Rainbow domination in the lexicographic product of graphs

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Abstract

A *k-rainbow dominating function* of a graph G is a map f from $V(G)$ to the set of all subsets of $\{1, 2, \ldots, k\}$ such that $\{1, \ldots, k\} = \bigcup_{u \in N(v)} f(u)$ whenever v is a vertex with $f(v) = \emptyset$. The *k-rainbow domination number* of G is the invariant $\gamma_{rk}(G)$, which is the minimum sum (over all the vertices of G) of the cardinalities of the subsets assigned by a k-rainbow dominating function. We focus on the 2-rainbow domination number of the lexicographic product of graphs and prove sharp lower and upper bounds for this number. In fact, we prove the exact value of $\gamma_{r2}(G \circ H)$ in terms of domination invariants of G except for the case when $\gamma_{r2}(H) = 3$ and there exists a minimum 2-rainbow dominating function of H such that there is a vertex in H with the label $\{1, 2\}$.

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1 Introduction

When a graph is used to model locations or objects which can exchange some resource along its edges, the study of ordinary domination is an optimization problem to determine the minimum number of locations to store the resource in such a way that each location either has the resource or is adjacent to one where the resource resides. Imagine a computer network in which some of the computers will be servers and the others clients. There are k distinct resources, and we wish to determine the optimum set of servers each hosting a non-empty subset of the resources so that any client (i.e., any computer on the network that is not a server) is directly connected to a subset of servers that together contain all k resources. Assuming all resources have the same cost, we seek to minimize the total

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number of copies of the k resources. This model leads naturally to the notion of k-rainbow domination.

In general we follow the notation and graph theory terminology in [5]. Specifically, let G be a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex g in G, the *open neighborhood of* g, written $N(g)$, is the set of vertices adjacent to g. The *closed neighborhood* of g is the set $N[g] = N(g) \cup \{g\}$. If $A \subset V(G)$, then $N(A)$ (respectively, $N[A]$) denotes the union of open (closed) neighborhoods of all vertices of A. (In the event that the graph G under consideration is not clear we write $N_G(g)$, and so on.) Whenever $N[A] = V(G)$ we call A a *dominating set* of G. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. If G has no isolated vertices, then the *total domination number*, $\gamma_t(G)$, is the minimum cardinality of a *total dominating set* in G (that is, a subset $S \subseteq V(G)$ such that $N(S) = V(G)$). It is clear that $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$ when $\gamma_t(G)$ is defined. The *maximum degree* of a graph G is denoted by $\Delta(G)$.

For graphs G and H, the *Cartesian product* $G \,\Box\, H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either $g_1 = g_2$ and $h_1h_2 \in E(H)$, or $h_1 = h_2$ and $g_1g_2 \in E(G)$. The *lexicographic product* of G and H is the graph $G \circ H$ with vertex set $V(G) \times V(H)$. In $G \circ H$ two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either $g_1g_2 \in E(G)$, or $g_1 = g_2$ and $h_1h_2 \in E(H)$. We use π_G to denote the projection map from $G \circ H$ onto G defined by $\pi_G(g, h) = g$. The projection map π_H onto H is defined in an analogous way.

Fix a vertex g of G. The subgraph of $G \circ H$ induced by $\{(g, h) : h \in V(H)\}$ is called an H-*layer* and is denoted ${}^gH$. If $h \in V(H)$ is fixed, then G^h, the subgraph induced by $\{(g, h) : g \in V(G)\}$, is a G-*layer*. Note that every G-layer of $G \circ H$ is isomorphic to G and every H-layer of $G \circ H$ is isomorphic to H. It is also helpful to remember that if g_1 and g_2 are adjacent in G, then the subgraph of $G \circ H$ induced by ${}^{g_1}H \cup {}^{g_2}H$ is isomorphic to the join of two disjoint copies of H.

For a positive integer k we denote the set $\{1, 2, \ldots, k\}$ by $[k]$. The *power set* (that is, the set of all subsets) of $[k]$ is denoted by $2^{[k]}$. Let G be a graph and let f be a function that assigns to each vertex a subset of integers chosen from the set $[k]$; that is, $f \colon V(G) \to 2^{[k]}$. The *weight*, $\|f\|$, of f is defined as $\|f\| = \sum_{v \in V(G)} |f(v)|$. The function f is called a k-*rainbow dominating function* of G if for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ it is the case that

$$\bigcup_{u \in N(v)} f(u) = \{1, \ldots, k\}\,.$$

Given a graph G, the minimum weight of a k-rainbow dominating function is called the k-*rainbow domination number of* G, which we denote by $\gamma_{rk}(G)$.

The notion of k-rainbow domination in a graph G is equivalent to domination of the Cartesian product $G \,\Box\, K_k$. There is a natural bijection between the set of k-rainbow dominating functions of G and the dominating sets of $G \,\Box\, K_k$. Indeed, if the vertex set of K_k

is $[k]$ and f is a k-rainbow dominating function of G, then the set

$$D_f = \bigcup_{v \in V(G)} \Big(\bigcup_{i \in f(v)} \{(v,i)\} \Big),$$

is a dominating set of $G \,\square\, K_k$. By reversing this one easily sees how to complete the one-to-one correspondence. This proves the following result from [1] where the concept of rainbow domination was introduced.

Proposition 1.1 ([1]) *For $k \geq 1$ and for every graph G, $\gamma_{\mathrm{r}k}(G) = \gamma(G \,\square\, K_k)$.*

Earlier, Hartnell and Rall had investigated $\gamma(G \,\square\, K_k)$. See [6]. The main focus for them was properties shared by graphs G for which $\gamma_{\mathrm{r}2}(G) = \gamma(G)$. In particular, they proved that for any tree T, $\gamma(T) < \gamma_{\mathrm{r}2}(T)$. In addition, they proved a lower bound for $\gamma_{\mathrm{r}k}(G)$ that implies $\gamma(G) < \gamma_{\mathrm{r}k}(G)$ for every graph G whenever $k \geq 3$. Expressed in terms of rainbow domination their result yields the following sharp bounds.

Theorem 1.2 ([6]) *If G is any graph and $k \geq 2$, then*

$$\min\{|V(G)|, \gamma(G) + k - 2\} \leq \gamma_{\mathrm{r}k}(G) \leq k\gamma(G)\,.$$

From the algorithmic point of view, rainbow domination was first studied in [1] where a linear algorithm for determining a minimum 2-rainbow dominating set of a tree was presented. Brešar and Kraner Šumenjak proved that the 2-rainbow domination problem is NP-complete even when restricted to chordal graphs or bipartite graphs [2]. Both mentioned results were later generalized for the case of k-rainbow domination problem by Chang, Wu and Zhu [3].

2 Upper bounds in general case

Let $f : V(G) \to 2^{[k]}$ be a k-rainbow dominating function of G. For each $A \in 2^{[k]}$ we define V_A by $V_A = \{x \in V(G) \,:\, f(x) = A\}$ (we will write also V_A^f to avoid confusion when more functions are involved). This allows us to speak of the natural partition induced by f on $V(G)$ instead of working with the function f itself. For small values of k we may, for example, abbreviate $V_{\{1,2,3\}}$ by V_{123}. Thus, for $k = 3$, the partition of $V(G)$ would be $(V_\emptyset, V_1, V_2, V_3, V_{12}, V_{13}, V_{23}, V_{123})$, and for example, for convenience we say that a vertex in V_{13} is labeled $\{1,3\}$. If this partition arises from a 3-rainbow dominating function f of minimum weight, then

$$\gamma_{\mathrm{r}3}(G) = \|f\| = |V_1| + |V_2| + |V_3| + 2(|V_{12}| + |V_{13}| + |V_{23}|) + 3|V_{123}|\,.$$

Proposition 2.1 *For any graph H, any graph G without isolated vertices and every positive integer k,*

$$\gamma_{\mathrm{r}k}(G \circ H) \leq k\gamma_t(G)\,.$$

Proof. Fix a vertex h in H and a minimum total dominating set D of G. Define $f : V(G \circ H) \to 2^{[k]}$ by $f((g,x)) = [k]$ if $g \in D$ and $x = h$. Otherwise, $f((g,x)) = \emptyset$. Clearly, f is a k-rainbow dominating function of $G \circ H$ and $\|f\| = k\gamma_t(G)$. Therefore, $\gamma_{\mathrm{r}k}(G \circ H) \leq k\gamma_t(G)$. ■

The upper bound from Proposition 2.1 can be improved if H has domination number 1. If D is a minimum dominating set of G and h is a vertex that dominates all of H, then the partition $V_{[k]} = \{(u,h) : u \in D\}$ and $V_\emptyset = V(G \circ H) \setminus V_{[k]}$ verifies this improved bound.

Proposition 2.2 *If G is any graph without isolated vertices and H is a graph such that $\gamma(H) = 1$, then $\gamma_{\mathrm{r}k}(G \circ H) \leq k\gamma(G)$.*

In [7] the concept of dominating couples was introduced that enabled the authors to establish the Roman domination number of the lexicographic product of graphs. We can use that concept to improve the upper bound from Proposition 2.1 in the case $|V(H)| \geq k$.

We say that an ordered couple (A,B) of disjoint sets $A, B \subseteq V(G)$ is a *dominating couple* of G if for every vertex $x \in V(G) \setminus B$ there exists a vertex $w \in A \cup B$, such that $x \in N_G(w)$.

Proposition 2.3 *If H is a graph such that $|V(H)| \geq k$ and G is a non-trivial graph, then*

$$\gamma_{\mathrm{r}k}(G \circ H) \leq \min\{k|A| + \gamma_{\mathrm{r}k}(H)|B| : (A,B) \text{ is a dominating couple of } G\}.$$

Proof. Let (A,B) be a dominating couple of G. Let $\widehat{f}$ be a k-rainbow dominating function of H with $\|\widehat{f}\| = \gamma_{\mathrm{r}k}(H)$ such that $\bigcup_{v \in V(H)} \widehat{f}(v) = [k]$ ($\widehat{f}$ exists since $|V(H)| \geq k$). Fix a vertex h in H and define $f : V(G \circ H) \to 2^{[k]}$ as follows: $f((g,h)) = [k]$ if $g \in A$; $f((g,x)) = \emptyset$ if $g \in A$ and $x \neq h$; $f((g,x)) = \widehat{f}(x)$ if $g \in B$ and x is any vertex of H; $f((g,x)) = \emptyset$ otherwise. Clearly, f is a k-rainbow dominating function of $G \circ H$. ■

One can observe that $(A, \emptyset)$ is a dominating couple if and only if A is a total dominating set. Thus, if $|V(H)| \geq k$, Proposition 2.1 is a corollary of Proposition 2.3.

Consider the lexicographic product $P_7 \circ H$, where P_7 is a path of order 7 and H is a graph consisting of two 4-cycles that have one vertex in common. In Figure 1 this product is presented in such a way that one can comprehend which H-layer corresponds to which vertex of P_7, but we omit edges between H-layers for the reason of clarity. Proposition 2.1 gives the upper bound $\gamma_{\mathrm{r}2}(P_7 \circ H) \leq 8$ while using the dominating couple $(A,B) = (\{a,b\},\{c\})$ of P_7 and 2-rainbow dominating function of $P_7 \circ H$ depicted in Figure 1 we obtain $\gamma_{\mathrm{r}2}(P_7 \circ H) \leq 7$ (one can check that in fact $\gamma_{\mathrm{r}2}(P_7 \circ H) = 7$).

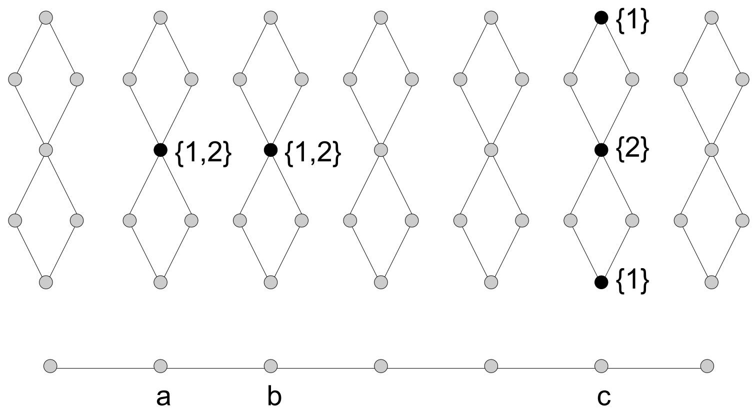

Figure 1: Dominating couple of P_7 and 2-rainbow dominating function of $P_7 \circ H$.

3 2-rainbow dominating number

We now focus on the $k = 2$ case and prove that $2\gamma(G)$ is a lower bound for $\gamma_{r2}(G \circ H)$, unless H has order 1. If H has order at least 2 and $G = K_1$, then $\gamma_{r2}(G \circ H) = \gamma_{r2}(H) \geq 2 = 2\gamma(G)$. To prove that $2\gamma(G)$ is a lower bound also when the order of G is greater than 1 we need the following observations.

Lemma 3.1 *Let G and H be non-trivial, connected graphs such that $|V(H)| \geq 3$ and suppose that $(V_\emptyset, V_1, V_2, V_{12})$ is the partition of $V(G \circ H)$ that arises from a 2-rainbow dominating function of minimum weight. It follows that $\pi_G(V_1 \cup V_{12})$ and $\pi_G(V_2 \cup V_{12})$ each dominate G.*

Proof. Suppose that $A = \pi_G(V_1 \cup V_{12})$ does not dominate G. Fix an arbitrary vertex $x \notin N_G[A]$. No vertex of the H-layer ${}^xH$ belongs to $V_1 \cup V_{12}$ or is adjacent to a vertex in $V_1 \cup V_{12}$. This implies that ${}^xH \subseteq V_2$.

Since H is connected and has order at least 3, it follows that H has a vertex of degree 2 or more. Let a be such a vertex of H. Let $W_\emptyset = (V_\emptyset \cup (\{x\} \times N_H(a))$, let $W_1 = V_1$, let $W_2 = V_2 \setminus (\{x\} \times N_H[a])$ and let $W_{12} = V_{12} \cup \{(x, a)\}$.

It is easy to check that $(W_\emptyset, W_1, W_2, W_{12})$ is a partition of $V(G \circ H)$ induced by a 2-rainbow dominating function and yet $|W_1| + |W_2| + 2|W_{12}| < |V_1| + |V_2| + 2|V_{12}|$. This contradiction shows that $\pi_G(V_1 \cup V_{12})$ dominates G. Interchanging the roles of 1 and 2 proves the lemma. ■

Lemma 3.2 *Let G be a non-trivial, connected graph. There exists a partition $(V_\emptyset, V_1, V_2, V_{12})$ of $V(G \circ K_2)$ induced by a 2-rainbow dominating function of $G \circ K_2$ of minimum weight such that $\pi_G(V_1 \cup V_{12})$ and $\pi_G(V_2 \cup V_{12})$ each dominate G.*

Proof. As in the proof of Lemma 3.1 we take a 2-rainbow dominating partition $(V_\emptyset, V_1, V_2, V_{12})$ of $G \circ K_2$ of minimum weight. Let $A = \pi_G(V_1 \cup V_{12})$ and let $B = \pi_G(V_2 \cup V_{12})$. Suppose that A does not dominate G. Let x be a vertex of G not dominated by A. This implies that ${}^xK_2 \subseteq V_2$. Let $V(K_2) = \{h_1, h_2\}$.

We define the partition $(W_\emptyset, W_1, W_2, W_{12})$ as follows. Let $W_\emptyset = V_\emptyset \cup \{(x, h_2)\}$, let $W_1 = V_1$, let $W_2 = V_2 \setminus \{(x, h_1), (x, h_2)\}$ and let $W_{12} = V_{12} \cup \{(x, h_1)\}$. The partition $(W_\emptyset, W_1, W_2, W_{12})$ is a 2-rainbow dominating partition of $G \circ K_2$. In addition, $\pi_G(W_1 \cup W_{12})$ dominates more vertices in G than A does while $\pi_G(W_2 \cup W_{12})$ dominates the same subset of G that B dominates since $B = \pi_G(W_2 \cup W_{12})$.

If $\pi_G(W_1 \cup W_{12})$ does not dominate G, then we can repeat this process until we arrive at a 2-rainbow dominating partition such that the projection onto G of those vertices labeled $\{1\}$ or $\{1, 2\}$ dominates G while simultaneously B is the projection of those vertices labeled $\{2\}$ or $\{1, 2\}$. If B does not dominate G, then we continue on with this new partition and reverse the roles of 1 and 2. This will lead to a 2-rainbow dominating function that has the required property. ■

Theorem 3.3 *For every connected graph G and every non-trivial, connected graph H, $\gamma_{r2}(G \circ H) \geq 2\gamma(G)$.*

Proof. The inequality was proved for G of order 1 at the beginning of this section. Now, let $|V(G)| \geq 2$ and let $(V_\emptyset, V_1, V_2, V_{12})$ be a partition of $V(G \circ H)$ induced by a minimum 2-rainbow dominating function. Using Lemmas 3.1 and 3.2 we derive

$$\gamma_{r2}(G \circ H) = |V_1| + |V_2| + 2|V_{12}| \geq |\pi_G(V_1 \cup V_{12})| + |\pi_G(V_2 \cup V_{12})| \geq 2\gamma(G)\,.$$

■

There are large classes of graphs that each have equal domination number and total domination number. For example, see [4] for a constructive characterization of trees with this property. Combining the results in Proposition 2.1 and Theorem 3.3 we immediately get the following.

Corollary 3.4 *If G and H are non-trivial, connected graphs and $\gamma(G) = \gamma_t(G)$, then $\gamma_{r2}(G \circ H) = 2\gamma(G)$.*

We now show that with no assumption about the relationship of $\gamma(G)$ and $\gamma_t(G)$ we get the same value for the 2-rainbow domination number of $G \circ H$ as in Corollary 3.4 by instead assuming that $\gamma_{r2}(H) = 2$.

Theorem 3.5 *For every non-trivial, connected graph G and every graph H such that $\gamma_{r2}(H) = 2$,*

$$\gamma_{r2}(G \circ H) = 2\gamma(G).$$

Proof. It is easy to see that if $B \subseteq V(G)$, then $(\emptyset, B)$ is a dominating couple of G if and only if B is a dominating set of G. Appealing to Proposition 2.3 with $k = 2$ shows that $\gamma_{r2}(G \circ H) \le \gamma_{r2}(H)\gamma(G) = 2\gamma(G)$. The desired equality follows by Theorem 3.3. ∎

By Proposition 2.1, an upper bound for $\gamma_{r2}(G \circ H)$ is $2\gamma_t(G)$. We will prove that in the case when $\gamma_{r2}(H) \ge 4$, this bound is actually the exact value for $\gamma_{r2}(G \circ H)$. In what follows we say that a layer ${}^gH$ *contributes k (respectively, at least k)* to the weight of a 2-rainbow dominating function f of $(G \circ H)$ if $k = \sum_{h \in V(H)} |f(g,h)|$, (respectively, $k \le \sum_{h \in V(H)} |f(g,h)|$).

Theorem 3.6 *If G and H are non-trivial, connected graphs and $\gamma_{r2}(H) \ge 4$, then $\gamma_{r2}(G \circ H) = 2\gamma_t(G)$.*

Proof. By the above observation it suffices to prove that $\gamma_{r2}(G \circ H) \ge 2\gamma_t(G)$. Let $(V_\emptyset, V_1, V_2, V_{12})$ be the partition of $V(G \circ H)$ that arises from a 2-rainbow dominating function f of minimum weight with the property that the cardinality of $\pi_G(V_{12})$ is maximum.

We claim that $\pi_G(V_1 \cup V_{12})$ and $\pi_G(V_2 \cup V_{12})$ are total dominating sets of G (we already know that they are dominating sets by Lemma 3.1). Suppose to the contrary that one of the sets, say $\pi_G(V_1 \cup V_{12})$, is not a total dominating set of G. It follows that there exists $g \in \pi_G(V_1 \cup V_{12})$ such that ${}^{g'}H \subseteq V_\emptyset \cup V_2$ for every $g' \in N_G(g)$. Fix such a neighbor g' of g.

Suppose that $f((g,x)) \ne \emptyset$ for every vertex x in H. Since $\gamma_{r2}(H) \ge 4$ we see that ${}^gH$ contributes at least 4 to the weight of f. Let h be any vertex of H, and let $\widehat{f}$ be the function on $V(G \circ H)$, induced by the partition $(W_\emptyset, W_1, W_2, W_{12})$ of $V(G \circ H)$, where $W_\emptyset = V_\emptyset \cup (({}^gH \cup {}^{g'}H) \setminus \{(g,h),(g',h)\})$, $W_1 = V_1 \setminus ({}^gH \cap V_1)$, $W_2 = V_2 \setminus (({}^gH \cup {}^{g'}H) \cap V_2)$ and $W_{12} = (V_{12} \setminus {}^gH) \cup \{(g,h),(g',h)\}$. One can check that $\widehat{f}$ is a 2-rainbow dominating function of $G \circ H$, and by its definition $\|\widehat{f}\| \le \|f\|$. This implies that $\widehat{f}$ is a 2-rainbow dominating function of $G \circ H$ of minimum weight. This is a contradiction since $|\pi_G(W_{12})| > |\pi_G(V_{12})|$. It follows that $V_\emptyset \cap {}^gH \ne \emptyset$.

Now we distinguish the following two possibilities.

Case 1. If every vertex from $V_\emptyset \cap {}^gH$ is adjacent to a vertex in $(V_2 \cup V_{12}) \cap {}^gH$ (i.e. the 2-rainbow domination of the ${}^gH$-layer is assured within the layer), then the ${}^gH$-layer contributes at least 4 to the weight of f, since $\gamma_{r2}(H) \ge 4$. By defining $\widehat{f}$ as above we obtain a 2-rainbow dominating function on $V(G \circ H)$, the weight of which is less than or equal to the weight of f, a contradiction in either case (in the second case with f being a 2-rainbow dominating function with the maximum cardinality of $\pi_G(V_{12})$).

Case 2. Suppose that there exists $(g,h) \in V_\emptyset \cap {}^gH$ that is not adjacent to any vertex in $(V_2 \cup V_{12}) \cap {}^gH$ (and is adjacent to (g,h'') with $f((g,h'')) = \{1\}$). This implies that there exist $g' \in N_G(g)$ and $h' \in V(H)$ such that $f((g',h')) = \{2\}$.

First we show that there are at least two vertices in $(V_1 \cup V_{12}) \cap {}^gH$. If we assume to the contrary that $|(V_1 \cup V_{12}) \cap {}^gH| = 1$, then $V_{12} \cap {}^gH = \emptyset$ and there exists (g, h''') with $f((g, h''')) = \{2\}$ (otherwise H would have a universal vertex, but this is in contradiction with $\gamma_{r2}(H) \geq 4$). Moreover, there are at least two such vertices, since equalities $|V_1 \cap {}^gH| = |V_2 \cap {}^gH| = 1$ imply $\gamma_{r2}(H) \leq 3$, a contradiction. Let $(W_\emptyset, W_1, W_2, W_{12})$ be the following partition of $V(G \circ H)$: $W_\emptyset = V_\emptyset \cup \{(g, h''')\}$, $W_1 = V_1$, $W_2 = V_2 \setminus \{(g, h'''), (g', h')\}$ and $W_{12} = V_{12} \cup \{(g', h')\}$. One can observe that this partition induces a 2-rainbow dominating function $\widehat{f}$ on $V(G \circ H)$ with the same weight as f, and $|\pi_G(W_{12})| > |\pi_G(V_{12})|$, a contradiction. Hence there are at least two vertices in $(V_1 \cup V_{12}) \cap {}^gH$.

Now, the function $\widehat{f}$ induced by the partition $(W_\emptyset, W_1, W_2, W_{12})$ where $W_\emptyset = V_\emptyset \cup \{(g, h'')\}$, $W_1 = V_1 \setminus \{(g, h'')\}$, $W_2 = V_2 \setminus \{(g', h')\}$ and $W_{12} = V_{12} \cup \{(g', h')\}$ is a 2-rainbow dominating function on $V(G \circ H)$ with the same weight as f, and such that $|\pi_G(W_{12})| > |\pi_G(V_{12})|$, which is a final contradiction. The claim that $\pi_G(V_1 \cup V_{12})$ and $\pi_G(V_2 \cup V_{12})$ are both total dominating sets of G is proved.

From this we easily derive the desired result. Namely, since both of $\pi_G(V_1 \cup V_{12})$ and $\pi_G(V_2 \cup V_{12})$ are total dominating sets of G we get

$$\gamma_{r2}(G \circ H) = |V_1| + |V_2| + 2|V_{12}| \geq |\pi_G(V_1 \cup V_{12})| + |\pi_G(V_2 \cup V_{12})| \geq 2\gamma_t(G)\,.$$

■

In the cases $\gamma_{r2}(H) = 2$ and $\gamma_{r2}(H) \geq 4$ we obtained the exact values for $\gamma_{r2}(G \circ H)$. However, the case $\gamma_{r2}(H) = 3$ is the most challenging. Combining Proposition 2.3 and Theorem 3.3 we obtain the following sharp bounds.

Corollary 3.7 *If G and H are non-trivial, connected graphs such that $\gamma_{r2}(H) = 3$, then*

$$2\gamma(G) \leq \gamma_{r2}(G \circ H) \leq \min\{2|A| + 3|B| : (A, B) \text{ is a dominating couple of } G\}\,.$$

As we will see in the theorem that follows, the upper bound in the above corollary is actually the exact value provided that every minimum 2-rainbow dominating function of H enjoys a certain property.

Theorem 3.8 *Let H be a connected graph with $\gamma_{r2}(H) = 3$ and assume that for every minimum 2-rainbow dominating function φ of H, $\varphi(h) \neq \{1, 2\}$ for every vertex h of H. If G is any graph, then*

$$\gamma_{r2}(G \circ H) = \min\{2|A| + 3|B| : (A, B) \text{ is a dominating couple of } G\}\,.$$

Proof. If G is isomorphic to K_1, the claim from the theorem obviously holds. Hence we assume that G is a non-trivial graph. The graph H contains at least four vertices since no connected graph of order less than 4 has 2-rainbow domination number 3. Since no minimum weight 2-rainbow dominating function of H uses the label $\{1, 2\}$, it follows that every minimum weight 2-rainbow dominating function of H uses both of $\{1\}$ and $\{2\}$.

From among all minimum 2-rainbow dominating functions of the graph $G \circ H$ assume that f is chosen with the property that for every minimum 2-rainbow dominating function f_1 of $G \circ H$,

$$\left| \{x \in V(G) \,:\, \{1,2\} = \bigcup_{h \in V(H)} f(x,h)\, \} \right| \geq \left| \{x \in V(G) \,:\, \{1,2\} = \bigcup_{h \in V(H)} f_1(x,h)\, \} \right| .$$

Let $(V_\emptyset, V_1, V_2, V_{12})$ be the partition of $V(G \circ H)$ induced by f.

We now define a partition $(W_\emptyset, W_1, W_2, W_{12})$ of $V(G)$.

- $W_\emptyset = \{\, w \in V(G) \,:\, {}^wH \subseteq V_\emptyset \,\}$;
- $W_1 = \{\, w \in V(G) \,:\, {}^wH \subseteq V_1 \cup V_\emptyset$ and ${}^wH \cap V_1 \neq \emptyset \,\}$;
- $W_2 = \{\, w \in V(G) \,:\, {}^wH \subseteq V_2 \cup V_\emptyset$ and ${}^wH \cap V_2 \neq \emptyset \,\}$; and
- $W_{12} = \{\, w \in V(G) \,:\, \{1,2\} = \bigcup_{h \in V(H)} f(w,h) \,\}$.

First we prove that if $w \in W_1 \cup W_2$, then by the choice of the 2-rainbow dominating function f the layer ${}^wH$ contributes exactly 1 to the weight of f.

Claim 1 *If $x \in W_1$, then there is exactly one vertex in ${}^xH$ labeled $\{1\}$. If $x \in W_2$, then there is exactly one vertex in ${}^xH$ labeled $\{2\}$.*

Fix $x \in W_1$, and suppose there are distinct vertices h_1 and h_2 in H such that $f(x,h_1) = \{1\} = f(x,h_2)$.

If x is isolated in G, then f restricted to ${}^xH$ is a minimum weight 2-rainbow dominating function of ${}^xH$. However, since $x \in W_1$, it follows that every vertex in ${}^xH$ is labeled $\{1\}$, and so ${}^xH$ contributes at least 4 to the weight of f. This contradiction shows that x is not isolated in G. Let $v \in N_G(x)$.

We claim that there is a vertex with label $\emptyset$ in ${}^xH$. Suppose to the contrary that ${}^xH \subseteq V_1$. We infer that such an f cannot be a minimum weight 2-rainbow dominating function, since we can obtain a 2-rainbow dominating function of weight less than $\|f\|$ by replacing the label $\{1\}$ with $\emptyset$ on each vertex in ${}^xH$ except for one and relabeling one vertex in ${}^vH$ with $\{1,2\}$. Because of this contradiction it follows that at least one vertex in ${}^xH$ has label $\emptyset$.

Hence, there is a vertex y adjacent to x in G such that ${}^yH$ contains a vertex labeled $\{2\}$ or a vertex labeled $\{1,2\}$. However, by the minimality of the weight of f it follows that $\{1,2\} \neq \bigcup_{h \in V(H)} f(y,h)$ (for otherwise we could "relabel" (x,h_2) with $\emptyset$ and the result would be a 2-rainbow dominating function with smaller weight). Thus, suppose that $f(y,h_3) = \{2\}$ for some $h_3 \in V(H)$. Let $\widehat{f} : V(G \circ H) \to 2^{[2]}$ be defined by $\widehat{f}(x,h_2) = \emptyset$, $\widehat{f}(y,h_3) = \{1,2\}$ and $\widehat{f}(g,h) = f(g,h)$ for every other vertex $(g,h) \notin \{(x,h_2),(y,h_3)\}$. It is easy to see that $\widehat{f}$ is a 2-rainbow dominating function of $G \circ H$ having the same weight as f. However, under this new 2-rainbow dominating function, $\widehat{f}$, there are more H-layers containing both of the labels 1 and 2 than there are with f. This contradiction proves the first statement. The second statement is proved by interchanging the roles of 1 and 2. Hence, the claim is verified.

Note that $\gamma_{r2}(H) = 3$ implies that $\gamma(H) = 2$ or $\gamma(H) = 3$. Hence, if $w \in W_1 \cup W_2$, then Claim 1 implies that there exist $u, v \in N_G(w)$ and $h, k \in V(H)$ such that $1 \in f(u,h)$ and $2 \in f(v,k)$ (where it is possible that $u = v$ or $h = k$). It also follows from this claim that if $w \in V(G)$ and the layer ${}^wH$ contributes 2 or more to the weight of f, then $w \in W_{12}$. Indeed, $w \in W_{12}$ if and only if ${}^wH$ contributes at least 2 to the weight of f. Furthermore, suppose $w \in W_{12}$ and ${}^wH$ contributes 3 or more to the weight of f. Since $\gamma_{r2}(H) = 3$ and f is a minimum 2-rainbow dominating function of $G \circ H$, for this w we can assume that the restriction of f to ${}^wH$ is a 2-rainbow dominating function of the subgraph of $G \circ H$ induced by this H-layer. This follows by using the labels $\emptyset$, $\{1\}$ and $\{2\}$ in a minimum 2-rainbow dominating function φ of H and setting $f(w,h) = \varphi(h)$. We conclude that if $w \in W_{12}$, then the layer ${}^wH$ contributes precisely 2 or 3 to the weight of f.

We will need to be able to distinguish the following types of H-layers:

- The layer ${}^xH$ is of *Type 1* if for some $y \in N_G(x)$, $y \in W_{12}$.
- The layer ${}^xH$ is of *Type 2* if ${}^xH$ is not of Type 1, and there exist distinct vertices $y, z \in N_G(x)$ such that $y \in W_1$ and $z \in W_2$.
- The layer ${}^xH$ is of *Type 3* if ${}^xH$ is 2-rainbow dominated by f restricted to ${}^xH$.

First we prove the following claim.

Claim 2 *For any vertex x of G, the H-layer ${}^xH$ is of exactly one of the types listed above.*

To prove the claim we use the fact that $(W_\emptyset, W_1, W_2, W_{12})$ is a partition of $V(G)$.

Consider first a vertex x in $W_\emptyset$. Every vertex in ${}^xH$ must have a neighbor with a 1 in its label and a neighbor with a 2 in its label. That is, either x has a neighbor in W_{12}, or x has a neighbor in W_1 and a neighbor in W_2. In other words, ${}^xH$ is of Type 1 or of Type 2.

Suppose $x \in W_1$. By Claim 1 there is a unique $h \in V(H)$ such that $f(x,h) = \{1\}$, and $f(x,h') = \emptyset$ for every $h' \in V(H) \setminus \{h\}$. Since $\gamma(H) \geq 2$ there is some vertex, say (x,k), in ${}^xH$ that is not adjacent to (x,h) and such that $f(x,k) = \emptyset$. Now, (x,k) must have a neighbor with a 1 in its label and a neighbor with a 2 in its label. That is, either x has a neighbor in W_{12}, or x has a neighbor in W_1 and a neighbor in W_2. In other words, ${}^xH$ is of Type 1 or of Type 2. The case $x \in W_2$ is handled similar to this with the roles of 1 and 2 interchanged.

Finally, assume that $x \in W_{12}$. The layer ${}^xH$ contributes either 2 or 3 to the weight of f. Suppose ${}^xH$ contributes 3. By an earlier argument we may assume that f restricted to ${}^xH$ is a 2-rainbow dominating function of ${}^{x}H$. By our assumption on H this means that the label $\{1,2\}$ does not appear on any vertex in ${}^{x}H$. Assume that ${}^{x}H$ is of Type 1 or Type 2. In this case one of the labels in ${}^{x}H$ could be changed from $\{1\}$ to $\emptyset$ or from $\{2\}$ to $\emptyset$ (whichever one of $\{1\}$ or $\{2\}$ that occurs twice in ${}^{x}H$) and this would yield a 2-rainbow dominating function of $G \circ H$ having smaller weight than f. This contradiction implies that an H-layer of Type 3 is not also of Type 1 or Type 2.

Assume that ${}^{x}H$ contributes exactly 2 to the weight of f. If $\{1,2\}$ occurs as a label on some vertex (x,h) in ${}^{x}H$, then since $\gamma(H) \geq 2$ there is some vertex (x,k) in ${}^{x}H$ that is not adjacent to (x,h). This vertex (x,k) has a neighbor with a 1 in its label and a neighbor with a 2 in its label. These neighbors are not in ${}^{x}H$ and once again as above we conclude that ${}^{x}H$ is of Type 1 or of Type 2.

Now, suppose that ${}^{x}H$ contributes exactly 2 to the weight of f and there exist distinct vertices h_1 and h_2 in H such that $f(x,h_1) = \{1\}$ and $f(x,h_2) = \{2\}$. Suppose that ${}^{x}H$ is not of Type 1 nor of Type 2. Since $\gamma_{r2}(H) = 3$ there is a vertex (x,h) that is not adjacent to both (x,h_1) and (x,h_2). If (x,h) is adjacent to neither of them, then it has a neighbor with a 1 in its label and a neighbor with a 2 in its label and both of these neighbors lie outside of ${}^{x}H$. However, this contradicts our assumption that ${}^{x}H$ is not of Type 1 nor of Type 2. Thus, we may assume without loss of generality that (x,h) is adjacent to (x,h_1) but not to (x,h_2). It follows that there exists $x' \in N_G(x)$ such that $x' \in W_2$. Let $f(x',h') = \{2\}$. By Claim 1, $f(x',k) = \emptyset$ for every $k \in V(H) \setminus \{h'\}$. Since ${}^{x}H$ is not of Type 1 nor of Type 2, no neighbor of x belongs to $W_1 \cup W_{12}$. This means that every vertex in ${}^{x}H \setminus \{(x,h_1),(x,h_2)\}$ is adjacent to (x,h_1). Let g be the function defined on $V(H)$ by $g(h_1) = \{1,2\}$, $g(h_2) = \{2\}$ and $g(v) = \emptyset$ for every other vertex v of H. This function g is a 2-rainbow dominating function of H having weight 3 and also having a vertex labeled $\{1,2\}$. This contradiction shows that ${}^{x}H$ is of Type 1 or of Type 2 and finishes the proof of Claim 2.

We may assume without loss of generality that $|W_1| \geq |W_2|$. We now modify the function f to produce another minimum 2-rainbow dominating function p of $G \circ H$ which has the property that each H-layer that receives a non-empty label contributes either exactly 2 or exactly 3 to the weight of p. The general idea is that if $w \in W_\emptyset \cup W_{12}$, then the labeling under p for vertices in ${}^{w}H$ will be the same as it was under f. Thus, all H-layers that contribute 2 or 3 to the weight of f will also contribute that amount to the weight of p. On the other hand, some H-layers that contribute 1 to the weight of f will contribute 2 to the weight of p while others will contribute 0 to the weight of p.

We define p by specifying the partition $(U_\emptyset, U_1, U_2, U_{12})$ that p induces on $V(G \circ H)$. Let

$$U_1 = \{\, (w,k) \,:\, w \in W_{12} \text{ and } f(w,k) = \{1\}\,\},$$

$$U_2 = \{\, (w,k) \,:\, w \in W_{12} \text{ and } f(w,k) = \{2\}\,\},$$

$$U_\emptyset = V_\emptyset \cup \{\, (w,k) \,:\, w \in W_1 \,\},$$

and

$$U_{12} = V_{12} \cup \{\, (w,k) \,:\, w \in W_2 \text{ and } f(w,k) = \{2\}\,\}.$$

To prove that p is a 2-rainbow dominating function of $G \circ H$ let $(g,h) \in U_\emptyset$ (in other words, (g,h) is such that $f(g,h) = \emptyset$, or $g \in W_1$ and $f(g,h) = \{1\}$). All possibilities are covered in the following cases.

- Suppose ${}^gH$ is of Type 1. As noted above, g has a neighbor $g' \in W_{12}$. The vertex (g,h) is adjacent to every vertex in ${}^{g'}H$. By the definitions of U_1, U_2, and U_{12} it follows that (g,h) has a neighbor that (under p) contains 1 in its label and a neighbor that (under p) contains 2 in its label.

- Suppose ${}^gH$ is of Type 2. Now, g has a neighbor $y \in W_1$ and a neighbor $z \in W_2$. There exists $h' \in V(H)$ such that $f(z,h') = \{2\}$. By the definition of p, $p(z,h') = \{1,2\}$ and (g,h) is adjacent to (z,h').

- Suppose that ${}^gH$ is of Type 3. By the definition of p the labels in ${}^gH$ under p agree with those under f, and by our assumption about f, the vertex (g,h) with the property $f(g,h) = \emptyset$ has a neighbor in ${}^gH \cap U_1$ and a neighbor in ${}^gH \cap U_2$.

Hence we see that in all of the above, $\{1,2\} = \bigcup\{\, p(g',h') \,:\, (g',h') \in N(g,h)\}$. It follows that p is a 2-rainbow dominating function of $G \circ H$, and by its definition $\|p\| \le \|f\|$. Therefore, $\|p\| = \gamma_{r2}(G \circ H)$.

Let

$$A = \{\, x \in V(G) \,:\, {}^xH \text{ contributes 2 to the weight of } p\,\}, \quad \text{and}$$

$$B = \{\, x \in V(G) \,:\, {}^xH \text{ contributes 3 to the weight of } p\,\}.$$

The definition of p shows that $\|p\| = 2|A| + 3|B|$. It remains to show that (A,B) is a dominating couple of G. For this purpose let $g \in V(G) \setminus B$. If g does not belong to A, then ${}^gH \subseteq U_\emptyset$. Since p is a 2-rainbow dominating function it follows that g has a neighbor in $A \cup B$. Finally, assume that $g \in A$. This means that ${}^gH$ contributes 2 to the weight of p. Since $\gamma_{r2}(H) = 3$ there exists at least one vertex $(g,h) \in U_\emptyset$ such that $\{1,2\} \ne \bigcup\{p(g,k) \,:\, k \in N_H(h)\}$. (That is, (g,h) is not 2-rainbow dominated by p from within ${}^gH$.) It follows that (g,h) has a neighbor in some ${}^{g'}H$ such that ${}^{g'}H \subseteq ((A \setminus \{g\}) \cup B) \times V(H)$. Hence g and g' are adjacent in G, and $g' \in A \cup B$. Therefore, (A,B) is a dominating couple of G. ■

The factors of the lexicographic product represented in Figure 1 satisfy the conditions of the above theorem so this graph attains the upper bound of Corollary 3.7.

We were also able to improve the upper bound from this corollary in the case of the lexicographic product of paths and graphs H that do not satisfy the condition on H from Theorem 3.8. We would like to point out that the construction used in the proof of the next proposition enabled us also to find a family of graphs that attains the lower bound $2\gamma(G)$.

Proposition 3.9 *Let H be a connected graph with $\gamma_{r2}(H) = 3$ and the property that there exists a 2-rainbow dominating function of H of minimum weight such that there is a vertex* *in H labeled with $\{1,2\}$. It follows that*

$$\gamma_{r2}(P_n \circ H) \leq \begin{cases} 6\lfloor \frac{n}{7} \rfloor + k & , \quad n \equiv k \pmod 7 \text{ for } k = 0,3,4,5,6, \\ \\ 6\lfloor \frac{n}{7} \rfloor + k + 1 & , \quad n \equiv k \pmod 7 \text{ for } k = 1,2. \end{cases}$$

Proof. Let H be a connected graph with $\gamma_{r2}(H) = 3$ and suppose there exists a 2-rainbow dominating function f of H of minimum weight such that $V_{12} \neq \emptyset$. Let $u, v \in V(H)$ be the vertices with $f(u) = \{1,2\}$ and (without loss of generality) $f(v) = \{1\}$.

To end the proof it suffices to construct a 2-rainbow dominating function p on $P_n \circ H$ with desired weight for each case. We will represent p with a table of integers $0,1,2,3$ where these numbers denote subsets $\emptyset, \{1\}, \{2\}$ and $\{1,2\}$, respectively. Numbers in the first and second row correspond to the values of p in the ${}^uP_n$-layer and ${}^vP_n$-layer, respectively (we omit other P_n-layers, since only zeros appear in them).

One can check that for each $i = 2,3,4,5,6,7,8$, R_i depicted below represents a 2-rainbow dominating function on $P_i \circ H$.

R_2	R_3	R_4	R_5	R_6	R_7	R_8
30	030	0330	02120	030030	0210210	02102130
10	010	0000	01010	010010	0100020	01000200

To construct a 2-rainbow dominating function of $P_n \circ H$ for $n \geq 9$ we distinguish three cases.

If $n \equiv 0 \pmod 7$, i.e $n = 7t$ for some integer $t \geq 1$, then we obtain the table that corresponds to a desired function by taking t copies of R_7.

$$\begin{array}{|c|}\hline \underbrace{R_7R_7\ldots R_7}_{t} \\ \hline \end{array}$$

If $n = 7t + 1$ for some $t \geq 1$, then we take $t - 1$ copies of R_7 and one copy of R_8.

$$\begin{array}{|c|c|}\hline \underbrace{R_7R_7\ldots R_7}_{t-1} & R_8 \\ \hline \end{array}$$

For all other cases (when $n = 7t + i$, for $t \geq 1$ and $2 \leq i \leq 6$), we take t copies of R_7 and one copy of R_i.

$$\begin{array}{|c|c|}\hline \underbrace{R_7R_7\ldots R_7}_{t} & R_i \\ \hline \end{array}$$

Verification that in each case we obtain a 2-rainbow dominating function of desired weight is left to the reader. ∎

As we have seen in the proof, $\gamma_{r2}(P_7 \circ H) = 6 = 2\gamma(P_7)$, so the lower bound in Corollary 3.7 is attained. Using similar ideas we can also construct an infinite family of graphs that attain this lower bound.

Let H be a connected graph with $\gamma_{r2}(H) = 3$ and the property that there exists a 2-rainbow dominating function of H of minimum weight such that $V_{12} \neq \emptyset$. As above, let $u, v \in V(H)$ be the vertices with $f(u) = \{1, 2\}$ and, say $f(v) = \{1\}$. Let G be a graph obtained from m paths isomorphic to P_6 and n paths isomorphic to P_2 in such way that we glue them together along a pendant vertex in each path, see Figure 2. In this figure the tables as above represent the values of a 2-rainbow dominating function on $G \circ H$ only in the G^u-layer (above) and G^v-layer (below), since only zeros appear elsewhere. This construction gives us $\gamma_{r2}(G \circ H) \leq 4m + 2$. On the other hand, one can verify that $\gamma(G) = 2m + 1$. Thus, by Corollary 3.7, $\gamma_{r2}(G \circ H) = 2\gamma(G)$.

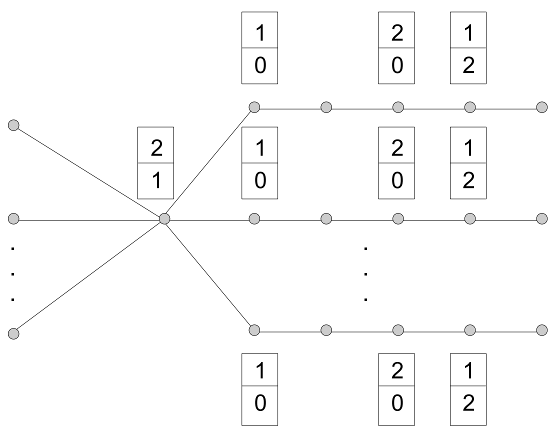

Figure 2: 2-rainbow dominating function of $G \circ H$.

4 Concluding remarks

By Proposition 3.9, $\gamma_{r2}(P_5 \circ P_4) \leq 5$, while the lower and the upper bounds from Corollary 3.7 are 4 and 6, respectively. In fact, it is a matter of case analysis to show that $\gamma_{r2}(P_5 \circ P_4) = 5$, but it is our conjecture that the bound from Proposition 3.9 is actually the exact value.

More generally, it remains an open problem to find the formula for $\gamma_{r2}(G \circ H)$ in the case when $\gamma_{r2}(H) = 3$ and there exists a minimum 2-rainbow dominating function of H such that there is a vertex in H with the label $\{1, 2\}$.

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