but is not equimatchable by Theorem 3. For $m \geq 3$, let $V(K_{1,m}) = \{y, y_1, \ldots, y_m\}$, and let $E(K_{1,m}) = \{yy_i : i \in [m]\}$. For $K_{1,n}$ as described in the paragraph above, let $M = \{(x_i, y)(x_i, y_1) : i \in [n]\}$. This set of edges is a matching in $K_{1,n} \Box K_{1,m}$, and $K_{1,n} \Box K_{1,m} - N_e[M]$ has a component isomorphic to the graph obtained from $m-1$ vertex disjoint copies of the star $K_{1,n}$ and an edge uv by adding $m-1$ edges making u adjacent to the centers of the disjoint stars. This graph is not equimatchable since it has maximal matchings of sizes $m-1$ and m. Having now checked all the possibilities, we conclude that $G = H = K_2$. Therefore, statement (a) implies (c). □

7 Open Questions

In their study of connected, equimatchable graphs of girth at least 5, Frendrup, Hartnell and Vestergaard [7] characterized the connected, well-edge-dominated graphs of girth at least 5. In particular, they proved the following result.

Theorem 7. ([7]) *If G is a connected graph with $g(G) \geq 5$, then G is well-edge-dominated if and only if $G \in \{K_2, C_5, C_7\}$ or G is bipartite with partite sets V_1 and V_2 such that V_1 is the set of all support vertices of G.*

In Theorem 1 of this paper we showed that only one additional graph, namely C_7^*, is added to the list of connected, well-edge-dominated graphs if the girth restriction is lowered to 4 but we now require that the graph be nonbipartite.

A natural problem now presents itself.

Problem 1. *Find a structural characterization of the class of connected, bipartite graphs of girth* 4 *that are well-edge-dominated.*

By Theorem 4 this class contains $K_{n,n}$, for any $n \geq 2$ and by Theorem 2 it does not contain any nontrivial Cartesian products other than $K_2 \Box K_2$.

For graphs that contain a triangle, we have characterized the connected, split graphs that are well-edge-dominated in Theorem 6. Determining the structure for arbitrary well-edge-dominated graphs of girth 3 is an interesting problem.

Problem 2. *Find a structural characterization of the class of connected graphs of girth* 3 *that are well-edge-dominated.*

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