

A characterization of well-dominated Cartesian products

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Abstract

A graph is well-dominated if all its minimal dominating sets have the same cardinality. In this paper we prove that at least one factor of every connected, well-dominated Cartesian product is a complete graph, which then allows us to give a complete characterization of the connected, well-dominated Cartesian products if both factors have order at least 2. In particular, we show that $G \square H$ is well-dominated if and only if $G \square H = P_3 \square K_3$ or $G \square H = K_n \square K_n$ for some $n \geq 2$.

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1 Introduction

A minimal dominating set in any finite graph can be found in linear time by processing its vertices sequentially and retaining only those vertices that are needed to dominate the graph. The invariant of interest in most applications is the domination number of the graph, and the related decision problem is a well-known NP-complete problem [4]. The class of *well-dominated* graphs, introduced by Finbow, Hartnell, and Nowakowski [3], are those for which every minimal dominating set has the same cardinality. Thus, a graph is well-dominated if and only if the simple algorithm described above always produces a dominating set of minimum cardinality. The structure of

well-dominated graphs is far from being known, but the collection of connected, well-dominated graphs within several classes of graphs have recently been determined. This list includes bipartite [3], girth larger than 4 [3], simplicial and chordal [11], and graphs without cycles of length 4 or 5 [10]. In addition, a characterization has been given of the well-dominated graphs within the classes of direct products [12], lexicographic products [5], and disjunctive products [1]. Several subclasses of well-dominated strong products and Cartesian products have also been exhibited.

In the initial attempt to determine the well-dominated Cartesian products, Anderson, Kuenzel and Rall [1] characterized the nontrivial, well-dominated Cartesian products when both factors are triangle-free.

Theorem 1. ([1, Theorem 2]) *Let G and H be nontrivial, connected graphs both of which have girth at least 4. The Cartesian product $G \square H$ is well-dominated if and only if $G = H = K_2$.*

The same paper by Anderson, et al. included the following result.

Theorem 2. ([1, Theorem 1]) *Let G and H be connected graphs. If the Cartesian product $G \square H$ is well-dominated, then G or H is well-dominated.*

However, in a private communication, Erika King and Michael O’Grady [9] pointed out a logical gap in the proof of Theorem 2.

Rall [12] determined the connected, well-dominated Cartesian products when one of the factors is a complete graph of order at least 2. In particular, he proved the following result.

Theorem 3. ([12, Theorem 3]) *Let m be a positive integer with $m \geq 2$ and let H be a nontrivial, connected graph. The Cartesian product $K_m \square H$ is well-dominated if and only if either $m \neq 3$ and $H = K_m$ or $m = 3$ and $H \in \{P_3, K_3\}$.*

In addition, he conjectured that every nontrivial, connected and well-dominated Cartesian product has a complete graph as a factor. Our main result in this paper is the following theorem that proves this conjecture and thus verifies Theorem 2.

Theorem 4. *If $G \square H$ is connected and well-dominated, then G or H is a complete graph.*

Applied together with Theorem 3, this completes the characterization of the class of nontrivial, connected, well-dominated Cartesian products.

In Section 2 we supply necessary definitions and provide additional background results in the study of well-dominated Cartesian products. In addition, we prove several preliminary lemmas that will be used in our proof of the main theorem. Section 3 is devoted to proving Theorem 4.

2 Definitions and Preliminary Results

In this paper we restrict our attention to finite, simple graphs. We say a graph G is *nontrivial* if its order, denoted by $n(G)$, is at least 2, and a Cartesian product is said to be *nontrivial* if both factors are nontrivial. In general we follow the definitions and notation of the book by Haynes, Hedetniemi and Henning [8]. For a positive integer n , the set $\{1, \dots, n\}$ will be denoted by $[n]$. If g is a vertex in a graph G , the set of vertices adjacent to g is denoted by $N_G(g)$ and is called the *open neighborhood* of g . The *closed neighborhood* of g is the set $N_G[g]$ defined by $N_G[g] = N_G(g) \cup \{g\}$. For a set $S \subseteq V(G)$, the open neighborhood of S is defined by $N_G(S) = \cup_{g \in S} N_G(g)$ and its closed neighborhood is the set $N_G[S] = N_G(S) \cup S$. We will omit the subscript on these neighborhood set names if the graph is clear from the context. A *dominating set* of G is a subset S of the vertex set of G such that $N[S] = V(G)$. A dominating set S is minimal if $S - \{x\}$ does not dominate G for every $x \in S$. This condition is equivalent to requiring $N[x] - N[S - \{x\}] \neq \emptyset$ for every $x \in S$. When $N[u] - N[A - \{u\}] \neq \emptyset$ for a vertex $u \in A$, we say that u has a *private neighbor with respect to A* . When the set A is clear from the context we may simply say that u has a private neighbor. If this is the case for each u in A , then we say that A is *irredundant*. If A is irredundant, then some of the private neighbors of a vertex $u \in A$ might belong to A (when $N(u) \cap A = \emptyset$) and others might belong to $V(G) - A$. If A is an irredundant set and every vertex $u \in A$ has a private neighbor with respect to A that does not belong to A , then we say A is *open irredundant*. A set $I \subseteq V(G)$ is *independent* if no pair of distinct vertices in I are adjacent.

The *domination number* of G is the cardinality of a smallest dominating set of G and is denoted $\gamma(G)$. A dominating set D of G with $|D| = \gamma(G)$ is called a $\gamma(G)$ -set. The *vertex independence number* of G is the cardinality of a largest independent set in G and is denoted $\alpha(G)$. A set $I \subseteq V(G)$ is called a $\alpha(G)$ -set if I is independent and has cardinality $\alpha(G)$. The graph G is said to be *well-dominated* if all of its minimal dominating sets have cardinality $\gamma(G)$, and G is *well-covered* if all of its maximal independent sets have cardinality $\alpha(G)$. One can greedily add vertices to any independent set in G to enlarge it to a maximal independent set. It follows that in a well-covered graph G any independent set is a subset of an $\alpha(G)$ -set. It is clear from the definition that a maximal independent set in G is a minimal dominating set of G . Consequently, a graph that is well-dominated is also well-covered.

If A is a nonempty set, then a collection of pairwise disjoint subsets of A whose union is A is called a *weak partition* of A . Let X and Y be graphs. The *Cartesian product*, $X \square Y$, is the graph whose vertex set is $V(X) \times V(Y)$. Two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $X \square Y$ if they are equal in one coordinate and adjacent in the other coordinate. The Cartesian product is associative, commutative and distributes over disjoint unions. For a fixed vertex $x \in V(X)$, the set $\{(x, y) \mid y \in V(Y)\}$ is called a *Y -layer*. Each Y -layer induces a subgraph of $X \square Y$ that is isomorphic to Y . Similarly,

for a fixed vertex y in $V(Y)$, the set $\{(x, y) \mid x \in V(X)\}$ is called a X -layer, and it induces a subgraph isomorphic to X .

We now list some preliminary results that will prove useful in the remainder of the paper. Note that if I is an independent set in a graph X , then $I \cup D$ is a minimal dominating set of X for every minimal dominating set D of $X - N[I]$. This implies the following observation as noted by Finbow, Nowakowski and Hartnell [3].

Observation 1. ([3]) *If X is a well-dominated graph and I is an independent set of X , then $X - N[I]$ is well-dominated.*

It is clear that the Cartesian product of two connected graphs is itself connected. Also, since the Cartesian product distributes over disjoint unions, we get the following useful fact.

Observation 2. *If $X \square Y$ is connected, then X and Y are connected.*

The following observation follows directly from the definitions.

Observation 3. *A graph is well-dominated if and only if each of its components is well-dominated.*

Although the well-covered Cartesian products have not been characterized, Hartnell and Rall reduced the characterization in the following result.

Theorem 5. ([6]) *If G and H are graphs such that $G \square H$ is well-covered, then at least one of G or H is well-covered.*

Since a graph is well-covered if it is well-dominated, we will use Theorem 5 to infer that at least one of the factors of a well-dominated Cartesian product is well-covered.

The following lemma established by Hartnell, Rall and Wash will be used to show that the Cartesian product of two triangle-free connected graphs is not well-dominated if both have order at least 3.

Lemma 1. ([7, Lemma 4]) *If G and H are connected graphs both having order at least 3 and girth at least 4, then $G \square H$ is not well-covered.*

Bollobás and Cockayne [2] showed that every connected graph has a minimum dominating set that is open irredundant. Using this fact, Rall proved the following proposition by showing that $D \times V(Y)$ is a minimal dominating set of $X \square Y$ for any graph Y if D is an open irredundant dominating set of X .

Proposition 1. ([12, Proposition 16]) *Let X and Y be nontrivial connected graphs. If $X \square Y$ is well-dominated, then $\gamma(X \square Y) = \gamma(X)n(Y) = \gamma(Y)n(X)$.*

The method we will use to prove Theorem 4 involves assuming that $G \square H$ is a minimal counterexample. By removing the closed neighborhood of a maximal independent set in $G \square H$, we then will analyze the Cartesian product of smaller factors that may be disconnected. To determine the structure of these product graphs we need the following three lemmas.

Lemma 2. *Let X be a connected graph of order at least 3. If $P_3 \square X$ is well-dominated, then $X = K_3$.*

Proof. Let $V(P_3) = \{a, b, c\}$ with $\deg(b) = 2$, and suppose $P_3 \square X$ is connected and well-dominated. It is clear that the set $\{b\} \times V(X)$ is a minimal dominating set of $P_3 \square X$, and thus $\gamma(P_3 \square X) = n(X)$. Since $P_3 \square X$ is well-covered, we infer that X contains a triangle by applying Lemma 1. Suppose first that X contains a vertex x of degree at least 3. Let

$$S = (\{a, b, c\} \times \{x\}) \cup (\{b\} \times (V(X) - N_X[x])).$$

It is easy to see that S dominates $P_3 \square X$. Also, $|S| = 3 + n(X) - (\deg(x) + 1) < n(X)$. This is a contradiction. Therefore, $\Delta(X) \leq 2$, and this implies that $X = K_3$. \square

Lemma 3. *Let X be a connected graph of order at least 3 and let r and s be positive integers. If $2 \leq r \leq s$ or if $r = 1$ and $s \geq 3$, then $K_{r,s} \square X$ is not well-dominated.*

Proof. Suppose to the contrary that there exists some connected graph X with order at least 3 such that $K_{r,s} \square X$ is well-dominated. Label the vertices of the two partite sets of $K_{r,s}$ as $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_s\}$.

Assume first that $2 \leq r \leq s$. Let D_X be any minimal dominating set of X . Since X is connected and has order at least 3, it follows that $V(X) - D_X$ is also a dominating set of X . We claim that

$$D_1 = (\{a_1\} \times (V(X) - D_X)) \cup (\{a_2, \dots, a_r\} \times D_X)$$

is a minimal dominating set of $K_{r,s} \square X$. Let $(u, x) \in V(K_{r,s} \square X) - D_1$. If $x \in V(X) - D_X$ and $u \notin \{a_2, \dots, a_r\}$, then (u, x) is dominated by (a_1, x) . If $x \in V(X) - D_X$ and $u = a_i$ for $2 \leq i \leq r$, then (u, x) is dominated by (a_i, x') for some $x' \in D_X$. If $x \in D_X$ and $u \neq a_1$, then (u, x) is dominated by (a_2, x) . If $x \in D_X$ and $u = a_1$, then (u, x) is dominated by (a_1, x') for some $x' \in V(X) - D_X$ since $V(X) - D_X$ is a dominating set of X . Thus, D_1 is in fact a dominating set of $K_{r,s} \square X$. To show that D_1 is a minimal dominating set of $K_{r,s} \square X$ we will show that each vertex in D_1 has a private neighbor with respect to D_1 . For any $x \in V(X) - D_X$, the vertex (b_1, x) is a private neighbor of (a_1, x) with respect to D_1 . Each vertex of $\{a_2, \dots, a_r\} \times D_X$ has a private neighbor with respect to D_1 in its X -layer since D_X is a minimal dominating set of X and $\{a_1, \dots, a_r\}$ is an independent set in $K_{r,s}$. Therefore, D_1 is a minimal dominating set of $K_{r,s} \square X$ of cardinality $n(X) + (r - 2)|D_X|$.

One can easily verify that $D_2 = \{a_1, b_1\} \times V(X)$ is a minimal dominating set of $K_{r,s} \square X$. Since $K_{r,s} \square X$ is well-dominated, it follows that

$$n(X) + (r - 2)|D_X| = |D_1| = |D_2| = 2n(X),$$

and hence $(r - 2)|D_X| = n(X)$. Since D_X is an arbitrary minimal dominating set of X , this implies that X is well-dominated and $\gamma(X) = \frac{n(X)}{r-2}$. By Proposition 1,

$$\gamma(K_{r,s} \square X) = \gamma(K_{r,s})n(X) = \gamma(X)n(K_{r,s}) = \frac{r+s}{r-2}n(X),$$

which implies that $r = s + 4$. This contradiction shows that if $2 \leq r \leq s$, then $K_{r,s} \square X$ is not well-dominated.

Finally, suppose $r = 1$ and $s \geq 3$ and let D_X be an $\alpha(X)$ -set. Using arguments similar to those above, it follows that both $\{a_1\} \times V(X)$ and $(\{b_1\} \times (V(X) - D_X)) \cup (\{b_2, \dots, b_s\} \times D_X)$ are minimal dominating sets of $K_{1,s} \square X$. Therefore,

$$n(X) = n(X) - |D_X| + (s - 1)|D_X|,$$

which implies $s = 2$, another contradiction. \square

Finally, we identify one more class of Cartesian products that are not well-dominated. Let \mathcal{F}_1 denote the class of all graphs of order at least 4 obtained by attaching some finite number of leaves to the vertices of a complete graph of order 3. Let \mathcal{F}_2 be the class of all graphs of order at least 5 constructed from a complete graph of order 4 by attaching some finite number of leaves to at most three of its vertices.

Lemma 4. *Let $F_1 \in \mathcal{F}_1$ and let $F_2 \in \mathcal{F}_2$. If X is any connected graph of order 3 or more, then neither $F_1 \square X$ nor $F_2 \square X$ is well-dominated.*

Proof. Suppose for the sake of obtaining a contradiction that $F_1 \square X$ is well-dominated. Let y_1, y_2 and y_3 be the vertices of F_1 that have degree at least 2. We let L_i represent the set of leaves adjacent to y_i for $i \in [3]$ and let D_X be a minimum dominating set of X . By the definition of \mathcal{F}_1 , at least one of y_1, y_2, y_3 is adjacent to a leaf. The argument is divided into three parts depending on how many of the sets L_1, L_2, L_3 are nonempty.

Suppose first that $L_i \neq \emptyset$ for every $i \in [3]$. Note that $\{y_1, y_2, y_3\}$ is an open irredundant dominating set of F_1 and therefore $D_1 = \{y_1, y_2, y_3\} \times V(X)$ is a minimal dominating set of $F_1 \square X$. Next, consider $D_2 = (\{y_1\} \times V(X)) \cup ((L_2 \cup L_3) \times D_X)$. We claim that D_2 is a minimal dominating set of $F_1 \square X$. To see this, note that (g, h) is dominated by (y_1, h) if $h \in V(X)$ and $g \notin L_2 \cup L_3$. If $g \in L_2 \cup L_3$, then (g, h) is dominated by some (g, h') where $h' \in D_X$. Moreover, (x, h) is a private neighbor of (y_1, h) where $x \in L_1$ and each vertex of $(L_2 \cup L_3) \times D_X$ has a private neighbor in its

X -layer since $L_2 \cup L_3$ is an independent set in F_1 . Since $F_1 \square X$ is well-dominated, it follows that

$$3n(X) = |D_1| = |D_2| = n(X) + (|L_2| + |L_3|)|D_X|.$$

Furthermore, $D'_2 = (\{y_2\} \times V(X)) \cup ((L_1 \cup L_3) \times D_X)$ and $D''_2 = (\{y_3\} \times V(X)) \cup ((L_1 \cup L_2) \times D_X)$ are also minimal dominating sets of $F_1 \square X$. Thus, $|L_1| = |L_2| = |L_3|$. Therefore, $3n(X) = n(X) + 2|L_1||D_X| = n(X) + 2|L_1|\gamma(X)$ from which it follows that $n(X)/\gamma(X) = |L_1|$. On the other hand, since $\gamma(F_1) = 3$, it follows by Proposition 1 that $\gamma(X)n(F_1) = \gamma(F_1)n(X) = 3n(X)$, or equivalently, $n(X)/\gamma(X) = n(F_1)/3$. Hence,

$$|L_1| = \frac{n(X)}{\gamma(X)} = \frac{n(F_1)}{3} = |L_1| + 1,$$

which is clearly a contradiction. Therefore, this case cannot occur.

Next, suppose that $L_1 = \emptyset$ and $L_i \neq \emptyset$ for $i \in \{2, 3\}$. Note that $A_1 = (\{y_2\} \times V(X)) \cup (L_3 \times D_X)$, $A_2 = (\{y_1\} \times (V(X) - D_X)) \cup ((L_2 \cup L_3) \times D_X)$, and $A_3 = (\{y_3\} \times V(X)) \cup (L_2 \times D_X)$ are all minimal dominating sets of $F_1 \square X$. Therefore, using calculations similar to the first case above, we get $|A_1| = |A_2|$, meaning $|L_2| = 1$ and similarly, $|L_3| = 1$. However, $\{y_2, y_3\} \times V(X)$ is also a minimal dominating set and now

$$n(X) + |D_X| = |A_1| = 2n(X)$$

which cannot be for any nontrivial connected graph X .

Finally, assume $L_1 = L_2 = \emptyset$ and $L_3 \neq \emptyset$. As above, we know $\{y_3\} \times V(X)$ and $(\{y_2\} \times V(X)) \cup (L_3 \times D_X)$ are minimal dominating sets of $F_1 \square X$. It follows that

$$n(X) = n(X) + |L_3||D_X|,$$

which implies $|L_3| = 0$, another contradiction. Therefore, $F_1 \square X$ is not well-dominated.

Now suppose there exists a connected graph X of order at least 3 such that $F_2 \square X$ is well-dominated. Next, let $\{y_1, y_2, y_3, y_4\}$ induce a complete subgraph of F_2 and let $\deg(y_4) = 3$. Let L_i be the set of leaves adjacent to y_i for $i \in [3]$ and let D_X be a minimum dominating set of X .

Suppose first that $L_i \neq \emptyset$ for $i \in [3]$. Note that $\{y_1, y_2, y_3\}$ is an open irredundant set of F_2 and therefore $D_1 = \{y_1, y_2, y_3\} \times V(X)$ is a minimal dominating set of $F_2 \square X$. By an argument similar to the first case above when we were analyzing $F_1 \square X$, it follows that $|L_1| = |L_2| = |L_3|$, $n(X)/\gamma(X) = |L_1|$, and $n(X)/\gamma(X) = n(F_2)/3$. Hence,

$$|L_1| = \frac{n(X)}{\gamma(X)} = \frac{n(F_2)}{3} = \frac{4 + 3|L_1|}{3} = |L_1| + \frac{4}{3},$$

which is clearly a contradiction. Therefore, this case cannot occur.

Next, suppose that $L_1 = \emptyset$ and $L_i \neq \emptyset$ for $i \in \{2, 3\}$. Note that $(\{y_1\} \times V(X)) \cup ((L_2 \cup L_3) \times D_X)$ and $(\{y_3\} \times V(X)) \cup (L_2 \times D_X)$ are both minimal dominating sets of $F_2 \square X$. This implies that $|L_3| = 0$, which is a contradiction.

Therefore, we may assume $L_1 = L_2 = \emptyset$ and $L_3 \neq \emptyset$. It follows that $\{y_3\} \times V(X)$ and $(\{y_2\} \times V(X)) \cup (L_3 \times D_X)$ are minimal dominating sets of $F_2 \square X$, which implies $L_3 = \emptyset$. This final contradiction completes the proof. \square

3 The Characterization

In this section we prove Theorem 4, which is restated below for the reader's convenience. A fact that we will use a number of times is the following.

Observation 4. *If I_G is a maximal independent set in G and I_H is a maximal independent set in H , then $I_G \times I_H$ is independent in $G \square H$ and*

$$G \square H - N_{G \square H}[I_G \times I_H] = (G - I_G) \square (H - I_H).$$

If in addition $G \square H$ is well-dominated, then by Observation 1 it follows that $(G - I_G) \square (H - I_H)$ is well-dominated.

Theorem 4 *If $G \square H$ is connected and well-dominated, then G or H is a complete graph.*

Proof. Suppose to the contrary that there exists a connected, well-dominated Cartesian product $G \square H$ such that neither G nor H is a complete graph. By Observation 2, both G and H are connected. Among all such counterexamples, choose one where the order of $G \square H$ is minimal. Since $G \square H$ is also well-covered, it follows from Theorem 5 that at least one of G or H is well-covered. Without loss of generality, we assume that G is well-covered. Moreover, by Lemma 2 we may assume that G and H each have order at least 4. Let I_G be any maximal independent set of G and let I_H be any maximal independent set of H . By Observation 4, $(G - I_G) \square (H - I_H)$ is well-dominated. To simplify notation we will let $G' = G - I_G$ and $H' = H - I_H$. By Observation 3 and the choice of $G \square H$, the conclusion of the theorem holds for each component of $G' \square H'$. In particular, let G_1, \dots, G_ℓ be the components of G' and let H_1, \dots, H_n be the components of H' . Thus, for each $(i, j) \in [\ell] \times [n]$, G_i or H_j is a complete graph. Furthermore, by Theorem 3 this implies that if the orders of G_i and H_j are both at least 2, then $G_i \square H_j$ is isomorphic to $K_m \square K_m$ for some $m \geq 2$ or to $P_3 \square K_3$.

We claim that we need not consider the case when all components of H' are isolated vertices. For suppose I_H is a maximal independent set such that $H' = H - I_H$ consists entirely of isolated vertices. This implies that H is bipartite. Since G is connected

and G and H both have order at least 4, it follows by Lemma 3 that H is not a complete bipartite graph, which allows us to select a maximal independent set J in H so that $H - J$ contains an edge. Thus we may assume I_H has been chosen so that $H' = H - I_H$ contains a nontrivial component. Similarly, we may assume that the maximal independent set I_G of G has been chosen so that $G' = G - I_G$ contains a nontrivial component.

By Theorem 3 it follows that each component of H' is either K_1 , P_3 , or K_m for some $m \geq 2$. The argument is now split into four cases depending on the components of H' .

Case 1. [H' has P_3 as a component.]

By Lemma 2, every component of G' is a complete graph of order 3. Furthermore, if I is any maximal independent set of G , then every component of $G - I$ is complete of order 3. For each $i \in [\ell]$ let $V(G_i) = \{x_1^i, x_2^i, x_3^i\}$. Let $i \in [\ell]$. Extend $\{x_1^i\}$ to a maximal independent set I of G . The edge $x_2^i x_3^i$ is in the graph $G - I$, which implies that there is a vertex $v \in V(G - I)$ such that $\{vx_2^i, vx_3^i\} \subseteq E(G)$. Note that $v \in I_G$, for otherwise the subgraph of G induced by $\{x_1^i, x_2^i, x_3^i, v\}$ is a subgraph of $G - I_G$. Now extend $\{x_2^i\}$ to a maximal independent set I' of G . Since $\{x_1^i x_3^i, vx_3^i\} \subseteq E(G - I')$, we infer that $vx_1^i \in E(G)$. Therefore, for every $i \in [\ell]$, there exists $y_i \in I_G$ such that $\{y_i, x_1^i, x_2^i, x_3^i\}$ induces a complete graph of order 4 in G . Note that we have also proved here that if a vertex of I_G is adjacent to two vertices of a triangle in G , then it is also adjacent to the other vertex of the triangle. (See the argument for v .)

We claim that $G = K_4$. To see this, note that since G is connected, there exists a path from x_1^1 to x_1^2 . Suppose first that for some $i \in \{2, \dots, \ell\}$, $j \in [3]$, and $k \in [3]$ that x_j^1 is adjacent to some vertex $z \in I_G - \{y_1\}$ and z is also adjacent to x_k^i . Reindexing if necessary, we may assume $j = k = 1$. Extend $\{x_2^1, x_1^i\}$ to a maximal independent set I_1 of G . Now the set $\{y_1, x_1^1, x_3^1, z\}$ belongs to a single component of $G - I_1$, which is a contradiction. Therefore, if $w \in (I_G - \{y_1\}) \cap N(V(G_1))$, then $N(w) \subseteq V(G_1)$. Next, suppose y_1 has a neighbor x_k^i for some $k \in [3]$ and $i \in \{2, \dots, \ell\}$. Reindexing if necessary, we may assume $k = 1$ and $i = 2$. Extend $\{x_1^1, x_2^2\}$ to a maximal independent set I_2 of G . Now the set $\{x_2^1, x_3^1, y_1, x_1^2\}$ belongs to a single component of $G - I_2$, another contradiction. It follows that $N_G[y_1] = V(G_1) \cup \{y_1\}$ and $G = K_4$. This is a contradiction.

Case 2. [Some component of H' is a complete graph of order m for some $m \geq 4$.]

Applying Theorem 3 we see that each component of G' is either an isolated vertex or a complete graph of order m . Furthermore, by our choice of the maximal independent set I_G , at least one component of G' is isomorphic to K_m .

Subcase 2.1. [G' contains an isolated vertex.]

We write $G' = G_1 \cup \dots \cup G_\alpha \cup G_{\alpha+1} \cup \dots \cup G_\ell$ where $G_i = K_1$ for $i \in [\alpha]$ and $G_i = K_m$ for $\alpha + 1 \leq i \leq \ell$. For each i with $\alpha + 1 \leq i \leq \ell$, let $V(G_i) = \{x_1^i, \dots, x_m^i\}$ and let $V(G_j) = \{x_1^j\}$ for $1 \leq j \leq \alpha$. Let J be the maximal independent set of G defined by $J = C \cup \bigcup_{i=1}^\ell \{x_1^i\}$ where $C = I_G - N_G[\bigcup_{i=1}^\ell \{x_1^i\}]$. Since G is well-covered, $|J| = |I_G|$, which implies $\ell \leq |I_G|$. Using the same argument as above $G'' = G - J$ is the disjoint union of components each of which is either K_1 or K_m . Note that $I_G - C \subseteq V(G) - J = V(G'')$. It follows that each vertex $x \in I_G - C$ is either an isolate in G'' or x is adjacent to every vertex in $V(G_i) - \{x_1^i\}$ for some $\alpha + 1 \leq i \leq \ell$ and x is not adjacent to any vertex of $V(G_j) - \{x_1^j\}$ for every $j \in \{\alpha + 1, \dots, \ell\} - \{i\}$.

Let $\{Y_G, Z_G, M_G\}$ be the weak partition of I_G defined as follows.

- Y_G is the set of all $g \in I_G$ such that $V(G_i) \cup \{g\}$ induces in G a complete graph (of order $m + 1$) for some i with $\alpha + 1 \leq i \leq \ell$,
- Z_G is the set of all $g \in I_G$ that are isolated in G'' , and
- $M_G = I_G - (Y_G \cup Z_G)$.

First we claim that $|Y_G| = \ell - \alpha$. Let i be an index with $\alpha + 1 \leq i \leq \ell$. Since each component of G'' is either K_1 or K_m and the subgraph of G'' induced by $V(G_i) - \{x_1^i\}$ is a complete graph of order $m - 1$, we see that there exists a vertex $y \in I_G$ that is adjacent to every vertex in $V(G_i) - \{x_1^i\}$ and is not adjacent to any other vertex in G'' . Suppose y is not adjacent to x_1^i . Extend $\{x_1^i\}$ to a maximal independent set I of G . In $G - I$ the set $(V(G_i) - \{x_1^i\}) \cup \{y\}$ induces a subgraph of order m that is not a complete graph. This is a contradiction, and thus $y \in Y_G$. This also shows that $|Y_G| \geq \ell - \alpha$. Furthermore, suppose there exist distinct vertices u and v in Y_G such that $V(G_j) \subseteq N_G(u) \cap N_G(v)$ for some $\alpha + 1 \leq j \leq \ell$. By enlarging $\{x_1^j\}$ to a maximal independent set I' of G , we see that u and v belong to a component of order $m + 1$ of $G - I'$. This contradiction shows that $|Y_G| \leq \ell - \alpha$. We thus denote Y_G as $\{y_1, \dots, y_t\}$, where $t = \ell - \alpha$ and such that $V(G_{\alpha+i}) \subseteq N_G(y_i)$ for each $1 \leq i \leq t$.

Next, we claim that $M_G \subseteq C$. To see this, suppose $x \in M_G$. Then, x is not isolated in G'' , and it follows that x is adjacent to a vertex x_j^i for some $2 \leq j \leq m$ and some $\alpha + 1 \leq i \leq \ell$ since G is connected. Reindexing if necessary, we may assume x is adjacent to $x_2^{\alpha+1}$. Since $x \notin Y_G$, there exists $k \in [m] - \{2\}$ such that $xx_k^{\alpha+1} \notin E(G)$. Suppose x is adjacent to a vertex in $V(G_{\alpha+1}) - \{x_2^{\alpha+1}\}$. When we extend $\{x_2^{\alpha+1}\}$ to a maximal independent set I of G , the set $(V(G_{\alpha+1}) - \{x_2^{\alpha+1}\}) \cup \{x\}$ is contained in a component that is complete, which implies x is adjacent to every vertex of $V(G_{\alpha+1})$, a contradiction. Thus, x is not adjacent to $x_1^{\alpha+1}$. Moreover, if x is adjacent to x_1^m for some $m \in [\ell] - \{\alpha + 1\}$, then when we extend $\{x_3^{\alpha+1}, x_1^m\}$ to a maximal independent set I , there exists a component in $G - I$ containing $\{x\} \cup (V(G_{\alpha+1}) - \{x_3^{\alpha+1}\})$, which is a contradiction. Thus, $x \in I_G - N_G[\bigcup_{i=1}^\ell \{x_1^i\}] = C$.

Let $Q_2 = (\bigcup_{i=1}^{\alpha} \{x_1^i\}) \cup (\bigcup_{i=\alpha+1}^{\ell} \{x_2^i\})$ and let $M_2 = I_G - N_G[Q_2]$. The set Q_2 is independent and the set T_2 defined by $T_2 = M_2 \cup Q_2$ is a maximal independent set of G . Let $G''' = G - T_2$. Recall that H' has a component isomorphic to K_m . Using the same argument as above, we see that each component of G''' is a complete graph of order 1 or m , and at least one of these components has order m since $x_1^{\alpha+1}x_3^{\alpha+1}$ is an edge in G''' .

Suppose $Z_G \neq \emptyset$. Let $z \in Z_G$. Since z is an isolate in G'' , it follows that z is not adjacent in G to any vertex in $V(G_k) - \{x_1^k\}$, for $\alpha + 1 \leq k \leq \ell$. Also, z is not adjacent in G to x_1^k for any k with $\alpha + 1 \leq k \leq \ell$ since z is not adjacent to x_3^k , for otherwise z is contained in a component of G''' that is not a complete graph of order m . Thus, $N_G(z) \subseteq \{x_1^1, \dots, x_1^{\alpha}\}$.

Since G is connected, there is a path in G from z to y_i for each $i \in [t]$. Without loss of generality we may assume that a shortest such path, say P , is a z, y_1 -path. Recall, as we proved above, that $M_G \subseteq C$, which implies that $N_G(v) \cap (\bigcup_{i=1}^{\ell} \{x_1^i\}) = \emptyset$ for every $v \in M_G$. Recall also that $N_G(z) \subseteq \{x_1^1, \dots, x_1^{\alpha}\}$. From this we infer that

$$V(P) \subseteq \{y_1\} \cup Z_G \cup \{x_1^1, \dots, x_1^{\alpha}\}.$$

Reindexing $\{x_1^1, \dots, x_1^{\alpha}\}$ if necessary, we may assume without loss of generality that $y_1x_1^1$ and $z'x_1^1$ are both edges of G for some $z' \in Z_G$. Since $N_G(z') \subseteq \{x_1^1, \dots, x_1^{\alpha}\}$, expand the independent set $\{z', x_2^{\alpha+1}\}$ to a maximal independent set I of G . Now we see that $\{x_1^1, y_1\} \cup (V(G_{\alpha+1}) - \{x_2^{\alpha+1}\})$ does not induce a complete graph of order m in $G - I$. This is a contradiction, and therefore $Z_G = \emptyset$. As a result

$$|I_G| = |Y_G| + |M_G| \leq \ell + |M_G| \leq \ell + |C| = |J| = |I_G|.$$

This implies that $|Y_G| = \ell$, which in turn implies that G' has no isolated components. This contradicts our assumption.

Subcase 2.2. [Every component of G' is a complete graph of order m .]

Since G is well-covered,

$$|M_2| + \ell = |M_2| + |Q_2| = |T_2| = |I_G| = |M_G| + \ell.$$

Thus, $|M_2| = |M_G|$. Moreover, $M_2 \subseteq M_G$ as each vertex w in M_2 is not adjacent to any vertex of the form x_2^k for $k \in [\ell]$ and so $w \notin Y_G$. It follows that $M_G = M_2$. In fact, if for $3 \leq j \leq m$ we define

$$Q_j = \bigcup_{i=1}^{\ell} \{x_j^i\} \quad \text{and} \quad M_j = I_G - N_G[Q_j],$$

then $T_j = M_j \cup Q_j$ is a maximal independent set of G from which - as we did above - we may conclude that $M_j = M_G$. That is, M_G consists of vertices which are not adjacent

to any vertex of $G_1 \cup \dots \cup G_\ell$. Since G is connected, it follows that $M_G = \emptyset$. Moreover, we know from above that y_i cannot have a neighbor in $G_{\alpha+j}$ when $i \neq j$. Therefore, I_G consists of a single vertex adjacent to every vertex in G' . That is, $G = K_{m+1}$, which contradicts the choice of $G \square H$.

Case 3. [Some component of H' is K_3 .]

We may assume that for every maximal independent set I of H the graph $H - I$ does not contain a component isomorphic to a path of order 3 or a complete graph of order at least 4, for otherwise we are in Case 1 or Case 2. Since H' contains a K_3 , it follows from Theorem 3 that every component of G' is either K_1 , K_3 , or P_3 . Furthermore, by the choice of I_G , at least one component of G' is nontrivial. By Theorem 3 we infer that no component of H' is K_2 .

Subcase 3.1. [H' contains an isolated vertex.]

We write $H' = H_1 \cup \dots \cup H_\sigma \cup H_{\sigma+1} \cup \dots \cup H_n$ where $H_i = K_1$ for $i \in [\sigma]$ and $H_i = K_3$ for $\sigma + 1 \leq i \leq n$. For each i with $\sigma + 1 \leq i \leq n$, let $V(H_i) = \{x_1^i, x_2^i, x_3^i\}$ and let $V(H_j) = \{x_1^j\}$ for $1 \leq j \leq \sigma$. The set J defined by $J = C \cup \bigcup_{i=1}^n \{x_1^i\}$, where $C = I_H - N_H[\bigcup_{i=1}^n \{x_1^i\}]$, is a maximal independent set in H . Using the same argument as above, the components of $H'' = H - J$ are each complete graphs of order 1 or 3 and at least one of these components has order 3 since $x_2^{\sigma+1}x_3^{\sigma+1}$ is an edge in H'' . It follows that each vertex $u \in I_H - C$ is either an isolate in H'' or u is adjacent to every vertex in $V(H_i) - \{x_1^i\}$ for some $\sigma + 1 \leq i \leq n$ and u is not adjacent to any vertex of $V(H_j) - \{x_1^j\}$ for every $j \in \{\sigma + 1, \dots, n\} - \{i\}$.

Let $\{Y_H, Z_H, M_H\}$ be the weak partition of I_H defined as follows.

- Y_H is the set of vertices h in I_H such that $V(H_i) \cup \{h\}$ induces a complete graph of order 4 in H for some $\sigma + 1 \leq i \leq n$,
- Z_H is the set of vertices in I_H that are isolated in H'' , and
- $M_H = I_H - (Y_H \cup Z_H)$.

First we claim that $Y_H \neq \emptyset$. To see this, recall that H'' has a component isomorphic to K_3 and every nontrivial component of H'' is a complete graph of order 3. For each $i \in [n] - [\sigma]$, the vertices x_2^i and x_3^i are adjacent in H'' , and thus there exists a vertex $y \in I_H$ that is adjacent to both x_2^i and x_3^i . Suppose y is not adjacent to x_1^i . Extend $\{x_2^i\}$ to a maximal independent set I of H . In $H - I$ the set $\{y, x_1^i, x_3^i\}$ induces a path of order 3. This is a contradiction, and the claim is established. Moreover, this shows that for each $\sigma + 1 \leq i \leq n$, there exists $y \in I_H$ such that $V(H_i) \cup \{y\}$ induces a complete graph of order 4 in H . Hence, $|Y_H| \geq n - \sigma$. By definition every vertex in Y_H is in H'' . Since each component of H'' is a complete graph of order 3 or an isolated

vertex, $|Y_H| = n - \sigma$. We thus denote Y_H as $\{y_1, \dots, y_t\}$, where $t = n - \sigma$ and such that $V(H_{\sigma+i}) \subseteq N_H(y_i)$ for each $1 \leq i \leq t$.

Let $Q_2 = (\bigcup_{i=1}^{\sigma} \{x_1^i\}) \cup (\bigcup_{i=\sigma+1}^n \{x_2^i\})$ and let $M_2 = I_H - N_H[Q_2]$. The set Q_2 is independent and the set T_2 defined by $T_2 = M_2 \cup Q_2$ is a maximal independent set in H . Let $H''' = H - T_2$. Recall that G' has a component isomorphic to K_3 or P_3 . Using the same argument as above, we see that each component of H''' is a complete graph of order 1 or 3, and at least one of these components is isomorphic to K_3 .

We claim that $N_H(M_H) \cap \{x_1^1, \dots, x_1^{\sigma}\} = \emptyset$. Suppose this is not the case, and let $x \in M_H$ such that x has a neighbor, say x_1^s , where $s \in [\sigma]$. Since x is not isolated in H'' , there exists a neighbor of x in $\cup_{i=\sigma+1}^n \{x_2^i, x_3^i\}$. Without loss of generality we assume $xx_2^{\sigma+1} \in E(H)$. Since $x \notin Y_H$, there exists $k \in \{1, 3\}$ such that $xx_k^{\sigma+1} \notin E(H)$. Let I' be a maximal independent set of H that contains $\{x_1^s, x_k^{\sigma+1}\}$. Then $\{x, y_1\} \cup (V(G_{\sigma+1}) - \{x_k^{\sigma+1}\})$ belongs to a component of order at least 4 in $H - I'$, which is a contradiction. Therefore, $N_H(M_H) \cap \{x_1^1, \dots, x_1^{\sigma}\} = \emptyset$.

Suppose $Z_H \neq \emptyset$. Let $z \in Z_H$. Since z is an isolate in H'' , it follows that z is not adjacent in H to any vertex in $V(H_k) - \{x_1^k\}$, for $\sigma+1 \leq k \leq n$. Also, z is not adjacent in H to x_1^k for any k with $\sigma+1 \leq k \leq n$ since z is not adjacent to x_3^k , for otherwise z is contained in a component of H''' that is not a complete graph of order 3. Thus, $N_H(z) \subseteq \{x_1^1, \dots, x_1^{\sigma}\}$.

Since H is connected, there is a path in H from z to y_i for each $i \in [t]$. Without loss of generality we may assume that a shortest such path, say P , is a z, y_1 -path. Recall from above that $N_H(M_H) \cap \{x_1^1, \dots, x_1^{\sigma}\} = \emptyset$. Recall also that $N_H(z) \subseteq \{x_1^1, \dots, x_1^{\sigma}\}$. From this we infer that

$$V(P) \subseteq \{y_1\} \cup Z_H \cup \{x_1^1, \dots, x_1^{\sigma}\}.$$

Reindexing $\{x_1^1, \dots, x_1^{\sigma}\}$ if necessary, we may assume without loss of generality that $y_1 x_1^1$ and $z' x_1^1$ are both edges of H for some $z' \in Z_H$. Since z' is not adjacent to all the vertices of $H_{\sigma+1}$ we can expand $\{z', x_1^{\sigma+1}\}$ to a maximal independent set I of H for some $x_j^{\sigma+1} \notin N_H(z')$. We now infer that $\{x_1^1, y_1\} \cup (V(H_{\sigma+1}) - \{x_j^{\sigma+1}\})$ induces a subgraph of order 4 in $H - I$. This is a contradiction, and therefore $Z_H = \emptyset$.

We claim that if $M_H \neq \emptyset$, then each vertex in M_H has exactly one neighbor in H and that neighbor belongs to $\cup_{i=\sigma+1}^n \{x_2^i, x_3^i\}$. Suppose there exists $h \in M_H$. Recall that $N_H(h) \cap \{x_1^1, \dots, x_1^{\sigma}\} = \emptyset$. Since h is not isolated in H'' and H is connected, we see that h has a neighbor in $\cup_{i=\sigma+1}^n \{x_2^i, x_3^i\}$. We may assume without loss of generality that $hx_2^{\sigma+1} \in E(H)$. If h has another neighbor in $H_{\sigma+1}$, by choosing a maximal independent set I of H containing $x_2^{\sigma+1}$ it follows that $H - I$ has a component of order at least 4 that contains $\{y_1, h, x_1^{\sigma+1}, x_3^{\sigma+1}\}$. This is a contradiction. Therefore, h has at most one neighbor in H_i for each $i \in [n] - [\sigma]$. Suppose next that h has a neighbor $w \in V(H_j)$ for some j with $\sigma+1 < j \leq n$. Let I be a maximal independent set in H such that

$\{w, x_1^{\sigma+1}\} \subseteq I$. Now the set $\{h, y_1, x_2^{\sigma+1}, x_3^{\sigma+1}\}$ induces a connected subgraph of $H - I$. This is a contradiction. Therefore, if $h \in M_H$, then $\deg_H(h) = 1$ and the only neighbor of h is in the set $\cup_{i=\sigma+1}^n \{x_2^i, x_3^i\}$.

Suppose some vertex in Y_H , say y_1 , is adjacent in H to a vertex that does not belong to $V(H_{\sigma+1})$. Let $L = \{u \in V(H') - V(H_{\sigma+1}) : u \in N_H(y_1)\}$. From what was shown earlier we know that $L \subseteq \cup_{i=1}^{\sigma} \{x_1^i\}$. Suppose there exists a vertex, say $x_1^k \in L$, such that $\deg_H(x_1^k) \geq 2$. Since no vertex of M_H is adjacent to x_1^k by the claim in the paragraph above, let $y_j \in N_H(x_1^k) - \{y_1\}$. Enlarge $\{y_j, x_1^{\sigma+1}\}$ to a maximal independent set I of H . Then $\{x_1^k, y_1, x_2^{\sigma+1}, x_3^{\sigma+1}\}$ induces a connected subgraph in $H - I$, which is again a contradiction, and it follows that every vertex in L is a leaf in H . Since H is connected, this implies that $H \in \mathcal{F}_2$, and thus $G \square H$ is not well-dominated by Lemma 4. This is a contradiction.

Subcase 3.2. [Every component of H' is a complete graph of order 3.]

Note that from above we have each vertex $u \in I_H - C$ that is adjacent to every vertex of $V(H_i) - \{x_1^i\}$ for some $i \in [n]$ is not adjacent to any vertex of $V(H_j) - \{x_1^j\}$ for $i \neq j$. Moreover, our arguments that $Y_H \neq \emptyset$ and $Z_H = \emptyset$ did not rely on H' containing isolates. Therefore, we may assume that there exists $y_1 \in Y_H$ such that $N(y_1) = V(H_1)$. Furthermore, our argument that if $M_H \neq \emptyset$ implies M_H has exactly one neighbor in H that belongs to $\cup_{i=\sigma+1}^n \{x_2^i, x_3^i\}$ did not rely on H' containing isolates. However, since H is connected, this leads us to $H \in \mathcal{F}_2$ or $H = K_4$, both of which are contradictions, one by Lemma 4 and one by our choice of $G \square H$.

Case 4. [Some component of H' is K_2 .]

We may assume that each component of H' is a complete graph of order 1 or 2 since the other possibilities have been argued in the first three cases. Since $G' \square H'$ is well-dominated, it follows from Theorem 3 that each component of G' is K_1 or K_2 . Recall that we may also assume that G' has at least one component that is a complete graph of order 2.

Subcase 4.1. [G' contains an isolated vertex.]

Let $G' = G_1 \cup \dots \cup G_{\alpha} \cup G_{\alpha+1} \cup \dots \cup G_{\ell}$ where $G_i = K_1$ for $i \in [\alpha]$ and $G_i = K_2$ for $\alpha + 1 \leq i \leq \ell$. For $\alpha + 1 \leq i \leq \ell$, let $V(G_i) = \{x_1^i, x_2^i\}$ and for each $G_i = K_1$, label the vertex x_1^i . Let J be the maximal independent set of G defined by $J = C \cup \bigcup_{i=1}^{\ell} \{x_1^i\}$, where $C = I_G - N_G[\bigcup_{i=1}^{\ell} \{x_1^i\}]$. Using the same argument as above, $G'' = G - J$ is the disjoint union of components each of which is either K_1 or K_2 . It follows that each vertex $x \in I_G - C$ is either an isolate in G'' or is adjacent to a vertex x_2^i for some $\alpha + 1 \leq i \leq \ell$. Note that if $x \in I_G - C$, then x has at most one neighbor in the set $\bigcup_{i=\alpha+1}^{\ell} \{x_2^i\}$, for otherwise x belongs to a component of order at least 3 in G'' .

Suppose G contains a triangle. This means there exists a vertex $g \in I_G - C$ and an index $k \in [\ell] - [\alpha]$ such that $\{g, x_1^k, x_2^k\}$ induces a triangle in G . Without loss of generality we may assume $k = \alpha + 1$. Suppose $x_1^{\alpha+1}$ has a neighbor $w \in I_G - \{g\}$ that is not a leaf in G . If w is adjacent to $x_2^{\alpha+1}$, then we can pick a maximal independent set I of G containing $x_2^{\alpha+1}$. However, $G - I$ contains the path induced by $\{w, g, x_1^{\alpha+1}\}$, which is a contradiction. If w is adjacent to some x_s^j for $j \neq \alpha + 1$ and $s \in \{1, 2\}$, then we can pick a maximal independent set I containing x_s^j and $x_2^{\alpha+1}$ and again we have a contradiction. Therefore, $N_G(x_1^{\alpha+1}) - \{g, x_2^{\alpha+1}\}$ is either empty or consists of leaves in G . A similar argument can be used to show that $N_G(x_2^{\alpha+1}) - \{g, x_1^{\alpha+1}\}$ is either empty or consists of leaves in G . Finally, suppose that g is adjacent to some x_r^j where $j \neq \alpha + 1$, $r \in \{1, 2\}$ and x_r^j is not a leaf in G . Since no vertex in $I_G - C$ has more than one neighbor in $\cup_{i=\alpha+1}^{\ell} \{x_2^i\}$, we infer that $r = 1$. Also, $j \notin [\ell] - [\alpha]$ for otherwise $\{g, x_1^{\alpha+1}, x_1^j\}$ induces a path in $G - I$ where I is a maximal independent set of G that contains $\{x_2^j, x_2^{\alpha+1}\}$. Since x_1^j is not a leaf in G , we have $zx_1^j \in E(G)$ for some vertex $z \in I_G$. It follows that z is not adjacent to $x_2^{\alpha+1}$ since $N_G(x_2^{\alpha+1}) - \{g, x_1^{\alpha+1}\}$ consists of leaves in G . Therefore, we can choose a maximal independent set I containing $\{z, x_2^{\alpha+1}\}$ and now $G - I$ contains the path of order 3 induced by $\{x_1^j, g, x_1^{\alpha+1}\}$, which is a contradiction. Therefore, $N_G(g) - \{x_1^{\alpha+1}, x_2^{\alpha+1}\}$ is either empty or consists entirely of leaves in G . Combining these conclusions about the neighbors of $g, x_1^{\alpha+1}$ and $x_2^{\alpha+1}$ and the fact that G is connected, we see that $G \in \mathcal{F}_1$ or $G = K_3$. If $G \in \mathcal{F}_1$, it follows from Lemma 4 that $G \square H$ is not well-dominated. On the other hand, $G \neq K_3$ as we have assumed G is not complete.

Therefore, G does not contain a triangle. Furthermore, the same argument can be used to prove that H does not contain a triangle. Hence, G and H are both of order at least 3 and girth at least 4. By Lemma 1, $G \square H$ is not well-covered and therefore not well-dominated. This is a contradiction.

Subcase 4.2. [Every component of G' is K_2 .]

Now, suppose a vertex $g \in I_G - C$ belongs to a triangle in G induced by $\{g, x_1^k, x_2^k\}$ for some $k \in [\ell]$. Using the argument in the paragraph above it follows that the two sets $N_G(x_1^k) - \{g, x_2^k\}$ and $N_G(x_2^k) - \{g, x_1^k\}$ are either empty or consist of leaves in G . On the other hand, $N_G(g) - \{x_1^k, x_2^k\} = \emptyset$ since G' does not contain an isolated vertex. Once again it follows that $G \in \mathcal{F}_1$ or $G = K_3$, both of which lead to a contradiction. Therefore G does not contain a triangle and as above we infer by Lemma 1 that $G \square H$ is not well-covered and therefore not well-dominated. This is a contradiction.

We have shown that every possible case concerning the components of H' leads to a contradiction. This finishes the proof. \square

Using Theorem 3 together with Theorem 4 we now have a complete characterization of nontrivial, connected Cartesian products that are well-dominated.

Corollary 1. *A nontrivial, connected Cartesian product $G \square H$ is well-dominated if and only if $G \square H = P_3 \square K_3$ or $G \square H = K_n \square K_n$ for some $n \geq 2$.*

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