Graphs with a unique maximum open packing

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Abstract

A set S of vertices in a graph is an open packing if the open neighborhoods of any two distinct vertices in S are disjoint. In this paper, we consider the graphs that have a unique maximum open packing. We characterize the trees with this property by using four local operations such that any non-trivial tree with a unique maximum open packing can be obtained by a sequence of these operations starting from P_2. We also prove that the decision version of the open packing number is NP-complete even when restricted to graphs of girth at least 6. Finally, we show that the recognition of the graphs with a unique maximum open packing is polynomially equivalent to the recognition of the graphs with a unique maximum independent set, and we prove that the complexity of both problems is not polynomial, unless P=NP.

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1   Introduction

A set D of vertices in a graph G is a *total dominating set of G* if every vertex of G has a neighbor in D. The minimum cardinality of a total dominating set is denoted $\gamma_t(G)$ and called the total domination number. This is one of the most studied invariants in domination theory, and has been surveyed in a monograph of Henning and Yeo [9]. As pointed out in [6], open packings are the dual object, as seen from an integer programming standpoint, of total dominating sets. An *open packing* in a graph G is a set of vertices whose open neighborhoods are pairwise disjoint. (The *open neighborhood $N(x)$* of a vertex $x \in V(G)$ is the set $\{v \in V(G) : xv \in E(G)\}$. Whenever the term neighborhood is used in this paper we mean the open

neighborhood.) By $\rho^{\mathrm{o}}(G)$ we denote the maximum cardinality of an open packing in G, and call it the *open packing number* of G.

It follows from definitions of both invariants that $\gamma_t(G) \geq \rho^{\mathrm{o}}(G)$ for an arbitrary graph G with no isolated vertices. Rall studied the relationship of these two parameters and proved the following:

Theorem 1 (Rall, [13]). *If T is a tree, then $\rho^{\mathrm{o}}(T) = \gamma_t(T)$.*

He also used this result to prove a formula for the total domination number of a direct product of a non-trivial tree T with an arbitrary graph H, which reads $\gamma_t(T \times H) = \gamma_t(T)\gamma_t(H)$. For other relations between the open packing number and the total domination number see [3, 4, 8].

Graphs which possess a unique minimum total dominating set were studied by Haynes and Henning in [7], who characterized the trees having this property. In spite of the fact that $\rho^{\mathrm{o}}(T) = \gamma_t(T)$ when T is a tree, there is seemingly no connection between the trees with a unique maximum open packing and the trees with a unique minimum total dominating set. For instance, the path P_4 is a tree with a unique minimum total dominating set, while there are several pairs of vertices in P_4 that form a maximum open packing. On the other hand, the path P_6 is a small example of a tree that has a unique maximum open packing but has more than one minimum total dominating set.

Trees with a unique maximum independent set were studied by Hopkins and Staton [10] and by Gunther et al. [5]. Recently, Jaume and Molina [11] provided a characterization of such trees from an algebraic point of view, which can be used for efficient recognition of trees with a unique maximum independent set; see Section 3. Trees that have a unique maximum 2-packing (where 2-packing is a set of vertices having pairwise disjoint closed neighborhoods) were also characterized recently [1]. In this paper, we begin the study of graphs which have a unique maximum open packing, and we denote the set of such graphs by $\mathcal{C}_{\rho^{\mathrm{o}}}$. In particular, we show that given a graph $G \in \mathcal{C}_{\rho^{\mathrm{o}}}$, we can perform one of four operations on G to create a larger graph G' which also belongs to $\mathcal{C}_{\rho^{\mathrm{o}}}$. We then characterize all trees in $\mathcal{C}_{\rho^{\mathrm{o}}}$ by showing that any such non-trivial tree can be obtained by a sequence of these four operations from P_2.

We also study complexity issues in identifying graphs in $\mathcal{C}_{\rho^{\mathrm{o}}}$. In doing so, we are able to show that the open packing number is related to two well-known graph invariants. Recall that the maximum cardinality of an independent set of vertices in a graph G is the *independence number* of G and is denoted by $\alpha(G)$. By $\alpha'(G)$ we denote the maximum cardinality of a matching in G (where a *matching* is a set of independent edges). The *subdivision of G* is the graph $S(G)$ obtained by subdividing every edge of G by exactly one vertex. We will prove that $\rho^{\mathrm{o}}(S(G)) = \alpha(G) + \alpha'(G)$.

The remainder of this paper is organized as follows. In Section 2, we characterize all trees that have a unique maximum open packing. We do so by showing that every non-trivial tree with a unique maximum open packing can be obtained by starting

with a single edge and performing a series of operations. In Section 3, we show that $\rho^o(S(G)) = \alpha(G) + \alpha'(G)$ and we discuss the complexity of identifying whether an arbitrary graph, or an arbitrary tree, contains a unique maximum open packing. In particular, we prove that the problems of recognizing the graphs with a unique maximum open packing, a unique maximum 2-packing, and a unique maximum independent set, respectively, are all polynomially equivalent problems (i.e., the complexity of recognizing the graphs in these classes is essentially the same). Finally, we prove that these recognition problems are not polynomial, unless P=NP.

2   Trees with unique maximum open packings

For any $G \in \mathcal{C}_{\rho^o}$, we let $U(G)$ represent the unique maximum open packing in G and we refer to $U(G)$ as the $\rho^o(G)$-*set*. In this section, we will characterize all trees in $\mathcal{C}_{\rho^o}$. To that end, we begin with the following preliminary results that apply to all graphs (not just trees) in $\mathcal{C}_{\rho^o}$. Recall that $x \in V(G)$ is a *leaf* in G if $\deg(x) = 1$ (where the degree, $\deg(x)$, of x is the cardinality of its neighborhood), and we refer to the neighbor of x as its *support vertex*. Additionally, we say that v is a *strong support vertex* if v is adjacent to more than one leaf.

Lemma 2. *If $G \in \mathcal{C}_{\rho^o}$, then every leaf of G belongs to $U(G)$.*

Proof. Let x be a leaf and y be its support vertex. We may assume there exists a vertex $z \in U(G)$ that is adjacent to y for otherwise $U(G) \cup \{x\}$ is a larger open packing of G. If $z \neq x$, then $U' = (U(G) - \{z\}) \cup \{x\}$ is an open packing, contradicting the uniqueness of $U(G)$. $\square$

Corollary 3. *If $G \in \mathcal{C}_{\rho^o}$, then G has no strong support vertices.*

Proof. Let u and v be leaves in G. By Lemma 2, $U(G)$ contains both u and v. Since $U(G)$ is an open packing, $N(u) \cap N(v) = \emptyset$. We infer that u and v are not adjacent to the same support vertex. $\square$

Lemma 4. *Let G be a graph. If ℓ is a leaf in G with support vertex x and z is a leaf with support vertex y such that $N(y) = \{z, x\}$, then $G \notin \mathcal{C}_{\rho^o}$.*

Proof. Suppose $G \in \mathcal{C}_{\rho^o}$. By Lemma 2, $z, \ell \in U(G)$, which implies $x \notin U(G)$ and $y \notin U(G)$. Hence, $U' = (U(G) - \{\ell\}) \cup \{y\}$ is another open packing of size $\rho^o(G)$, which contradicts the fact that $U(G)$ is the unique maximum open packing of G. $\square$

Lemma 5. *Let G be a graph. If ℓ_1 and ℓ_2 are leaves of G with support vertices y_1 and y_2, respectively, $N(y_1) = \{\ell_1, x\}$ and $N(y_2) = \{\ell_2, x\}$, then $G \notin \mathcal{C}_{\rho^o}$.*

Proof. Suppose $G \in \mathcal{C}_{\rho^o}$. By Lemma 2, $\{\ell_1, \ell_2\} \subseteq U(G)$, which implies that $x \notin U(G)$. The open neighborhood of x intersects $U(G)$, for otherwise $U(G) \cup \{y_1\}$ is a

larger open packing. If $z \in U(G) \cap N(x)$ and $z \notin \{y_1, y_2\}$, then $(U(G) - \{z\}) \cup \{y_1\}$ is another maximum open packing, which is a contradiction. Thus, assume without loss of generality that $U(G) \cap N(x) = \{y_1\}$. But now $(U(G) - \{y_1\}) \cup \{y_2\}$ is a different maximum open packing, which again contradicts the assumption that $G \in \mathcal{C}_{\rho^o}$. $\qquad\square$

We now define four operations on a graph G that belongs to $\mathcal{C}_{\rho^o}$. These operations will be used in the remainder of the paper to build additional graphs in $\mathcal{C}_{\rho^o}$. By *appending* a path P_n to a vertex $x \in V(G)$ we mean make x adjacent to a leaf of a (new) path of order n. An edge $e \in E(G)$ is called a *cut edge* of G if e does not lie on any cycle, and hence the endvertices of e belong to two distinct connected components in $G - e$.
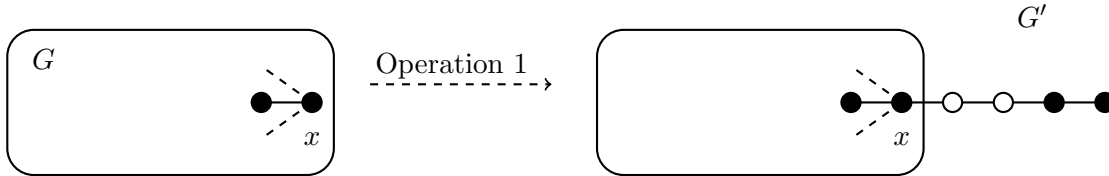

Figure 1: Operation 1; $G \in \mathcal{C}_{\rho^o}$, x and one of its neighbors are in $U(G)$; vertices of the unique maximum open packing are black.
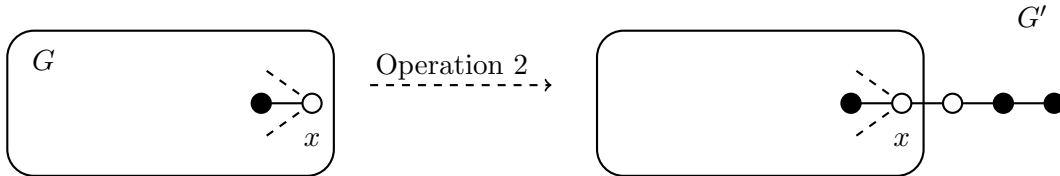

Figure 2: Operation 2; $G \in \mathcal{C}_{\rho^o}$, $x \notin U(G)$, but one of its neighbors belongs to $U(G)$.

- Operation 1: Let $x \in V(G)$ such that x and some neighbor of x both belong to $U(G)$. Append a P_4 to x. The resulting graph G' is said to be obtained by Operation 1 at x.

- Operation 2: Let $x \in V(G)$ such that x does not belong to $U(G)$, but some neighbor of x is in $U(G)$. Append a P_3 to x. The resulting graph G' is said to be obtained by Operation 2 at x.

- Operation 3: Let $x \in V(G)$ such that x belongs to $U(G)$, some neighbor of x belongs to $U(G)$, and there exists exactly one open packing of G of size $\rho^o(G) - 1$ that does not contain x. (Equivalently, the third condition states

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that $G - x \in \mathcal{C}_{\rho^\circ}$.) Append a P_3 to x and then append another P_3 to x. The resulting graph G' is said to be obtained by Operation 3 at x.

- Operation 4: Let xy be a cut edge of G such that neither x nor y belongs to $U(G)$, but both x and y have a neighbor that belongs to $U(G)$. Subdivide the edge xy and add a leaf adjacent to the new vertex of the subdivided edge. The resulting graph G' is said to be obtained by Operation 4 at xy.
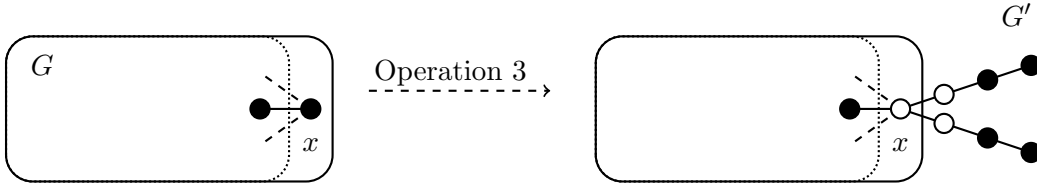

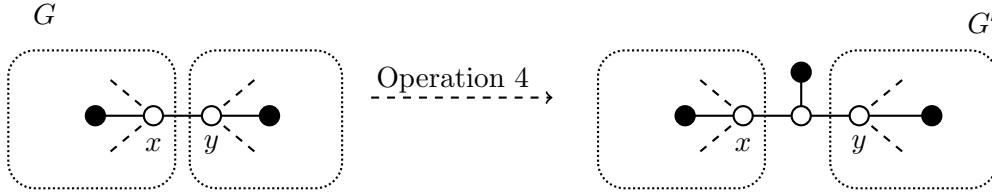Figure 3: Operation 3; both G and $G - x$ belong to $\mathcal{C}_{\rho^\circ}$.

Figure 4: Operation 4; $G \in \mathcal{C}_{\rho^\circ}$, and xy is a cut edge in G.

We first show that if $G \in \mathcal{C}_{\rho^\circ}$ and G' is obtained from G by performing one of the above operations, then $G' \in \mathcal{C}_{\rho^\circ}$.

Lemma 6. *Let $G \in \mathcal{C}_{\rho^\circ}$ and assume that there exists a vertex x such that $x \in U(G)$ and a neighbor y of x also belongs to $U(G)$. If G' is the graph obtained from G by Operation 1 at x, then $G' \in \mathcal{C}_{\rho^\circ}$.*

Proof. Label the vertices of the appended P_4 a, b, c and d so that $xa \in E(G')$. We let $\rho^\circ(G) = k$. Since $U(G) \cup \{c, d\}$ is an open packing of G', it follows that $\rho^\circ(G') \geq k + 2$. Let S be a maximum open packing of G'. Note that $|S \cap \{a, b, c, d\}| \leq 2$, which implies that $S \cap V(G)$ is an open packing of G of size at least k. Hence $|S \cap \{a, b, c, d\}| = 2$ and $S \cap V(G) = U(G)$. Hence $x, y \in S$, which implies that $S \cap \{a, b, c, d\} = \{c, d\}$, and S is unique. Therefore, $G' \in \mathcal{C}_{\rho^\circ}$. ☐

Lemma 7. *Let $G \in \mathcal{C}_{\rho^\circ}$ and assume that there exists a vertex x such that $x \notin U(G)$, but a neighbor y of x belongs to $U(G)$. If G' is the graph obtained from G by Operation 2 at x, then $G' \in \mathcal{C}_{\rho^\circ}$.*

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Proof. Label the vertices of the appended P_3 a, b, and c so that $xa \in E(G')$. We let $\rho^o(G) = k$. Since $U(G) \cup \{b, c\}$ is an open packing of G', it follows that $\rho^o(G') \geq k+2$. If M is a $\rho^o(G')$-set, then $|M \cap V(G)| \leq k$, which implies that $\rho^o(G') \leq k + 2$. Therefore, $\rho^o(G') = k+2$. To see that $G' \in \mathcal{C}_{\rho^o}$, suppose there exists another $\rho^o(G')$-set T different from $U(G) \cup \{b, c\}$. We may assume $|T \cap \{a, b, c\}| \leq 1$ for otherwise $T \cap V(G)$ is a $\rho^o(G)$-set different from $U(G)$. However, this implies $|T \cap V(G)| \geq k+1$, contradicting the assumption that $\rho^o(G) = k$. Therefore, no such T exists and $G' \in \mathcal{C}_{\rho^o}$. $\qquad\square$

Lemma 8. *Let $G \in \mathcal{C}_{\rho^o}$ and assume that there is a vertex $x \in U(G)$ such that $N(x) \cap U(G) \neq \emptyset$. Moreover, suppose that G has exactly one open packing of cardinality $\rho^o(G) - 1$ that does not contain x. If G' is the graph obtained from G by Operation 3 at x, then $G' \in \mathcal{C}_{\rho^o}$.*

Proof. Label the vertices of the two appended paths of order 3 as a, b, and c, and a', b' and c', respectively, where $xa \in E(G')$ and $xa' \in E(G')$. We let $\{y\} = U(G) \cap N(x)$ and we assume that $\rho^o(G) = k$. We claim that $\rho^o(G') = k+3$. Certainly $(U(G) - \{x\}) \cup \{b, b', c, c'\}$ is an open packing of G', and hence $\rho^o(G') \geq k + 3$. Suppose that $\rho^o(G') \geq k+4$. Let T be an open packing in G' of cardinality $k+4$. If $|T \cap \{a, a', b, b', c, c'\}| \leq 3$, then $|T \cap V(G)| \geq k+1$, which contradicts our assumption that $\rho^o(G) = k$. Thus, $|T \cap \{a, a', b, b', c, c'\}| = 4$. If $a \in T$, then $T \cap V(G)$ is another $\rho^o(G)$-set different from $U(G)$ since $y \notin T \cap V(G)$. Therefore, $\{b, b', c, c'\} \subset T$ and $T \cap V(G)$ is a $\rho^o(G)$-set different from $U(G)$ as $x \notin T \cap V(G)$. Therefore, no such set T exists and $\rho^o(G') = k + 3$.

Let M be a maximum open packing of G'. Notice that $|M \cap \{a, a', b, b', c, c'\}| \geq 3$ for otherwise $|M \cap V(G)| \geq k + 1$, which is a contradiction. Thus, $\{b, b'\} \cap M \neq \emptyset$. This implies that $x \notin M$. Now, if a or a' belongs to M, then $y \notin M$, which implies that $M \cap V(G)$ is an open packing of G of size at least $k - 1$ such that $M \cap \{x, y\} = \emptyset$. This contradicts the assumptions on G since the open packing of G of size $\rho^o(G)$ is unique and since there is only one open packing of G of size $\rho^o(G) - 1$ that does not contain x. Thus, neither a nor a' is in M. This implies that $\{b, b', c, c'\} \subset M$, for otherwise M is not maximum. Therefore, $|M \cap V(G)| = k - 1$ and hence $M \cap V(G)$ is unique, which implies that M is unique. Thus, $G' \in \mathcal{C}_{\rho^o}$. $\qquad\square$

Lemma 9. *Let $G \in \mathcal{C}_{\rho^o}$ and let xy be a cut edge in G such that neither x nor y belongs to $U(G)$ yet $N(x) \cap U(G) \neq \emptyset$ and $N(y) \cap U(G) \neq \emptyset$. If G' is the graph obtained from G by Operation 4 at xy, then $G' \in \mathcal{C}_{\rho^o}$.*

Proof. Let b be the vertex of the subdivided edge and let a be the leaf adjacent to b. We shall assume that $\rho^o(G) = k$. Also assume that $N(x) \cap U(G) = \{w\}$ and $N(y) \cap U(G) = \{z\}$. Note that $\rho^o(G') \geq k+1$ since $U(G) \cup \{a\}$ is an open packing of G'. Suppose there exists an open packing M of G' of size $k+2$. If $|M \cap \{x, y, a, b\}| = 2$, then clearly $b \in M$, but then $M \cap V(G)$ is a different open packing from $U(G)$ of

size k, which is a contradiction. Therefore, $|M \cap \{x, y, a, b\}| \leq 1$. This implies that $|M \cap (V(G') - \{x, y, a, b\})| \geq k + 1$. Note that $M \cap (V(G') - \{x, y, a, b\})$ is an open packing also in G, which contradicts $\rho^o(G) = k$. Therefore, no such M exists and we conclude that $\rho^o(G') = k + 1$. Clearly, $T = U(G) \cup \{a\}$ is an open packing of size $\rho^o(G')$.

To see that $G' \in \mathcal{C}_{\rho^o}$, suppose S is a $\rho^o(G')$-set of cardinality $k + 1$. Let L be the component of $G - y$ containing x and let R be the component of $G - x$ containing y. Note that $U(G) \cap V(L)$ is the unique maximum open packing of L that does not contain x, and $U(G) \cap V(R)$ is the unique maximum open packing of R that does not contain y. Let $|V(L) \cap U(G)| = \ell$ and $|V(R) \cap U(G)| = r$. We claim that $|S \cap V(L)| \leq \ell$. Otherwise, $(S \cap V(L)) \cup (U(G) \cap (V(R) - \{z\}))$ is an open packing of G of cardinality at least k different from $U(G)$, a contradiction. In a similar way we can prove that $|S \cap V(R)| \leq r$.

Suppose $x \in S$. Then $\{a, y\} \cap S = \emptyset$. We have two cases with respect to b belonging to S. If $b \notin S$, then $|S \cap V(R)| \geq r + 1$, which contradicts the above observation. On the other hand, if $b \in S$, then $S - \{b\}$ is an open packing of G of size k different from $U(G)$, which contradicts the assumption that $G \in \mathcal{C}_{\rho^o}$. This implies that $x \notin S$. In a similar way one proves that $y \notin S$.

Finally, we claim that $b \notin S$. Suppose $b \in S$. From this we derive that $N(y) \cap S = \{b\} = N(x) \cap S$. Hence, $S \cap V(L)$ is an open packing of L that does not contain x. Since $S \cap V(L)$ also does not contain w, we infer that $|S \cap V(L)| < \ell$, because there is just one open packing of size ℓ in L that does not contain x, and it contains w. Similarly, $|S \cap V(R)| < r$. But now, we infer that $k + 1 = |S| \leq (\ell - 1) + (r - 1) + 2 \leq k$, a contradiction, which yields that $b \notin S$. Since S is a maximum open packing of G', we infer that $|V(L) \cap S| = \ell$, $|V(R) \cap S| = r$ and $a \in S$. This readily implies that $S = T$.

Therefore, $U(G) \cup \{a\}$ is the only $\rho^o(G')$-set, and hence $G' \in \mathcal{C}_{\rho^o}$. $\qquad\square$

From this point on we concentrate on the class of trees in $\mathcal{C}_{\rho^o}$. For this purpose we construct the class $\mathcal{O}$ as follows. A tree T is in $\mathcal{O}$ if and only if $T = P_1$, $T = P_2$ or T can be constructed from P_2 by a finite sequence of steps, each one of which is Operation 1, Operation 2, Operation 3, or Operation 4.

Let $\mathcal{T}_{\rho^o}$ be the class of all trees that have a unique maximum open packing. Lemmas 6, 7, 8 and 9 show that $\mathcal{O} \subseteq \mathcal{T}_{\rho^o}$. To prove $\mathcal{T}_{\rho^o} \subseteq \mathcal{O}$ we will use induction on the order of a tree. We will show that $T' \in \mathcal{T}_{\rho^o}$ implies that T' can be obtained from a smaller tree, which is in $\mathcal{T}_{\rho^o}$, by using one of the four operations. The induction hypothesis implies that this smaller tree is also in $\mathcal{O}$, which finally implies that $T' \in \mathcal{O}$.

The basis of induction with trees P_1 and P_2 is clear by definition of $\mathcal{O}$. Let T' be a tree in $\mathcal{T}_{\rho^o}$ with $|V(T')| > 2$. If $\Delta(T') = 2$, then T' is a path. In the following result we determine which paths belong to $\mathcal{T}_{\rho^o}$.

Lemma 10. *If $T' = P_n$, $n \geq 2$, then $T' \in \mathcal{T}_{\rho^o}$ if and only if $n \equiv 2 \pmod 4$.*

Proof. One can easily verify that P_3, P_4, and P_5 are not in $\mathcal{T}_{\rho^\circ}$ yet P_6 is in $\mathcal{T}_{\rho^\circ}$. Suppose $n \geq 7$. Let $P_n = x_1 x_2 \cdots x_n$. Recall by Theorem 1 that $\gamma_t(P_n) = \rho^\circ(P_n)$ for any positive integer $n \geq 2$. Since the total domination numbers of the paths are known, we use this below to know when an open packing of a path is maximum. Consider the following subsets of $V(P_n)$.

- $S_1 = \{x_k : k \equiv 1 \pmod 4 \text{ or } k \equiv 2 \pmod 4\}$,

- $S_2 = \{x_k : k \equiv 0 \pmod 4 \text{ or } k \equiv 1 \pmod 4\}$,

- $S_3 = \{x_k : k \equiv 2 \pmod 4 \text{ or } k \equiv 3 \pmod 4\}$,

For $n \equiv 0 \pmod 4$, $\rho^\circ(P_n) = n/2$ and it is easy to check that S_1 and $V(P_n) - S_1$ are two distinct maximum open packings of P_n, which implies that $P_n \notin \mathcal{T}_{\rho^\circ}$. For $n \equiv 1 \pmod 4$, $\rho^\circ(P_n) = (n+1)/2$ and it is easy to check that S_1 and S_2 are two distinct maximum open packings of P_n, which implies that $P_n \notin \mathcal{T}_{\rho^\circ}$. If $n \equiv 2 \pmod 4$, then it follows from repeated application of Lemma 6 starting with P_2 that $P_n \in \mathcal{T}_{\rho^\circ}$. Finally, if $n \equiv 3 \pmod 4$, then $\rho^\circ(P_n) = (n+1)/2$ and it is easy to check that S_1 and S_3 are two distinct maximum open packings of P_n, which shows that $P_n \notin \mathcal{T}_{\rho^\circ}$. $\qquad\square$

Hence, the case when T' is a path is settled. That is, $P_n \in \mathcal{T}_{\rho^\circ}$ implies $n \equiv 2 \pmod 4$, which in turn implies $P_n \in \mathcal{O}$ since P_n can be constructed from P_2 by applying Operation 1 a finite number of times.

Now, let $\Delta(T') \geq 3$. We define a graph $L(T')$ as follows. Vertices of $L(T')$ are vertices of T' of degree at least 3, and two vertices p, q of $L(T')$ are adjacent if $pq \in E(T')$ or if the internal vertices in the unique path from p to q in T' are of degree 2. It is clear that $L(T')$ is also a tree. Indeed, $L(T')$ is connected, and the eventual existence of a cycle C in $L(T')$ would immediately imply the existence of a cycle in T' in which every edge of C that connects two vertices which are not adjacent in T' is replaced by the corresponding path in T' between these two vertices. In each of the subsequent lemmas we consider a vertex $x \in V(T')$, which is a leaf in $L(T')$. Note that x is a vertex in T' with $\deg_{T'}(x) \geq 3$, and since x has only one neighbor in $L(T')$, all components of $T' - x$, with only one possible exception, are paths.

Lemma 11. *Let $T' \in \mathcal{T}_{\rho^\circ}$ and let x be a leaf in $L(T')$. Suppose there is a path $x = u_0, u_1, \ldots, u_k$ in T', where $\deg_{T'}(u_i) = 2$ for $i \in [k-1]$, $\deg_{T'}(u_k) = 1$, and $k \geq 4$. The tree $T = T' - \{u_{k-3}, \ldots, u_k\}$ belongs to $\mathcal{T}_{\rho^\circ}$. In addition, T' is obtainable from T by Operation 1 at u_{k-4}.*

Proof. Let $U(T')$ be the unique $\rho^\circ(T')$-set. By Lemma 2, $u_k \in U(T')$. If $u_{k-3} \in U(T')$, then $U'' = (U(T') - \{u_{k-3}\}) \cup \{u_{k-1}\}$ is $\rho^\circ(T')$-set different from $U(T')$, which contradicts the uniqueness of $U(T')$. Hence, $\{u_{k-1}, u_k\} \subset U(T')$. Let $M = U(T') \cap V(T)$. If u_{k-4} is not totally dominated by M, then $U'' = (U(T') - \{u_{k-1}\}) \cup$

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$\{u_{k-3}\}$ is another $\rho^o(T')$-set different from $U(T')$, another contradiction. Therefore, $N_T(u_{k-4}) \cap M \neq \emptyset$. Denote by z the vertex in $N_T(u_{k-4}) \cap M$. If $u_{k-4} \notin U(T')$, then $U''' = (U(T') - \{u_k\}) \cup \{u_{k-2}\}$ is another $\rho^o(T')$-set. Therefore, $u_{k-4} \in M$. Note that M is a $\rho^o(T)$-set, and that M contains u_{k-4} and its neighbor z.

Finally, to see that $T \in \mathcal{T}_{\rho^o}$, suppose that S is a $\rho^o(T)$-set different from M. Then $S \cup \{u_{k-1}, u_k\}$ and $M \cup \{u_{k-1}, u_k\}$ are two different $\rho^o(T')$-sets, which is a contradiction. Therefore, $T \in \mathcal{T}_{\rho^o}$ and note that T' is obtained from T by Operation 1 at u_{k-4}. $\qquad\square$

Let x be a leaf in $L(T')$. By using Lemma 11 (multiple times if necessary) we may assume that $k \leq 3$ for any path $x = u_0, u_1, \ldots, u_k$, where $\deg_{T'}(u_i) = 2$ for $i \in [k-1]$ and $\deg_{T'}(u_k) = 1$. In addition, by Corollary 3 and Lemmas 4 and 5, we infer that at most one such path is of length $k \in \{1, 2\}$, and all other such paths are of length $k = 3$ (since $\deg_{T'}(x) \geq 3$ there is at least one such path of length 3).

Lemma 12. *Let $T' \in \mathcal{T}_{\rho^o}$ and let x be a leaf in $L(T')$. Suppose there are two paths x, a, b, c and x, a', b', c', where $\deg_{T'}(a) = \deg_{T'}(a') = \deg_{T'}(b) = \deg_{T'}(b') = 2$, and $\deg_{T'}(c) = \deg_{T'}(c') = 1$. If $T_1 = T' - \{a, b, c\}$ and $T = T' - \{a, b, c, a', b', c'\}$, then at least one of the trees T, T_1 is in $\mathcal{T}_{\rho^o}$. In addition, T' is obtainable from T by Operation 3 at x, or T' is obtainable from T_1 by Operation 2 at x.*

Proof. By Lemma 2, $\{c, c'\} \subset U(T')$. If $a \notin N(U(T'))$, then $U(T')$ is not maximum because $U(T') \cup \{b\}$ is also an open packing of T'. Therefore, either $x \in U(T')$ or $b \in U(T')$. If $x \in U(T')$, then $(U(T') - \{x\}) \cup \{b, b'\}$ is an open packing of T' of cardinality larger than $\rho^o(T')$, a contradiction. Thus, $\{b, b'\} \subset U(T')$.

Let $T_1 = T' - \{a, b, c\}$. Let $\rho^o(T') = k$ and note that $U(T') - \{b, c\}$ is an open packing of T_1. Hence $\rho^o(T_1) \geq k - 2$. Suppose that $\rho^o(T_1) \geq k - 1$ and let X be a $\rho^o(T_1)$-set. It follows that $x \in X$, for otherwise $X \cup \{b, c\}$ is an open packing of T' of cardinality $k + 1$. However, $X \cup \{c\}$ is then a different maximum open packing of T', which is a contradiction. Therefore, $\rho^o(T_1) = k - 2$.

Next note that if x is not in $N(U(T'))$, then $(U(T') - \{c'\}) \cup \{a'\}$ is a different $\rho^o(T')$-set. Therefore, x is totally dominated by $U(T')$ and there exists $y \in N_T(x) \cap U(T')$. Note, that if there exist two different $\rho^o(T_1)$-sets that do not contain x, then there exist two different $\rho^o(T')$-sets, which is a contradiction. This implies that $W_1 = U(T') - \{b, c\}$ is the unique maximum open packing of T_1 that does not contain x. Now, if $T_1 \in \mathcal{T}_{\rho^o}$, then T' is obtained from T_1 using Operation 2 at x.

Thus, we may assume $T_1 \notin \mathcal{T}_{\rho^o}$ and all $\rho^o(T_1)$-sets except W_1 contain x. Let P be a $\rho^o(T_1)$-set that contains x. We know that P contains precisely one of a' or c'. Therefore, $\rho^o(T) \geq k - 3$ since $P - \{a', c'\}$ is an open packing of T. On the other hand, if there exists an open packing W of T, of cardinality $k - 2$, then $W \cup \{c, c'\}$ is a different $\rho^o(T')$-set. Therefore $\rho^o(T) = k - 3$. Let $R = V(T) \cap P$ and $R' = (R - \{x\}) \cup \{b, b', c, c'\}$. We infer that $R' = U(T')$, for otherwise there would be two different $\rho^o(T')$-sets. This implies that there is exactly one $\rho^o(T)$-set that

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contains x. Moreover, every $\rho^o(T)$-set S contains x, for otherwise $S \cup \{b, b', c, c'\}$ would be an open packing of T' of size $k+1$. This in turn implies that $T \in \mathcal{T}_{\rho^o}$ and the unique $\rho^o(T)$-set is R. In addition, $R - \{x\}$ is a unique open packing of T that does not contain x and is of size $\rho^o(T) - 1$ (because $(R - \{x\}) \cup \{b, b', c, c'\}$ is $U(T')$, which contains a neighbor of x). So we can obtain T' from T by using Operation 3 at x. $\qquad\square$

Applying Lemma 12 repeatedly we may assume that $k = 3$ for at most one path $x = u_0, u_1, \ldots, u_k$, where $\deg_{T'}(u_i) = 2$ for $i \in [k-1]$ and $\deg_{T'}(u_k) = 1$. In addition, by Corollary 3 and Lemmas 4 and 5, we may assume that there is one such path with $k = 3$, for otherwise $\deg_{T'}(x) < 3$ and x is therefore not a leaf of $L(T')$. For the same reason (i.e., $\deg_{T'}(x)$ must be at least 3), there is another path, which is either x, e or x, d, e, where $\deg_{T'}(d) = 2$ and $\deg_{T'}(e) = 1$. In the next lemma we deal with the former.

Lemma 13. *Let $T' \in \mathcal{T}_{\rho^o}$ and let x be a leaf in $L(T')$ such that $\deg_{T'}(x) = 3$. Suppose there is a path x, a, b, c and a path x, e, where $\deg_{T'}(a) = \deg_{T'}(b) = 2$, and $\deg_{T'}(c) = \deg_{T'}(e) = 1$. If $\{y\} = N(x) - \{a, e\}$, and T is the tree obtained from T' by removing the vertices x and e and adding the edge ay, then T is in $\mathcal{T}_{\rho^o}$. In addition, T' is obtainable from T by Operation 4 at ay.*

Proof. Assume $\rho^o(T') = k$. Note that $\{c, e\} \subseteq U(T')$, which implies that neither a nor y is in $U(T')$. If $x \in U(T')$, then $(U(T') - \{x\}) \cup \{b\}$ is an open packing of T' of size k, which contradicts the uniqueness of $U(T')$. Therefore $b \in U(T')$ and $x \notin U(T')$. Note that $U(T') - \{e\}$ is an open packing of T, hence $\rho^o(T) \geq k - 1$. Suppose there exists a $\rho^o(T)$-set M of size k. Then M is a $\rho^o(T')$-set which is different from $U(T')$. Thus, we infer $\rho^o(T) = k - 1$.

Let S be a $\rho^o(T)$-set and let $H = T - \{a, b, c\}$. Suppose $|S \cap V(H)| \geq k - 2$. Then $(S \cap V(H)) \cup \{b, c\}$ is a maximum open packing of T' different from $U(T')$, a contradiction. Therefore, $|S \cap V(H)| = k - 3$. If $y \in S$, then $|S \cap \{a, b, c\}| \leq 1$, which implies $|S| \leq k - 2$, a contradiction. Therefore, $y \notin S$, which implies that $b \in S$. We claim that y is totally dominated by vertices in $S \cap V(H)$. If this claim were not true, then $(S \cap V(H)) \cup \{e, x, c\}$ and $(S \cap V(H)) \cup \{e, b, c\}$ are two different $\rho^o(T')$-sets. This contradiction says that y is totally dominated by vertices in $S \cap V(H)$, which in turn implies $a \notin S$. Also, $S \cap V(H) = U(T') \cap V(H)$, because $|S \cap V(H)| = |U(T') \cap V(H)| = k - 3$, none of the sets contains y, and $U(T')$ is unique. Since S is a maximum open packing of T we conclude that $\{b, c\} \subset S$, which makes S the unique $\rho^o(T)$-set of size $k - 1$. Thus, $T \in \mathcal{T}_{\rho^o}$, and $\{a, y\} \cap U(T) = \emptyset$. Hence, T' can be obtained from T by Operation 4 at ay. $\qquad\square$

Finally, we are in the case when $\deg_{T'}(x) = 3$, there are two paths x, a, b, c and x, d, e with $\deg_{T'}(a) = \deg_{T'}(b) = \deg_{T'}(d) = 2$ and with c and e both leaves of T'.

Lemma 14. *Let $T' \in \mathcal{T}_{\rho^o}$ and let x be a leaf in $L(T')$ such that $N_{T'}(x) = \{a, d, y\}$. Suppose there is a path x, a, b, c and a path x, d, e, where $\deg_{T'}(a) = \deg_{T'}(b) = \deg_{T'}(d) = 2$, and $\deg_{T'}(c) = \deg_{T'}(e) = 1$. If $T_1 = T' - \{a, b, c\}$, then T_1 is in $\mathcal{T}_{\rho^o}$. In addition, T' is obtainable from T_1 by Operation 2 at x.*

Proof. Let $\rho^o(T') = k$. Let $T_2 = T' - \{a, b, c, d, e, x\}$, and let $W = U(T') \cap V(T_2)$. By Lemma 2, we know $\{c, e\} \subset U(T')$. Thus, neither a nor x is in $U(T')$. It follows that $b \in U(T')$. Furthermore, $d \in U(T')$, for otherwise $y \in U(T')$ and then $(U(T') - \{y\}) \cup \{d\}$ is another $\rho^o(T')$-set, which is a contradiction. Hence, $\{b, c, d, e\} \subseteq U(T')$, $x \notin U(T')$, $y \notin U(T')$ and W is the only open packing of T_2 of cardinality $k - 4$ that does not contain y. Since $U(T') - \{b, c\}$ is an open packing of T_1, we know that $\rho^o(T_1) \geq k - 2$. Suppose there exists an open packing M of T_1 of cardinality at least $k - 1$. If $x \notin M$, then $M \cup \{b, c\}$ is an open packing of T' of cardinality at least $k + 1$ which is a contradiction. On the other hand, if $x \in M$, then $(M - \{x\}) \cup \{b, c, e\}$ is an open packing of T' of cardinality at least $k + 1$, which is also a contradiction. Therefore, no such M exists, and $\rho^o(T_1) = k - 2$. Let $X = V(T_1) \cap U(T') = U(T') - \{b, c\}$. The set X is a maximum open packing of T_1. If $T_1 \in \mathcal{T}_{\rho^o}$, then $X = U(T_1)$ and T' can be obtained from T_1 by Operation 2 at x. Hence, it remains to prove that $T_1 \in \mathcal{T}_{\rho^o}$.

We claim that $X = W \cup \{d, e\}$ is the only open packing of T_1 of cardinality $k - 2$. Let S be any open packing of T_1 such that $|S| = k - 2$. Suppose that $y \in S$. Since S is an open packing, $d \notin S$. Since S is a maximum open packing of T_1, we infer that $|\{x, e\} \cap S| = 1$. Thus $S \cap (V(T_2) - \{y\})$ is an open packing of T_2 of cardinality $k - 4$ that does not contain y. From above this means that $S \cap (V(T_2) - \{y\}) = W$. But then $W \cup \{y, e, b, c\}$ is a $\rho^o(T')$-set that is different from $U(T')$, which is a contradiction. Therefore, $y \notin S$. Since $|S \cap \{x, d, e\}| \leq 2$, it follows that $|S \cap V(T_2)| \geq k - 4$ and $S \cap V(T_2)$ is an open packing of T_2 that does not contain y. Since W is the only such open packing of T_2, we get that $S = W \cup \{d, e\}$, which coincides with X. That is, $T_1 \in \mathcal{T}_{\rho^o}$, and T' is obtainable from T_1 by Operation 2 at x. $\qquad \square$

Combining the above lemmas, we infer the following, our main result.

Theorem 15. *Let T be an arbitrary tree. Then T has a unique maximum open packing if and only T is P_1, P_2, or can be obtained from P_2 by a sequence of Operations 1-4.*

3 Computational complexity of related problems

Given a graph G, consider the subdivision $S(G)$ of G. The vertices of $S(G)$ that are the result of the subdivision are *subdivided vertices*, while other vertices of $S(G)$ (which correspond to vertices of G) are *original vertices*. If $e \in E(G)$, then by v_e we denote the subdivided vertex of $S(G)$ that corresponds to the edge e. Clearly, v_e is

in $S(G)$ adjacent only to two original vertices that correspond to the endvertices of e in G.

Theorem 16. *If G is a graph, then $\rho^o(S(G)) = \alpha(G) + \alpha'(G)$.*

Proof. First, we prove that $\rho^o(S(G)) \geq \alpha(G) + \alpha'(G)$. Let A be a maximum independent set and M be a maximum matching in a graph G. Denote by A' the set of original vertices in $S(G)$ that correspond to the vertices of A, and let M' be the subdivided vertices of $S(G)$ that correspond to the edges of M. We claim that $A' \cup M'$ is an open packing of $S(G)$. Note that any two original vertices in A' are at distance at least 4 in $S(G)$. The same holds for any two subdivided vertices in M'; indeed, since two edges $e, e' \in M$ have no common endvertex, the distance between the vertices v_e and $v_{e'}$ in $S(G)$ is at least 4. Now, if $a \in A'$ and $v_e \in M'$, then we have two possibilities. If $av_e \in E(S(G))$, then $N(a) \cap N(v_e) = \emptyset$, because the only neighbor of v_e different from a is an original vertex b, hence a and b are not adjacent in $S(G)$. But if a and v_e are not adjacent, then their distance is at least 3. By this we conclude that $A' \cup M'$ is indeed an open packing of $S(G)$, thus $\rho^o(S(G)) \geq \alpha(G) + \alpha'(G)$.

For the reversed inequality, consider a maximum open packing U of $S(G)$, and let A' be the set of original vertices in U and M' be the set of subdivided vertices in U. Now, let A be the set of original vertices that correspond to A' and M the set of edges that correspond to vertices in M'. It is clear that A must be an independent set in G, because for two adjacent vertices $x, y \in V(G)$ the corresponding original vertices in $S(G)$ share a common neighbor, which is the vertex v_{xy}. Hence, $\alpha(G) \geq |A| = |A'|$. In a similar way, we deduce $\alpha'(G) \geq |M| = |M'|$, because the edges of M must form a matching, since M' is an open packing. Combining the latter inequalities, we get $\alpha(G) + \alpha'(G) \geq |A'| + |M'| = |U| = \rho^o(S(G))$. $\qquad\square$

By a classical result of Edmonds [2], a maximum matching of an arbitrary graph G, and hence also $\alpha'(G)$, can be obtained in polynomial time. On the other hand, the decision version of the independence number is a well-known NP-complete problem (its dual invariant vertex cover number appeared in Karp's list of NP-complete problems [12]). This implies that the decision version of ρ^o is an NP-complete problem. Due to our reduction, it is NP-complete even in subdivision graphs (that is, the class of graphs that can be realized as the subdivision, $S(G)$, of some graph G), and consequently also in graphs of girth at least 6. We therefore derive the following result.

Corollary 17. *Let G be a graph, and k a positive integer. The problem of deciding whether $\rho^o(G) \geq k$ is NP-complete, even if G is restricted to subdivision graphs.*

We next focus on the computational aspects with respect to the classes $\mathcal{C}_{\rho^o}$ and $\mathcal{T}_{\rho^o}$. To the best of our knowledge, graphs that have a unique maximum open packing have not been studied before. On the other hand, graphs with a unique maximum

independent set appeared in several contexts. Recently, Jaume and Molina [11] considered the trees with a unique maximum independent set in the context of the so-called null decomposition of trees. This is a decomposition of a tree into two types of trees, one of which are the trees with a unique maximum independent set. In addition, these trees are characterized through a property of their adjacency matrix as follows. The null space $\mathcal{N}(T)$ of a tree T is defined as the null space of its adjacency matrix, and its *support*, $supp(\mathcal{N}(T))$, is the set of coordinates that are non-zero for at least one of the generators of $\mathcal{N}(T)$. Now, a tree T has a unique maximum independent set if and only if the set $supp(\mathcal{N}(T))$ is a dominating set of T. Since the null space of a tree (and therefore also $supp(\mathcal{N}(T))$) can be computed in polynomial time, and it is trivial to check if $supp(\mathcal{N}(T))$ is a dominating set of T, we infer that the class of trees with a unique maximum independent set can be recognized efficiently.

Since the independent set and open packing are related problems we wonder whether a similar algebraic approach could work for graphs with unique maximum open packing. In addition, could a more involved analysis of Operations 1-4 from the previous section, result in an efficient algorithm for recognition of the trees in $\mathcal{T}_{\rho^\circ}$? We pose this as a question and suspect it has a positive answer.

Question 1. *Is there an algorithm to efficiently recognize the trees in $\mathcal{T}_{\rho^\circ}$?*

We now focus on the problem of recognizing the graphs in $\mathcal{C}_{\rho^\circ}$, which we relate to two similar problems. Recall that a *2-packing* in a graph G is a set of vertices with the property that their closed neighborhoods are pairwise disjoint. (The *closed neighborhood* of a vertex $x \in V(G)$ is defined as $N[x] = N(x) \cup \{x\}$.) Alternatively, S is a 2-packing in G if the distance between any two distinct vertices in S is at least 3. The *2-packing number*, $\rho(G)$, of G is the maximum cardinality of a 2-packing in G.

We denote by $\mathcal{C}_\rho$ the class of graphs G that have a unique maximum 2-packing, and by $\mathcal{C}_\alpha$ the class of graphs with a unique maximum independent set. We next prove that the problems of recognizing the graphs in classes $\mathcal{C}_{\rho^\circ}, \mathcal{C}_\rho$ and $\mathcal{C}_\alpha$, respectively, are polynomially equivalent problems. In particular, a polynomial algorithm that recognizes graphs from one of the classes implies the existence of a polynomial algorithm to recognize graphs from the other two classes.

We start by showing how the problem of recognizing whether G is a graph with a unique maximum 2-packing translates to the problem of recognizing whether a graph is in $\mathcal{C}_{\rho^\circ}$. For this purpose, we need some more definitions. The *distance* in a graph G between two vertices x and y is defined as the length of a shortest path between x and y, and we denote it by $d_G(x,y)$. The *diameter* of G, $\mathrm{diam}(G)$, is the maximum distance between vertices in G.

Starting from an arbitrary connected graph G, we construct the graph G' by adding some vertices and edges to G as follows. For each edge $e = xy$ in G we add a new vertex v_e, and connect v_e by an edge with x and with y. Finally, for every

two distinct edges $e, f \in E(G)$ we make v_e and v_f adjacent in G'. Note that vertices of the set $C = \{v_e \: : \: e \in E(G)\}$ induce a clique in G', and for any two vertices $x, y \in V(G)$, we have $d_{G'}(x, y) = \min\{d_G(x, y), 3\}$. If $x \in V(G)$ and $e \in E(G)$, then $d_{G'}(x, v_e) \leq 2$, and $d_{G'}(x, v_e) = 1$ if and only if x is an endvertex of e. We also infer that if $\mathrm{diam}(G) > 2$, then $\mathrm{diam}(G') = 3$. This construction is illustrated using a small example in Figure 5.
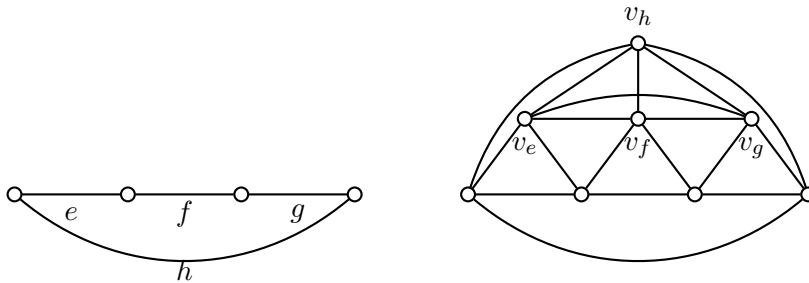

Figure 5: Example of graph G (left) and G' (right)

Lemma 18. *If G is a connected graph, then $\rho^\circ(G') = \rho(G)$. In addition, G is in $\mathcal{C}_\rho$ if and only if G' is in $\mathcal{C}_{\rho^\circ}$.*

Proof. Let G' be obtained by the described construction from a (non-trivial) connected graph G, and let S be a maximum open packing of G' (the case when G is isomorphic to K_1 is clear). Note that any two adjacent vertices $x, y \in V(G)$ have v_{xy} as their common neighbor in G'. Hence at most one of the adjacent vertices x, y belongs to S.

Suppose that $C \cap S \neq \emptyset$, where C is the set of vertices that are not in $V(G)$ (and arise from edges of G). If $v \in C \cap S$, then note that v is the only vertex of S. Hence, $\rho^\circ(G') = 1$, which is only possible when $\mathrm{diam}(G) \leq 2$. But when $\mathrm{diam}(G) \leq 2$, we have $\rho^\circ(G') = 1 = \rho(G)$, G has more than one maximum 2-packing, and also G' has more than one maximum open packing. In this case, the proof is done.

Now, we may assume that $C \cap S = \emptyset$ and $\mathrm{diam}(G) > 2$. Note that S is a (maximum) open packing of G' if and only if S is a (maximum) 2-packing of G. From this we infer $\rho^\circ(G') = \rho(G)$, and the second statement of the lemma also readily follows. $\qquad \square$

For the reversed translation we provide the following construction. Let G be an arbitrary graph on n vertices. By adding additional vertices and edges to G we construct the graph G^+ as follows. For each vertex $u \in V(G)$ we add six additional vertices denoted as $u_1, u_2, u_3, u'_1, u'_2, u'_3$. Connect with an edge all pairs u_i, u_j when $i \neq j$; also for each $i \in \{1, 2, 3\}$, connect u_i with u and with u'_i. Finally, for every

edge $uv \in E(G)$, add an edge between u'_1 and v, and add an edge between v'_1 and u. Note that $|V(G^+)| = 7n$, and for any $u \in V(G)$ we have $\deg_{G^+}(u) = 2\deg_G(u) + 3$, $\deg_{G^+}(u_i) = 4$, $\deg_{G^+}(u'_1) = \deg_G(u) + 1$, and $\deg_{G^+}(u'_2) = 1 = \deg_{G^+}(u'_3)$. In Figure 6 we illustrate this construction with $G = P_3$.
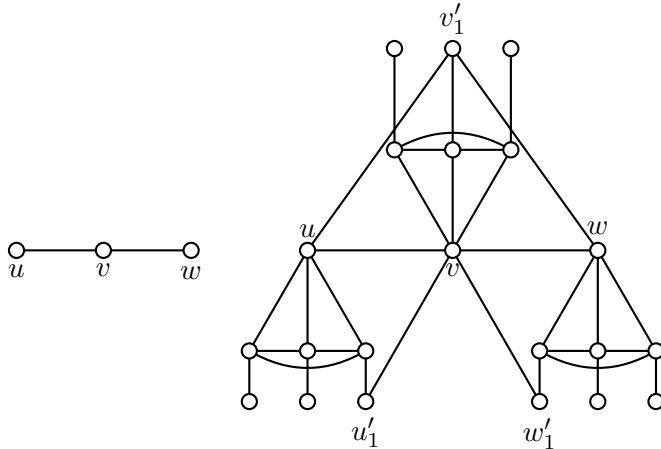

Figure 6: Example of graph G (left) and G^+ (right)

Lemma 19. *If G is a graph on n vertices, then $\rho(G^+) = 2n + \rho^o(G)$. In addition, G is in $\mathcal{C}_{\rho^o}$ if and only if G^+ is in $\mathcal{C}_\rho$.*

Proof. Let S be a maximum 2-packing of G^+. Note that for any $u \in V(G)$, there can be at most three vertices in $S \cap \{u_1, u_2, u_3, u'_1, u'_2, u'_3\}$, namely, u'_1, u'_2, u'_3, which form a 2-packing. Also note that $u \notin S$, because u is at distance at most 2 from any other vertex in $\{u_1, u_2, u_3, u'_1, u'_2, u'_3\}$; for the same reason, $u_i \notin S$ for any $i \in [3]$. Next, it is easy to see that for any $u \in V(G)$, we have $u'_2 \in S$ and $u'_3 \in S$. Now, if u and v have a common neighbor in G, then at most one of the vertices u'_1 and v'_1 can be in S. We infer that the set $\{u \in V(G) : u'_1 \in S\}$ is an open packing of G, and it is a maximum open packing of G, since S is maximum. Thus, $\rho(G^+) = 2n + \rho^o(G)$.

In the previous paragraph we established that any maximum 2-packing of G^+ is obtained by taking the union of $\{u'_2, u'_3 : u \in V(G)\}$ and $\{u'_1 : u \in P\}$, where P is a maximum open packing of G. Conversely, note that for any maximum open packing P of G, the union of $\{u'_2, u'_3 : u \in V(G)\}$ and $\{u'_1 : u \in P\}$ is a maximum 2-packing of G^+. Both statements imply that G is in $\mathcal{C}_{\rho^o}$ if and only if G^+ is in $\mathcal{C}_\rho$. $\square$

In the next two lemmas, we show that the recognition problems for graphs with a unique maximum independent set and graphs with a unique maximum 2-packing are polynomially equivalent.

Let G be an arbitrary graph, and let $V(G) = \{a_1, \ldots, a_n\}$. From G we construct the graph G^* as follows. First, we start with the Cartesian product $G \square K_n$, which is defined as follows. Let vertices of K_n be denoted by $\{v_1, \ldots, v_n\}$. Then, $V(G \square K_n) = V(G) \times V(K_n)$, and the edges of $G \square K_n$ are $(a_i, v_k)(a_j, v_k)$ whenever $a_i a_j \in E(G)$, and $(a_i, v_k)(a_i, v_l)$ for any $a_i \in V(G)$ and any distinct $v_k, v_l \in V(K_n)$. Now, for each $(i, j) \in [n] \times [n]$ such that $i \neq j$, let b_{ij} and c_{ij} be two new vertices of G^*. Make c_{ij} and b_{ij} adjacent in G^*, and also connect each b_{ij} with (a_i, v_j) in G^*. Finally, for each $i \in [n]$ make b_{ij} and b_{ik} adjacent whenever $j \neq k$.

Lemma 20. *If G is a graph on n vertices, then $\rho(G^*) = n(n-1) + \alpha(G)$. In addition, G is in $\mathcal{C}_\alpha$ if and only if G^* is in $\mathcal{C}_\rho$.*

Proof. Let $C = \{c_{ij} : (i, j) \in ([n] \times [n] - \{(i, i) : i \in [n]\})\}$ denote the set of vertices of degree 1 in G^*, and $B = \{b_{ij} : (i, j) \in ([n] \times [n] - \{(i, i) : i \in [n]\})\}$ the set of its neighbors.

Suppose S is a maximum independent set in G. Note that the set $C \cup \{(a_i, v_i) : a_i \in S\}$ is a 2-packing in G^* with size $n(n-1) + \alpha(G)$, which implies $\rho(G^*) \geq n(n-1) + \alpha(G)$. For the reversed inequality first observe that all vertices of C must belong to a maximum 2-packing P of G^*. Indeed, P should not contain any vertex of B, and $(a_i, v_j) \in P$ only if $i = j$. However, if $(a_i, v_i) \in P$, then for any a_j, which is adjacent to a_i in G, we have $(a_j, v_j) \notin P$. In other words, $\{a_i : P \cap \{(a_i, v_i) : i \in [n]\} \neq \emptyset\}$ is an independent set in G, which implies that $\rho(G^*) \leq n(n-1) + \alpha(G)$.

Since C is a subset of any maximum 2-packing P of G^*, this yields the structure of maximum 2-packings of G^*, which are formed as the union of C and the set of vertices (a_i, v_i), where vertices a_i form a maximum independent set I of G. Therefore, we get that P is a unique maximum 2-packing of G^* if and only if I is a unique maximum independent set of G. $\qquad\square$

Given a graph G, the *square G^2* of G is obtained from G by adding edges between any two vertices of G that are at distance 2.

Lemma 21. *If G is a graph, then $\alpha(G^2) = \rho(G)$. In addition, G is in $\mathcal{C}_\rho$ if and only if G^2 is in $\mathcal{C}_\alpha$.*

Proof. Let $S \subseteq V(G)$ be any set of vertices in G. Note that S is a 2-packing in G if and only if S is an independent set in G^2. From this we readily derive both statements of the lemma. $\qquad\square$

From the above lemmas and translations one infers that the computational complexity for recognizing one of the three classes of graphs implies polynomially equivalent computational complexity for recognizing the other two classes.

Theorem 22. *The recognition of the graphs from classes $\mathcal{C}_{\rho^\circ}, \mathcal{C}_\rho$ and $\mathcal{C}_\alpha$ are polynomially equivalent problems.*

We conclude by observing that the recognition of graphs in each of these classes of graphs is likely to be computationally hard.

Theorem 23. *The problem of recognizing the graphs in $\mathcal{C}_{\rho^\circ}$, respectively $\mathcal{C}_\rho$ and $\mathcal{C}_\alpha$, is not polynomial, unless P=NP.*

Proof. By Theorem 22, it suffices to show the statement of the theorem only for the class $\mathcal{C}_\alpha$. Suppose that there exists a polynomial time algorithm to decide whether an arbitrary graph is in the class $\mathcal{C}_\alpha$ or not. For a graph G and a positive integer r, let G_r be the graph $G \vee \overline{K_r}$, obtained as the join of G and the edgeless graph on r vertices. Note that the independence number $\alpha(G)$ of G equals the largest integer r such that G_r is not in $\mathcal{C}_\alpha$. Hence, checking whether $G_r \in \mathcal{C}_\alpha$ for positive integers r, starting with $r = |V(G)|$, and decreasing r by 1, until r reaches $\alpha(G)$, results in a polynomial time algorithm to determine $\alpha(G)$. Since the decision version of the independence number of a graph is an NP-complete problem, this is only possible if P=NP. $\qquad\square$

4   Concluding remarks

In this paper we studied the class, $\mathcal{C}_{\rho^\circ}$, of graphs that have a unique maximum open packing. It would be nice to give a complete structural characterization of the graphs in $\mathcal{C}_{\rho^\circ}$. While we were unable to do this, we give four operations that can be used to enlarge any graph $G \in \mathcal{C}_{\rho^\circ}$ to a new graph G' that also has a unique maximum open packing. The class of trees that have a unique maximum open packing are then characterized as P_1 and P_2 together with the trees of order at least 6 that can be built from P_2 by a finite sequence of these operations.

The computational complexity of recognizing the class $\mathcal{C}_{\rho^\circ}$ is shown to be polynomially equivalent to that of recognizing the graphs with a unique maximum independent set and also polynomially equivalent to the problem of recognizing the class of graphs that have a unique maximum 2-packing. In addition, we prove that these three recognition problems are not polynomial in time unless $P = NP$. One may wonder whether similar recognition properties hold for some other classes of graphs defined through uniqueness of sets which are optimal for some other (important) graph invariant. It might be interesting to investigate algorithmic and structural properties of some of these classes of graphs.

In definitions of the classes $\mathcal{C}_{\rho^\circ}$, $\mathcal{C}_\alpha$ and $\mathcal{C}_\rho$ we consider uniqueness of the corresponding sets in the strong sense of labelled graphs. Another interesting problem arises when we consider unlabelled graphs and uniqueness is considered up to an automorphism. For instance, one may consider the graphs in which for any two maximum open packings A and B there exists an automorphism of G which maps A onto B. Clearly, the class of such graphs contains $\mathcal{C}_{\rho^\circ}$, but might be much larger. The question whether one can characterize these graphs (or at least characterize them among trees) seems challenging.

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