On k-rainbow independent domination in graphs

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Abstract

In this paper, we define a new domination invariant on a graph G, which coincides with the ordinary independent domination number of the generalized prism $G \,\square\, K_k$, called the *k-rainbow independent domination number* and denoted by $\gamma_{\mathrm{ri}k}(G)$. Some bounds and exact values concerning this domination concept are determined. As a main result, we prove a Nordhaus-Gaddum-type theorem on the sum for 2-rainbow independent domination number, and show if G is a graph of order $n \geq 3$, then $5 \leq \gamma_{\mathrm{ri2}}(G) + \gamma_{\mathrm{ri2}}(\overline{G}) \leq n + 3$, with both bounds being sharp.

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1 Introduction

Domination in graphs has been an extensively researched branch of graph theory. Already in [11] more than 75 variations of domination were cited, and many more have been introduced since then. This is not surprising as many of these concepts found applications in different fields, for instance in facility location problems, monitoring communication or electrical networks, land surveying, computational biology, etc. Recent studies on domination and related concepts include also [1, 7, 12, 14, 17]. Although our original motivation for defining a new invariant arises from a desire to reduce the problem of computing the independent domination number of the generalized prism $G \,\square\, K_k$ to an integer labeling problem on a graph G, the new concept can also be seen as a model for a problem in combinatorial optimization.

In introducing this new concept we first mention that in general we follow the notation and graph theory terminology in [9]. Specifically, let G be a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. By $G\langle A\rangle$ we denote the subgraph of G induced by the vertex set $A \subseteq V(G)$. For a positive integer n with $n \geq 2$ we denote an empty graph on n vertices

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by N_n and a star of order n by S_n. The graph obtained from S_n by adding a single edge is denoted S_n^+. For any vertex g in G, the *open neighborhood of* g, written $N(g)$, is the set of vertices adjacent to g. The *closed neighborhood* of g is the set $N[g] = N(g) \cup \{g\}$. The vertex g is a *universal* vertex of G if $N[g] = V(G)$. If $A \subset V(G)$, then $N(A)$ (respectively, $N[A]$) denotes the union of the open (closed) neighborhoods of all vertices of A. (In the event that the graph G under consideration is not clear we write $N_G(g)$, and so on.)

The *Cartesian product*, $G\Box H$, of graphs G and H is a graph with $V(G\Box H) = V(G) \times V(H)$, where two vertices (g, h) and (g', h') are adjacent in $G\Box H$ whenever ($gg' \in E(G)$ and $h = h'$) or ($g = g'$ and $hh' \in E(H)$). For a fixed $h \in V(H)$ we call $G^h = \{(g, h) \in V(G\Box H) : g \in V(G)\}$ a *G-layer* in $G\Box H$. Similarly, an H-layer ${}^gH$ for a fixed $g \in V(G)$ is defined as ${}^gH = \{(g, h) \in V(G\Box H) : h \in V(H)\}$. Notice that the subgraph of $G\Box H$ induced by a G-layer or an H-layer is isomorphic to G or H, respectively.

If A and B are any two nonempty subsets of $V(G)$ we say A dominates B if $B \subseteq N[A]$. If A dominates $V(G)$ then A is a *dominating set* of G. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. An *independent set* of a graph is a set of vertices, no two of which are adjacent. An *independent dominating set* of G is a set that is both dominating and independent in G. This set is also called a *stable set* or a *kernel* of the graph G. The *independent domination number*, $i(G)$, of a graph G is the size of a smallest independent dominating set. The *independence number* of G, denoted by $\alpha(G)$, is the maximum size of an independent set in G. Observe that $\gamma(G) \le i(G) \le \alpha(G)$.

For a positive integer k we denote the set $\{1, 2, \ldots, k\}$ by $[k]$. In the remainder of this paper we will always assume the vertex set of the complete graph K_k is $[k]$. The *power set* (that is, the set of all subsets) of $[k]$ is denoted by $2^{[k]}$. Let G be a graph and let f be a function that assigns to each vertex a subset of integers chosen from the set $[k]$; that is, $f\colon V(G) \to 2^{[k]}$. The *weight*, $\|f\|$, of f is defined as $\|f\| = \sum_{v\in V(G)} |f(v)|$. The function f is called a *k-rainbow dominating function* (kRDF for short) of G if for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ it is the case that

$$\bigcup_{u\in N(v)} f(u) = [k]\,.$$

Given a graph G, the minimum weight of a k-rainbow dominating function is called the *k-rainbow domination number of* G, which we denote by $\gamma_{\mathrm{r}k}(G)$. Motivation for introducing this concept arose from the observation that for $k \ge 1$ and for every graph G, $\gamma_{\mathrm{r}k}(G) = \gamma(G \,\Box\, K_k)$. See [3]. In other words, the problem of finding the domination number of the Cartesian product $G \,\Box\, K_k$ is equivalent to an optimization problem involving a restricted "labeling" of $V(G)$ with subsets of $[k]$. In what follows we impose additional conditions on this labeling in order to represent an independent dominating set of $G \,\Box\, K_k$. Since the dominating set is independent and the K_k-layers are complete, vertices cannot be labeled with a subset of cardinality more than 1. This allows for each vertex of G to be labeled by a single integer and leads us to the following.

For a function $f : V(G) \to \{0, 1, 2, \ldots, k\}$ we denote by V_i the set of vertices to which the value i is assigned by f, i.e. $V_i = \{x \in V(G) : f(x) = i\}$. A function $f : V(G) \to$

$\{0,1,\ldots,k\}$ is called an *k-rainbow independent dominating function* (*kRiDF* for short) of G if the following two conditions hold:

1. V_i is independent for $i \in [k]$, and
2. for every $x \in V_0$ it follows that $N(x) \cap V_i \neq \emptyset$, for every $i \in [k]$.

Note that a k-rainbow independent dominating function f can be represented by the ordered partition $(V_0, V_1, \ldots, V_k)$ determined by f. Hence, when it is convenient we simply work with the partition. The weight of a kRiDF f is defined as $w(f) = \sum_{i=1}^{k} |V_i|$, or equivalently $w(f) = n - |V_0|$, where n is the order of the graph. The *k-rainbow independent domination number* of a graph, denoted by $\gamma_{\mathrm{ri}k}(G)$, is the minimum weight of a kRiDF of G. Note that $\gamma_{ri1}(G) = i(G)$. A $\gamma_{\mathrm{ri}k}(G)$*-function* is a kRiDF of G with weight $\gamma_{\mathrm{ri}k}(G)$, and a *kRiDF-partition* of G is an ordered partition $(V_0, V_1, \ldots, V_k)$ of $V(G)$ that represents a $\gamma_{\mathrm{ri}k}(G)$-function.

Chellali and Jafari Rad [5] introduced a graphical invariant using the name, independent 2-rainbow domination number, and later Shao et al. [17] independently presented a natural generalization to the invariant of Chellali and Jafari Rad for $k \geq 3$. Our new invariant is quite different from these. To explain these differences we provide their definition for the case $k = 2$ and then to illustrate the differences we present examples for the natural generalization for $k = 3$. In [5] and [17] a 2-rainbow dominating function $f\colon V(G) \to 2^{[2]}$ is an *independent 2-rainbow dominating function* if the set $\{x \in V(G) : f(x) \neq \emptyset\}$ is an independent subset of G. This induces a partition $(V_\emptyset, V_1, V_2, V_{12})$ of $V(G)$ (in which $V_{12} = f^{-1}(\{1,2\})$) such that $V_1 \cup V_2 \cup V_{12}$ is an independent dominating set of G, which has the additional requirement that every vertex in $V_\emptyset$ either has a neighbor in V_{12} or a neighbor in each of V_1 and V_2. The independent 2-rainbow domination number ([5], [17]) is defined by $i_{r2}(G) = \min\{|V_1| + |V_2| + 2|V_{12}|\}$, where the minimum is computed over all partitions $(V_\emptyset, V_1, V_2, V_{12})$ arising from independent 2-rainbow dominating functions of G. For $k \geq 2$, their independent k-rainbow domination number is defined similarly. Since f is a 2-rainbow dominating function, it does correspond to a dominating set of $G \,\square\, K_2$, but not always (in particular, when $V_{12} \neq \emptyset$) to an independent dominating set of $G \,\square\, K_2$. There are, of course, graphs G such that $i_{rk}(G) = \gamma_{\mathrm{ri}k}(G)$, but in general these invariants are incomparable. For example, $\gamma_{ri3}(S_7) = 6$ while $i_{r3}(S_7) = 3$. On the other hand, it is easy to verify that for the graph G shown in Figure 1, $\gamma_{ri3}(G) = 3$ and $i_{r3}(G) = 4$.

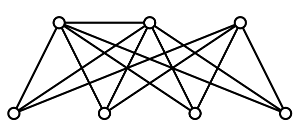

Figure 1: $\gamma_{ri3}(G) = 3$ and $i_{r3}(G) = 4$

Our motivation for defining the new invariant, the k-rainbow independent domination number, is to reduce the problem of computing the independent domination number of the generalized prism $G \,\square\, K_k$ to an integer labeling problem on G. Hence, we do not allow a vertex of G to receive a label containing more than one integer from $[k]$ nor do we require $V_1 \cup \cdots \cup V_k$ to be independent.

The rest of this paper is organized as follows. Section 2 is dedicated to basic properties and bounds concerning the k-rainbow independent domination. Then in Section 3 we study the 2-rainbow independent domination number on some standard families of graphs and prove that for any non-trivial tree the 2-rainbow independent domination number always exceeds the independent domination number. In Section 4 we establish the lower and the upper bound on the sum of 2-rainbow independent domination number of G and of $\overline{G}$ in terms of the number of vertices of G, and in the last section we pose a few open problems.

2 Basic properties and bounds

The following observation follows directly from the definition of the k-rainbow independent domination number.

Observation 2.1 *If G has order n and $n \leq k$, then $\gamma_{\mathrm{ri}k}(G) = n$.*

For a graph G of order larger than k, the definition also implies that $\gamma_{\mathrm{ri}k}(G) \geq k$. The following result characterizes those graphs that have k-rainbow independent domination number equal to k.

Proposition 2.2 *Let k and n be positive integers such that $n \geq k$. For any connected graph G of order n, $\gamma_{\mathrm{ri}k}(G) = k$ if and only if $n = k$ or G has a spanning subgraph isomorphic to $K_{k,n-k}$.*

Proof. Let G have order n and assume that $n \geq k$. We assume throughout that $V(G) = \{x_1, \ldots, x_n\}$. Suppose first that $n = k$. The function f defined by $f(x_j) = j$ for $j \in [k]$ is clearly a kRiDF of G. By Observation 2.1 it follows that f is a $\gamma_{\mathrm{ri}k}(G)$-function and thus $\gamma_{\mathrm{ri}k}(G) = k$. Now suppose that the order of G is $n > k$ and assume that $K_{k,n-k}$ is a spanning subgraph of G with partite sets $\{x_1, \ldots, x_k\}$ and $\{x_{k+1}, \ldots, x_n\}$. For $i \in [k]$, let $V_i = \{x_i\}$ and let $V_0 = \{x_{k+1}, \ldots, x_n\}$. It follows that $(V_0, V_1, \ldots, V_k)$ is a kRiDF-partition of G, and again $\gamma_{\mathrm{ri}k}(G) = k$. For the converse assume that $\gamma_{\mathrm{ri}k}(G) = k$ and that $n > k$. Let $(V_0, V_1, \ldots, V_k)$ be a kRiDF-partition of G. Suppose first that $V_0 = \emptyset$. This implies that $n = |V(G)| = \sum_{j=1}^{k} |V_j| = \gamma_{\mathrm{ri}k}(G) = k$, which is a contradiction. Therefore, $V_0 \neq \emptyset$. Each vertex in V_0 has at least one neighbor in V_j for each $j \in [k]$. Since $\gamma_{\mathrm{ri}k}(G) = k$, it follows that $|V_j| = 1$ for each $j \in [k]$, and consequently G has a spanning subgraph isomorphic to $K_{k,n-k}$. ■

Let $(V_0, V_1, \ldots, V_k)$ be a kRiDF-partition of G. If x is any vertex of G and $\deg(x) < k$, then $x \notin V_0$. Thus, if we let n_i denote the number of vertices in G that have degree i, then it follows immediately that $\gamma_{\mathrm{ri}k}(G) \geq \sum_{i=1}^{k-1} n_i$. In particular, if G has maximum degree Δ and $\Delta < k$, then $\gamma_{\mathrm{ri}k}(G) = |V(G)|$, which is the maximal possible value for the k-rainbow independent domination number of a graph.

Observation 2.3 *If k is a positive integer and G is a graph of order n such that $\Delta(G) < k$, then $\gamma_{\mathrm{ri}k}(G) = n$.*

The aforementioned observation, which lead to the introduction of the k-rainbow independent domination concept, is the following.

Proposition 2.4 *If k is a positive integer and G an arbitrary graph, then $\gamma_{\mathrm{ri}k}(G) = i(G \,\square\, K_k)$.*

Proof. Let $(V_0, V_1, \ldots, V_k)$ be a kRiDF-partition of G. We define a subset D of $V(G \square K_k)$ by $D = \bigcup_{i=1}^{k} \{(g, i) : g \in V_i\}$. It follows from the structure of the Cartesian product that D is an independent dominating set in $G \,\square\, K_k$. Thus $\gamma_{\mathrm{ri}k}(G) = |D| \geq i(G \,\square\, K_k)$.

Now assume that D is a smallest independent dominating set of $G \square K_k$. For $i \in [k]$, let $V_i = \{v \in V(G) : (v, i) \in D\}$, and let $V_0 = V(G) - (V_1 \cup \cdots \cup V_k)$. Note that the sets V_i are well defined since the independence of D implies that there is at most one vertex of D in each K_k-layer. Thus we have $V_i \cap V_j = \emptyset$ for $0 \leq i < j \leq k$ and $V_0 \cup V_1 \cup \cdots \cup V_k = V(G)$. In addition, for $i \in [k]$, V_i is an independent set (otherwise we obtain a contradiction with D being independent in $G \square K_k$). Since D is a dominating set in $G \square K_k$, we have that for every $i \in [k]$ and $x \in V_0$, (x, i) is adjacent to some $(g, i) \in V_i \times \{i\} \subseteq D$, which implies that $N(x) \cap V_i \neq \emptyset$. Therefore $(V_0, V_1, \ldots, V_k)$ represents a kRiDF of G and we have $\gamma_{\mathrm{ri}k}(G) \leq |D| = i(G \,\square\, K_k)$. ■

It follows immediately that

$$\gamma_{\mathrm{r}k}(G) = \gamma(G \,\square\, K_k) \leq i(G \,\square\, K_k) = \gamma_{\mathrm{ri}k}(G) \leq \alpha(G \,\square\, K_k).$$

For any positive integer k, the independent domination number and the independence number of a graph G have a natural relationship to $\gamma_{\mathrm{ri}k}(G)$.

Proposition 2.5 *Let k be a positive integer. If G is any graph, then*

$$i(G) \leq \gamma_{\mathrm{ri}k}(G) \leq k\alpha(G)\,.$$

Proof. The proposition is clearly true for $k = 1$. Suppose then that $k \geq 2$. Let $(V_0, V_1, \ldots, V_k)$ be any kRiDF-partition of G. By the definition of a kRiDF-partition it follows that V_1 dominates V_0 and that V_i is independent for each $i \in [k]$. Let $V_1' = V_1$. For $2 \leq i \leq k$, starting with $i = 2$ and proceeding through increasing values of i, let $V_i' = V_i - \{t \in V_i : N(t) \cap (V_1' \cup \cdots \cup V_{i-1}') \neq \emptyset\}$. Define W by $W = V_1' \cup \cdots \cup V_k'$. By its construction we see that W is independent. Let $x \in V(G) - W$. If $x \in V_0$, then x has a neighbor in V_1 (that is, a neighbor in V_1') and hence W dominates x. If $x \in V_j - W$ for some $j \geq 2$, then $x \in \{t \in V_j : N(t) \cap (V_1' \cup \cdots \cup V_{j-1}') \neq \emptyset\}$, and consequently x is dominated by $V_1' \cup \cdots \cup V_{j-1}'$ and hence also by W. This proves that W is a dominating set of G. We have shown that W is an independent dominating set of G. Furthermore, for each $i \in [k]$, V_i is independent. This implies

$$i(G) \le |W| = \sum_{i=1}^{k} |V_i'| \le \gamma_{\mathrm{ri}k}(G) = \sum_{i=1}^{k} |V_i| \le k\alpha(G)\,.$$

■

There are graphs G that show the lower bound in Proposition 2.5 is sharp. For example, for any positive integer $k \ge 2$ and any positive integer $m \ge 2$, let G be the complete multipartite graph $K_{k,n_2,\ldots,n_m}$ where $k \le n_2 \le \cdots \le n_m$. Let $\{a_1, \ldots, a_k\}$ be the partite set in G of size k. For each $i \in [k]$ let $V_i = \{a_i\}$ and let $V_0 = V(G) - \{a_1, \ldots, a_k\}$. It is now easy to verify that $(V_0, V_1, \ldots, V_k)$ is a kRiDF-partition of G and that $i(G) = k = \gamma_{\mathrm{ri}k}(G)$.

Another class of graphs that attain the lower bound in Proposition 2.5 for $k = 2$ were studied by Hartnell and Rall [10]. They gave a necessary and sufficient condition on a graph G for $\gamma(G) = \gamma_{\mathrm{r}2}(G)$. Such a graph has a minimum dominating set D that can be partitioned as $D_1 \cup D_2$ such that $V(G) - N[D_1] = D_2$ and $V(G) - N[D_2] = D_1$. In addition, each of D_1 and D_2 is a 2-packing (i.e. a subset of $V(G)$ in which all the vertices are in distance at least 3 from each other) and D is independent. This implies that $i(G) = \gamma(G)$ and gives a 2RiDF-partition $(V(G) - D, D_1, D_2)$ of G. Two examples of this class of graphs are shown in Figure 2. For $i \in [2]$ the set D_i consists of the vertices labeled i.

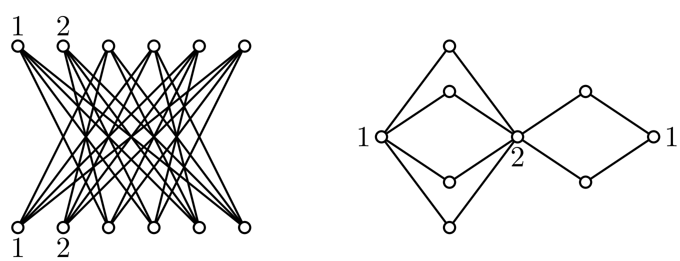

Figure 2: Examples with $i = \gamma_{\mathrm{ri}2}$.

By examining the proof of Proposition 2.5 we can see that more can be said about any graph that attains the lower bound. Indeed, for such a graph $V_i' = V_i$ for each $i \in [k]$. This gives the following corollary.

Corollary 2.6 *Let k be a positive integer. If G is a graph such that $i(G) = \gamma_{\mathrm{ri}k}(G)$, then for any kRiDF-partition $(V_0, V_1, \ldots, V_k)$ of G, the set $V_1 \cup \cdots \cup V_k$ is independent.*

We will have use of a concept first introduced by Fradkin [8] who defined a *greedy independent decomposition* (GID) of a graph G to be a partition $A_1, \ldots, A_t$ of $V(G)$ such that A_1 is a maximal independent set in G, and for each $2 \le i \le t$, the set A_i is a maximal independent set in the graph $G - (A_1 \cup \cdots \cup A_{i-1})$. The number of subsets in the partition, here t, is called its *length*. In what follows we do not require the last subset A_t in the sequence to be non-empty and when it is non-empty we do not require it to be independent. With these modifications we call such a sequence of subsets of $V(G)$ a *partial GID* of G.

Note that if $A_1, \ldots, A_t$ is a partial GID of G, then each of $A_1, \ldots, A_{t-1}$ is an independent set of G, and for each $i \in [t-1]$, A_i dominates A_t. This proves the correctness of the upper bound in the following proposition. We choose a labeling of the parts of the partition to fit with our notation.

Proposition 2.7 *If $V_1, V_2, \ldots, V_k, V_0$ is a partial GID of G, then*

$$\gamma_{\mathrm{ri}k}(G) \leq \sum_{i=1}^{k} |V_i| \, .$$

If $G = K_n$ for any $n \geq k$, then by noting as above that $\gamma_{\mathrm{ri}k}(G) = i(G \,\square\, K_k)$, it is easy to show that the bound in Proposition 2.7 is sharp. That is, if $n \geq k$, then $\gamma_{\mathrm{ri}k}(K_n) = i(K_n \,\square\, K_k) = \min\{n, k\} = k$. Indeed, in this case a complete graph of order n has a partial GID of length $k+1$, and all of the independent parts of the decomposition have order 1. However, if $k > n$, then K_n does not have a partial GID of length $k+1$, and thus Proposition 2.7 is not applicable. In this case, $\gamma_{\mathrm{ri}k}(K_n) = n$.

3 Some families of graphs and their 2RiDF

In this section we first give the value of the invariant γ_{ri2} on some common classes of graphs, and then we prove that $i(T) < \gamma_{\mathrm{ri2}}(T)$ for any nontrivial tree T.

Jacobson and Kinch [13] determined that for the path P_n on n vertices, $\gamma(P_n \,\square\, K_2) = \left\lceil \frac{n+1}{2} \right\rceil$, and Cortés [6] proved that $i\left(P_n \,\square\, K_2\right) = \left\lceil \frac{n+1}{2} \right\rceil$. In the language of rainbow domination this is equivalent to

$$\gamma_{r2}\left(P_n\right) = \left\lceil \frac{n+1}{2} \right\rceil = \gamma_{\mathrm{ri2}}\left(P_n\right) \, ,$$

(see [4] and Proposition 2.4). Pavlič and Žerovnik [16] proved that $\gamma_{\mathrm{ri2}}\left(C_n\right) = \lceil n/2 \rceil$ for n congruent to either 0 or 3 modulo 4, while $\gamma_{\mathrm{ri2}}\left(C_n\right) = \lceil n/2 \rceil + 1$ for n congruent to 1 or 2 modulo 4. In addition, the following values are easy to verify.

- For a star S_n with $n \geq 3$, $\gamma_{\mathrm{ri2}}\left(S_n\right) = n - 1$.
- For $n \geq 2$, $\gamma_{\mathrm{ri2}}\left(K_n\right) = 2$.
- For a complete multipartite graph $K_{r_1,\ldots,r_k}$, where $2 \leq r_1 \leq \ldots \leq r_k$, $\gamma_{\mathrm{ri2}}\left(K_{r_1,\ldots,r_k}\right) = r_1 = i\left(K_{r_1,\ldots,r_k}\right)$.

From Proposition 2.5 it follows that $i(G) \leq \gamma_{\mathrm{ri2}}(G)$. In the following we show that for trees we have strict inequality.

Lemma 3.1 *If x is a leaf of a tree T, then $i(T) = i(T-x)$ or $i(T) = i(T-x) + 1$.*

Proof. Let x be a leaf of a tree T and y its unique neighbor. For notational convenience let $T' = T - x$. To prove the lemma we show that $i(T') \le i(T) \le i(T') + 1$. Let M be an independent dominating set of T of cardinality $i(T)$. If $x \notin M$, then M dominates T' and is independent in T', which implies $i(T') \le |M| = i(T)$. Now suppose that $x \in M$. If $M \setminus \{x\}$ dominates T', then $i(T') \le |M| - 1 < i(T)$. If $M \setminus \{x\}$ does not dominate T', then $N(y) \cap M = \{x\}$ and $M' = (M \setminus \{x\}) \cup \{y\}$ is an independent dominating set of T', which shows $i(T') \le |M'| = |M| = i(T)$. This establishes the first inequality. To prove the second inequality, let J be an independent dominating set of T' such that $i(T') = |J|$. If $y \in J$, then J also dominates T and thus $i(T) \le |J| = i(T')$. If $y \notin J$, then $J \cup \{x\}$ is an independent dominating set of T, which implies $i(T) \le |J| + 1 = i(T') + 1$. Therefore, $i(T - x) \le i(T) \le i(T - x) + 1$. ■

Lemma 3.2 *Let x be a leaf of a nontrivial tree T. If $T' = T - x$, then $\gamma_{\text{ri2}}(T) = \gamma_{\text{ri2}}(T')$ or $\gamma_{\text{ri2}}(T) = \gamma_{\text{ri2}}(T') + 1$.*

Proof. Let T, x and T' be as in the statement of the lemma, and let y be the unique neighbor of x in T. Recall from Proposition 2.4 that $\gamma_{\text{ri2}}(G) = i(G \,\Box\, K_2)$ for any graph G. We will prove the lemma by showing that $i(T\Box K_2) = i(T'\Box K_2)$ or $i(T\Box K_2) = i(T'\Box K_2)+1$. First we shall prove that $i(T\Box K_2) \le i(T'\Box K_2) + 1$. Let J be an independent dominating set of $T'\Box K_2$ such that $|J| = i(T'\Box K_2)$. Since $(y, 1)$ and $(y, 2)$ are adjacent in $T'\Box K_2$, at least one of them, say $(y, 2)$, is not in J. The set $J \cup \{(x, 2)\}$ is independent in $T\Box K_2$ and dominates $T\Box K_2$. Hence $i(T\Box K_2) \le |J| + 1 = i(T'\Box K_2) + 1$.

Now we shall prove that $i(T'\Box K_2) \le i(T\Box K_2)$. Let M be an independent dominating set of $T\Box K_2$ of cardinality $i(T\Box K_2)$. Since $\{(y, 1), (y, 2)\}$ is not a subset of M, exactly one of $(x, 1)$ or $(x, 2)$ is in M. Without loss of generality we assume $(x, 1) \in M$. If $(y, 2) \in M$, then $M \setminus \{(x, 1)\}$ is an independent dominating set of $T'\Box K_2$ and thus $i(T'\Box K_2) \le |M| - 1 < i(T\Box K_2)$. If $(y, 2) \notin M$, then there exists $(z, 2) \in N_{T'\Box K_2}(y, 2) \cap M$. If there also exists $(w, 1) \in N_{T'\Box K_2}(y, 1) \cap M$, then $M \setminus \{(x, 1)\}$ is an independent dominating set of $T'\Box K_2$ and again $i(T'\Box K_2) \le |M| - 1 < i(T\Box K_2)$. On the other hand, if $N_{T'\Box K_2}(y, 1) \cap M = \emptyset$, then $(M \setminus \{(x, 1)\}) \cup \{(y, 1)\}$ is an independent dominating set of $T'\Box K_2$, which implies $i(T'\Box K_2) \le |M| = i(T\Box K_2)$, and the proof is complete. ■

Observation 3.3 *If G is a graph without isolated vertices and (V_0, V_1, V_2) is a 2RiDF-partition of G, then every leaf of G belongs to $V_1 \cup V_2$.*

The above results help us to show that for any non-trivial tree T the 2-rainbow independent domination number always exceeds the independent domination number.

Theorem 3.4 *If T is a non-trivial tree, then $i(T) < \gamma_{\text{ri2}}(T)$.*

Proof. The proof is by induction on the order of the tree. It is straightforward to show the statement holds for all trees of order 2, 3, or 4. Let n be an integer such that $n \ge 5$

and assume the theorem is true for all trees of order less than n. Let T be a tree of order n, let x be a vertex of degree 1 in T and let y be the unique neighbor of x. Let T' denote the subtree $T - x$ of T. By the induction hypothesis $i(T') < \gamma_{\text{ri2}}(T')$. Using Lemma 3.1 and Lemma 3.2 we get four cases.
Case 1. $i(T) = i(T')$ and $\gamma_{\text{ri2}}(T) = \gamma_{\text{ri2}}(T')$.
Case 2. $i(T) = i(T') + 1$ and $\gamma_{\text{ri2}}(T) = \gamma_{\text{ri2}}(T') + 1$.
Case 3. $i(T) = i(T')$ and $\gamma_{\text{ri2}}(T) = \gamma_{\text{ri2}}(T') + 1$.
Case 4. $i(T) = i(T') + 1$ and $\gamma_{\text{ri2}}(T) = \gamma_{\text{ri2}}(T')$.
It is easy to observe and is left to the reader to show that in the first three cases $i(T) < \gamma_{\text{ri2}}(T)$. Just using the equations in Case 4 it could happen that $i(T) = \gamma_{\text{ri2}}(T)$. We now show that this leads to a contradiction. Let $f : V(T) \to \{0, 1, 2\}$ be a γ_{ri2}-function and (V_0, V_1, V_2) the resulting 2RiDF-partition of T. By Observation 3.3, $x \in V_1 \cup V_2$ and by Corollary 2.6 the set $V_1 \cup V_2$ is independent. Without loss of generality we may assume that $x \in V_1$. This implies that $y \in V_0$ and also that there exists a vertex $z \in V_2 \cap N_T(y)$. Let $I = \{y\} \cup \{t \in V_1 \cup V_2 : d(y, t) \geq 2\}$. Clearly, I is independent and $|I| < |V_1 \cup V_2| = i(T)$. We claim that I dominates T. By the definition of I, it suffices to show that I dominates the set $\{t \in V(T) : d(y, t) = 2\}$. Let w be a vertex such that $d(y, w) = 2$ and $w \notin I$. Thus, $w \in V_0$. Since (V_0, V_1, V_2) is a 2RiDF-partition of T, w has a neighbor a in V_1 and a neighbor b in V_2. At most one of a or b is in $N_T(y)$ and it follows that w is dominated by I. This is a contradiction, and thus in all cases, $i(T) < \gamma_{\text{ri2}}(T)$. The theorem follows by induction. ■

4 A Nordhaus-Gaddum type of result for $\gamma_{\text{ri2}}(G)$

In 1956, Nordhaus and Gaddum [15] gave lower and upper bounds on the sum and the product of the chromatic number of a graph and its complement, in terms of the order of the graph [15]. Relations of a similar type have been proposed for many other graph invariants since then. See the survey [2] by Aouchiche and Hansen on this topic. An invariant not addressed in the survey was rainbow domination. Wu and Xing [18] proved sharp Nordhaus-Gaddum type bounds for the 2-rainbow domination number. In particular, they proved that if G is any graph of order n and $n \geq 3$, then $5 \leq \gamma_{r2}(G) + \gamma_{r2}(\overline{G}) \leq n + 2$.

In this section we explore this relation for the 2-rainbow independent domination number. Although our bounds are very similar to those given by Wu and Xing for (ordinary) 2-rainbow domination, the proof of our bounds turned out to be more involved than theirs because of the requirement that V_1 and V_2 are independent sets in the definition of a 2RiDF-partition of G. We will need the following lemmas to prove our main result Theorem 4.3.

Lemma 4.1 *For any graph G of order n, $\gamma_{\text{ri2}}(G) = n$ if and only if every connected component of G is isomorphic either to K_1 or K_2. In addition, if $\gamma_{\text{ri2}}(G) = n$, then $\gamma_{\text{ri2}}(\overline{G}) = 2$.*

Proof. If every connected component of a graph G of order n is isomorphic to K_1 or K_2, then clearly $\gamma_{\text{ri2}}(G) = n$. Since $\gamma_{\text{ri2}}(G) = \sum_{j=1}^{l} \gamma_{\text{ri2}}(G_j)$ if $G_1, G_2, \ldots, G_l$ are the connected components of G, then to prove the converse it suffices to show that $\gamma_{\text{ri2}}(G) < n$ when G is connected and $n \geq 3$. Thus, with the goal of obtaining a contradiction suppose that G is connected of order $n \geq 3$ and that $\gamma_{\rm ri2}(G) = n$. There exists $y \in V(G)$ of degree at least 2. Let f be a $\gamma_{\rm ri2}(G)$-function and let (V_0, V_1, V_2) be the 2RiDF-partition associated with f. Since $\gamma_{\rm ri2}(G) = n$, we have $V_0 = \emptyset$, and since for each $i \in [2]$, there is no edge between two vertices from V_i, G is a bipartite graph where V_1 and V_2 constitute a bipartition of $V(G)$. Assume (without loss of generality) that $y \in V_1$. Then there exist $x, z \in N(y) \cap V_2$. Let $D_i = \{v \in V(G) : d(v, y) = i\}$ and suppose the eccentricity of y (i.e., the largest distance of y to a vertex from $V(G)$) is t. Then $\{y\} \cup \bigcup_{i=1}^{t} D_i = V(G)$. Note that $E(G\langle D_i \rangle) = \emptyset$, and if $v \in D_i$, then $f(v) = 1$ in the case when i is even and $f(v) = 2$ when i is odd. We will show that a 2RiDF, $g : V(G) \to \{0, 1, 2\}$ with $w(g) < w(f)$ can be constructed as follows.

Let $g(y) = 0$, $g(x) = 1$ and $g(z) = 2$. Let $A = D_2 \cap (N(x) \setminus N(z))$, $B = D_2 \cap (N(z) \setminus N(x))$, and $C = D_2 \cap N(x) \cap N(z)$. Set $g(v) = 2$ if $v \in A$, $g(v) = 1$ if $v \in B$, and $g(v) = 0$ if $v \in C$. Next, we define the following sets: F is the set of vertices $v \in D_1 \setminus \{x, z\}$ such that v has a neighbor in A as well as a neighbor in B; D is the set of vertices $v \in D_1 \setminus \{x, z\}$ such that v has a neighbor in B, but no neighbor in A; and E is the set of vertices $v \in D_1 \setminus \{x, z\}$ such that v has a neighbor in A, but no neighbor in B. Set $g(v) = 2$ if $v \in D$, $g(v) = 1$ if $v \in E$, and $g(v) = 0$ if $v \in F$. Now consider vertices in $D_1 \setminus (\{x, z\} \cup D \cup E \cup F)$. None of these vertices is adjacent to a vertex in A. Hence we let $g(v) = 2$ for every $v \in D_1 \setminus (\{x, z\} \cup D \cup E \cup F)$. Next, we define g on $D_2 \setminus (A \cup B \cup C)$ according to the following procedure.

Suppose we have already assigned values of g to vertices of a set D_i, $i \in [t-1]$. Let $M = \{v \in D_i : g(v) = 1\}$, $N = \{v \in D_i : g(v) = 2\}$, and $O = \{v \in D_i : g(v) = 0\}$. Now consider vertices w from D_{i+1}. If w has a neighbor in M and a neighbor in N, then $g(w) = 0$. If w has a neighbor in M, but not in N, $g(w) = 2$. If w has a neighbor in N, but not in M, $g(w) = 1$. For the remaining vertices in D_{i+1} we set $g(w) = f(w)$. In this way we first assign values of g to all vertices of $D_2 \setminus (A \cup B \cup C)$, and then to all remaining vertices in $V(G)$.

It is straightforward to check that g is a 2RiDF on G. Since at least one vertex (that is y) has the value 0 under g, we have $w(g) \leq n - 1$, as desired.

For the proof of the second part of the lemma, note that if G is an empty graph on n vertices, then $\overline{G} = K_n$ and $\gamma_{\rm ri2}(\overline{G}) = 2$. If $\gamma_{\rm ri2}(G) = n$ and at least two vertices, say a and b, are adjacent in G, then $N_{\overline{G}}(a) = V(G) - \{a, b\}$ and $N_{\overline{G}}(b) = V(G) - \{a, b\}$. Thus $h : V(G) \to \{0, 1, 2\}$, $h(a) = 1$, $h(b) = 2$ and $h(c) = 0$ otherwise, is a 2RiDF with minimum weight. ■

Lemma 4.2 *If G is a graph of order $n \geq 3$ containing a universal vertex and $\gamma_{\rm ri2}(G) = n - 1$, then G is either isomorphic to S_n or S_n^+.*

Proof. Let u be a universal vertex of G and assume $\gamma_{\rm ri2}(G) = n - 1$. We claim there is a $\gamma_{\rm ri2}(G)$-function f for which $f(u) = 0$. For suppose that (W_0, W_1, W_2) is a 2RiDF-partition corresponding to a $\gamma_{\rm ri2}(G)$-function, and $W_0 = \{v\}$ for some $v \neq u$. Without loss of generality we may assume that $u \in W_1$. Since W_1 is independent and u is a universal vertex, it follows that $W_1 = \{u\}$. If $|N(v) \cap N(u)| = 1$, then $G = S_n^+$. Otherwise, let r and s be distinct vertices in $W_2 \cap N(v)$. Now, let $W_0' = \{u, v\}, W_1' = \{s\}$ and $W_2' = W_2 - \{s\}$. We see that (W_0', W_1', W_2') is a 2RiDF-partition of G, which contradicts our assumption that $\gamma_{\rm ri2}(G) = n - 1$. Thus, we may assume that such a $\gamma_{\rm ri2}(G)$-function f exists, and let (V_0, V_1, V_2) be the partition of $V(G)$ corresponding to f. Assume that $G\langle N(u)\rangle$ contains at least two edges. In the first case, let xy and yz be edges in $G\langle N(u)\rangle$, and suppose (without loss of generality) that $f(x) = f(z) = 1$. Then $f(y) = 2$. We will consider a function $g : V(G) \to \{0, 1, 2\}$, defined as follows. Let $A = V_2 - N_G(z)$, let B be the set of vertices in $V_2 \cap N(z)$ having another neighbor in V_1 different from z, and let C be the set of vertices in V_2 whose only neighbor in V_1 is z. Let $g(v) = 2$ for $v \in A \cup \{z\}$, $g(v) = 0$ for $v \in B \cup \{u\}$, and $g(v) = 1$ for $v \in C \cup (V_1 - \{z\})$. The function g is a 2RiDF with $w(g) \le n-2$ (since $g(u) = 0$ and $g(y) = 0$) a contradiction. This implies that any two edges from $G\langle N(u)\rangle$ are disjoint. Assume that $G\langle N(u)\rangle$ contains two such edges, xy and ab, and suppose $f(y) = f(a) = 1$ and $f(x) = f(b) = 2$. Let $g : V(G) \to \{0, 1, 2\}$ be defined by $g(u) = 1$, $g(v) = 0$ for every $v \in V_1$ that has a neighbor in V_2, $g(v) = 2$ for every $v \in V_1$ that does not have a neighbor in V_2, and $g(t) = f(t)$ otherwise. This function g is a 2RiDF, and since $g(y) = g(a) = 0$, we obtain $w(g) \le n-2$, a contradiction again with $\gamma_{\rm ri2}(G) = n-1$. This implies that $G\langle N(u)\rangle$ contains at most one edge, which means that G is either isomorphic to S_n or to S_n^+. ■

Theorem 4.3 *If G is a graph of order n where $n \ge 3$, then*

$$5 \le \gamma_{\rm ri2}(G) + \gamma_{\rm ri2}(\overline{G}) \le n + 3,$$

and the bounds are sharp.

Proof. Let G be a graph of order n such that $n \ge 3$. First we consider the lower bound. Clearly, $\gamma_{\rm ri2}(H) \ge 2$ for any graph H of order at least 2, and thus $\gamma_{\rm ri2}(G) + \gamma_{\rm ri2}(\overline{G}) \ge 4$. However, this bound cannot be achieved. Namely, if $\gamma_{\rm ri2}(G) = 2$ and (V_0, V_1, V_2) corresponds to a $\gamma_{\rm ri2}(G)$-function of G, where $V_1 = \{x\}$ and $V_2 = \{y\}$, then in $\overline{G}$ the set $\{x, y\}$ induces either N_2 or K_2, and none of the vertices from V_0 is adjacent to x or to y. Thus we have $\gamma_{\rm ri2}(\overline{G}) \ge 3$, and the lower bound follows.

For the upper bound, let a partition (V_0, V_1, V_2) of $V(G)$ correspond to a $\gamma_{\rm ri2}(G)$-function f of G (then $\{V_0, V_1, V_2\}$ is a set partition of $V(\overline{G})$). Thus $\gamma_{\rm ri2}(G) = n - |V_0|$. If $\gamma_{\rm ri2}(G) = 2$ then clearly $\gamma_{\rm ri2}(G) + \gamma_{\rm ri2}(\overline{G}) \le n + 2$. Now consider the case when at least one of V_1 and V_2, say V_1, contains at least two vertices. Let x and y be different vertices in V_1, which implies that $xy \in E(\overline{G})$. Let A be a maximal independent set in $\overline{G}$ containing x, and B a maximal independent set in $\overline{G} - A$ containing y. Since $A, B, V(\overline{G}) - (A \cup B)$ is a partial GID of $\overline{G}$, Proposition 2.7 implies that $\gamma_{\rm ri2}(\overline{G}) \le |A| + |B|$. Since V_1 induces a complete subgraph in $\overline{G}$, we have $(A \cup B) \cap V_1 = \{x, y\}$. Similarly, V_2 induces a complete subgraph in $\overline{G}$, which implies there is at most one vertex in $A \cap V_2$ and at most one vertex in $B \cap V_2$. Thus

$$\gamma_{\rm ri2}(\overline{G}) \le |A| + |B| \le |V_0| + 4,$$

which implies

$$\gamma_{\rm ri2}(G) + \gamma_{\rm ri2}(\overline{G}) \le n - |V_0| + |V_0| + 4 = n + 4. \tag{1}$$

Now suppose that $\gamma_{\rm ri2}(G)+\gamma_{\rm ri2}(\overline{G})=n+4$ which is by (1) equivalent to the fact that $\gamma_{\rm ri2}(\overline{G})=|V_0|+4$. The latter holds if and only if, for A and B defined as above, the function $g:V(\overline{G})\to\{0,1,2\}$, $g(a)=1$ for every $a\in A$, $g(a)=2$ for every $a\in B$, and $g(a)=0$ otherwise, is a $\gamma_{\rm ri2}(\overline{G})$-function and $|V_2\cap A|=|V_2\cap B|=1$. Let s and t be vertices in V_2 such that $s\in A$ and $t\in B$. This means that for every $v\in V_0$, $g(v)\neq 0$. (Note that V_0 is nonempty for otherwise $\gamma_{\rm ri2}(G)=n$, which by Lemma 4.1 implies that $\gamma_{\rm ri2}(\overline{G})=2$ and leads to a contradiction.) Also note that $\gamma_{\rm ri2}(\overline{G}\langle V_0\rangle)=|V_0|$, for otherwise we obtain a contradiction with g being a $\gamma_{\rm ri2}(\overline{G})$-function. Thus every connected component of $\overline{G}\langle V_0\rangle$ is either K_1 or K_2, by Lemma 4.1.

Suppose first that $\overline{G}\langle V_0\rangle$ contains at least one edge. We distinguish the following cases.

Case 1. Suppose there exist $u,v\in V_0$ such that $uv\in E(\overline{G})$ and $N_{\overline{G}}(\{u,v\})-\{u,v\}=\emptyset$. Then $N_G(u)=V(G)-\{u,v\}$ and $N_G(v)=V(G)-\{u,v\}$, which implies that $\gamma_{\rm ri2}(G)=2$, thus $\gamma_{\rm ri2}(G)+\gamma_{\rm ri2}(\overline{G})\le n+2$, a contradiction.

Case 2. Thus for each pair of vertices $u,v\in V_0$ such that $uv\in E(\overline{G})$ it follows $N_{\overline{G}}(\{u,v\})-\{u,v\}\neq\emptyset$. Fix such an edge uv in $\overline{G}\langle V_0\rangle$. There exists an edge $uz\in E(\overline{G})$ such that $z\in V_1\cup V_2$.

First we claim that there is at most one such edge uz. Suppose to the contrary that there are different vertices $z_1,z_2\in V_1\cup V_2$ such that $uz_1,uz_2\in E(\overline{G})$. There are two possibilities, either z_1 and z_2 belong to the same one of these two sets V_1 and V_2 or they do not.

In either case we will reach a contradiction by constructing an 2-rainbow independent dominating function $g':V(\overline{G})\to\{0,1,2\}$ with weight strictly less than $|V_0|+4$. In both cases we first let $g'(u)=0$, $g'(z_1)=1$ and $g'(z_2)=2$; next we assign values of g' to all vertices from $V_1\cup V_2-\{z_1,z_2\}$; and lastly to all vertices in $V_0-\{u\}$.

Now we describe the procedure of assigning values to vertices in $V_1\cup V_2-\{z_1,z_2\}$. First assume that z_1,z_2 belong to different sets V_1 and V_2, say $z_1\in V_1$ and $z_2\in V_2$. Consider vertices from $V_2-\{z_2\}$. If all of them are adjacent to z_1, then their value under g' will be 0. If there exist vertices in $V_2-\{z_2\}$ not adjacent to z_1, then one of them obtains the value 1 and all others are assigned the value 0. Analogously, we assign values of g' to vertices in $V_1-\{z_1\}$. Now assume that z_1,z_2 belong to the same set, say V_1. If there exist vertices $a,b\in V_2$ such that $az_1,bz_2\notin E(\overline{G})$, then $g'(a)=1$, $g'(b)=2$, and $g'(z)=0$ for every $z\in V_1\cup V_2-\{z_1,z_2,a,b\}$. If all vertices in V_2 are adjacent to z_2 and there exist vertices in V_2 not adjacent to z_1, then one of them, say a, obtains value 1 and all other vertices in $V_2-\{a\}$ as well as all vertices in $V_1-\{z_1,z_2\}$ obtain value 0. In an analogous manner, if all vertices in V_2 are adjacent to z_1 and there exist vertices in V_2 not adjacent to z_2, then one of them, say b, obtains value 2 and all other vertices in $V_2-\{b\}$ as well as all vertices in $V_1-\{z_1,z_2\}$ obtain value 0. In the last case, when all vertices from V_2 are adjacent to both z_1 and z_2, we assign value 0 to every vertex in $V_1\cup V_2-\{z_1,z_2\}$. Note that in all cases at most four vertices in $V_1\cup V_2$ have been assigned a nonzero value under g'.

Now we assign values of g' to all vertices in $V_0-\{u\}$. In the beginning let $C=\{z\in V_1\cup V_2:g'(z)=1\}$ and $D=\{z\in V_1\cup V_2:g'(z)=2\}$. During the procedure other vertices from V_0 may be added to the sets C and D. Order the vertices in $V_0-\{u\}$ arbitrarily.

By proceeding through this list we assign values under g', one vertex z at a time, in the following way. If z has a neighbor in C as well as in D, then $g'(z) = 0$. If z has a neighbor in D but not in C, then $g'(z) = 1$ and we add z to C. If z has a neighbor in C but not in D, then $g'(z) = 2$ and we add z to D. If z does not have a neighbor in $C \cup D$, then $g'(z) = 1$ and we add z to C.

The function g' is a 2RiDF of $\overline{G}$ and $w(g') < |V_0| + 4$. This proves the claim that if $u \in V_0$ and $N_{\overline{G}}(u) \cap V_0 \neq \emptyset$, then u has at most one neighbor in $V_1 \cup V_2$.

Suppose there exist $u, v \in V_0$ such that $uv \in E(\overline{G})$ and there exist $z, w \in V_1 \cup V_2$ such that $uz, vw \in E(\overline{G})$. Observe that $N_G(u) = V(G) - \{z, u, v\}$ and $N_G(v) = V(G) - \{w, u, v\}$. Assume that $z \neq w$. One can easily verify that $f' : V(G) \to \{0, 1, 2\}$, defined by $f'(u) = f'(z) = 1, f'(v) = f'(w) = 2$, and $f'(a) = 0$ otherwise, is a 2RiDF of G. This implies $\gamma_{\mathrm{ri2}}(G) \leq 4$, hence $\gamma_{\mathrm{ri2}}(\overline{G}) \geq n$, which in fact gives $\gamma_{\mathrm{ri2}}(\overline{G}) = n$. By Lemma 4.1, $\gamma_{\mathrm{ri2}}(G) = 2$, which leads to a contradiction. On the other hand, if $z = w$, then $f' : V(G) \to \{0, 1, 2\}$, defined by $f'(u) = f'(z) = 1, f'(v) = 2$, and $f'(a) = 0$ otherwise, is a 2RiDF of G. Hence $\gamma_{\mathrm{ri2}}(G) \leq 3$, and $\gamma_{\mathrm{ri2}}(G) + \gamma_{\mathrm{ri2}}(\overline{G}) \leq n + 3$, which is again a contradiction.

In the remaining case, suppose there exist vertices u and v in V_0 such that $uv \in E(\overline{G})$ and exactly one of u and v is adjacent in $\overline{G}$ to a vertex in $V_1 \cup V_2$. Without loss of generality we assume $uz \in E(\overline{G})$ and $z \in V_1 \cup V_2$. The function $f' : V(G) \to \{0, 1, 2\}$, defined by $f'(u) = f'(z) = 1, f'(v) = 2$, and $f'(a) = 0$ otherwise, is a 2RiDF of G. This implies $\gamma_{\mathrm{ri2}}(G) \leq 3$, which is a contradiction.

The above two cases lead us to the conclusion that each component of $\overline{G}\langle V_0 \rangle$ is an isolated vertex. That is, V_0 induces a complete subgraph in G. Here the following possibilities arise.

Case a. Assume that $N_{\overline{G}}(u) = \emptyset$, for every $u \in V_0$. If there are at least two vertices in V_0, then G has at least two universal vertices. Thus $\gamma_{\mathrm{ri2}}(G) = 2$, which is a contradiction since $\gamma_{\mathrm{ri2}}(\overline{G}) \leq n$. On the other hand, if $V_0 = \{u\}$, then u is a universal vertex in G and $\gamma_{\mathrm{ri2}}(G) = n - 1$. Hence G is either S_n or S_n^+ by Lemma 4.2. It follows that $\overline{G} - u$ has at least two universal vertices since V_1 and V_2 both contain at least two vertices, which implies $\gamma_{\mathrm{ri2}}(\overline{G}) \leq 3$. We conclude that $\gamma_{\mathrm{ri2}}(G) + \gamma_{\mathrm{ri2}}(\overline{G}) \leq n - 1 + 3 = n + 2$, which is a contradiction.

Case b. In the last case assume that there exists an edge in $\overline{G}$ joining a vertex in V_0 and a vertex in $V_1 \cup V_2$.

First we claim that each vertex in V_0 has at most one neighbor in $V_1 \cup V_2$. Suppose to the contrary that for some $u \in V_0$ there are distinct vertices $v_1, v_2 \in V_1 \cup V_2$ such that $uv_1, uv_2 \in E(\overline{G})$.

In the two possibilities that arise we will construct an 2-rainbow independent dominating function $g' : V(\overline{G}) \to \{0, 1, 2\}$ with weight strictly less than $|V_0| + 4$, which will lead to a contradiction. Similar to how we proceeded in Case 2 above, we first let $g'(u) = 0$, $g'(v_1) = 1$ and $g'(v_2) = 2$; next we assign values of g' to all vertices from $V_1 \cup V_2 - \{v_1, v_2\}$; and lastly to all vertices in $V_0 - \{u\}$.

Assume first that v_1, v_2 belong to different sets V_1 and V_2, say $v_1 \in V_1$ and $v_2 \in V_2$.

Consider vertices from $V_2 - \{v_2\}$. If all of them are adjacent to v_1, their value under g' will be 0. If there exist vertices in $V_2 - \{v_2\}$ not adjacent to v_1, one of them obtains the value 1 and all others the value 0. In a similar manner we assign values of g' to vertices in $V_1 - \{v_1\}$. Now assume that v_1, v_2 belong to the same set, say V_1. If there exist vertices $a, b \in V_2$ such that $av_1, bv_2 \notin E(\overline{G})$, then $g'(a) = 1$, $g'(b) = 2$, and $g'(w) = 0$ for every $w \in V_1 \cup V_2 - \{v_1, v_2, a, b\}$. If all vertices in V_2 are adjacent to v_2 and there exist vertices in V_2 not adjacent to v_1, then one of them, say a, obtains value 1 and all other vertices in $V_2 - \{a\}$ as well as all vertices in $V_1 - \{v_1, v_2\}$ obtain value 0. Similarly, if all vertices in V_2 are adjacent to v_1 and there exist vertices in V_2 not adjacent to v_2, then one of them, say b, obtains value 2 and all other vertices in $V_2 - \{b\}$ as well as all vertices in $V_1 - \{v_1, v_2\}$ obtain value 0. In the last case, when all vertices from V_2 are adjacent to both v_1 and v_2, we assign value 0 to every vertex in $V_1 \cup V_2 - \{v_1, v_2\}$.

What remains is to define g' on $V_0 - \{u\}$. Let $C = \{z \in V_1 \cup V_2 : g'(z) = 1\}$ and $D = \{w \in V_1 \cup V_2 : g'(w) = 2\}$. Fix a vertex v in $V_0 - \{u\}$. If v has a neighbor in C as well as in D, then $g'(v) = 0$. If v has a neighbor in D but not in C, then $g'(v) = 1$. If v has a neighbor in C but not in D, then $g'(v) = 2$. For each $w \in V_0 - \{u, v\}$ we let $g'(w) = g(w)$.

As in Case 2 above it is easy to see that the function g' is a 2RiDF of $\overline{G}$ and $w(g') < |V_0|+4$. This contradiction proves the claim that if $u \in V_0$, then u has at most one neighbor in $V_1 \cup V_2$.

Suppose there exist at least two vertices in V_0, say u and v, such that $uz, vw \in E(\overline{G})$ for some $z, w \in V_1 \cup V_2$. Observe that $N_G(u) = V(G) - \{u, z\}$ and $N_G(v) = V(G) - \{v, w\}$. Assume first that $z \neq w$. One can easily verify that $f' : V(G) \to \{0, 1, 2\}$ defined by $f'(u) = f'(z) = 1, f'(v) = f'(w) = 2$, and $f'(a) = 0$ otherwise, is a 2RiDF on G. This implies $\gamma_{\text{ri2}}(G) \leq 4$, hence $\gamma_{\text{ri2}}(\overline{G}) \geq n$ which in fact gives $\gamma_{\text{ri2}}(\overline{G}) = n$. By Lemma 4.1, $\gamma_{\text{ri2}}(G) = 2$, which leads to a contradiction. On the other hand, if $z = w$, then $f' : V(G) \to \{0, 1, 2\}$ defined by $f'(u) = f'(z) = 1, f'(v) = 2$, and $f'(a) = 0$ otherwise, is a 2RiDF of G. Hence $\gamma_{\text{ri2}}(G) \leq 3$, and $\gamma_{\text{ri2}}(G) + \gamma_{\text{ri2}}(\overline{G}) \leq n + 3$, which is again a contradiction.

Consequently, we are led to the situation in the graph $\overline{G}$ in which there is exactly one vertex u in V_0 that has (exactly one) neighbor, say v in V_1. However, if $|V_0| \geq 2$, then $f' : V(G) \to \{0, 1, 2\}$, $f'(u) = f'(v) = 1$, $f'(z) = 2$ for some $z \in V_0 \setminus \{u\}$, and $f'(a) = 0$ otherwise, is a 2RiDF of G. This is again a contradiction since $\gamma_{\text{ri2}}(G) \leq w(f') = 3$. Thus u is the only vertex in V_0.

Recall that $g(u) \neq 0$ and that there are vertices $x, y \in V_1$ and $s, t \in V_2$ such that $xy, st \in E(\overline{G})$ and $g(x) = g(s) = 1$ and $g(y) = g(t) = 2$. Thus $\gamma_{\text{ri2}}(\overline{G}) = 5$. Note also that $xs, yt \in E(G)$. Let $W = \{w \in V_2 : N_G(w) \cap V_1 \neq \emptyset\}$. Let $f' : V(G) \to \{0, 1, 2\}$ be defined by $f'(u) = 2$, $f'(w) = 0$ for every vertex $w \in W$, and $f'(v) = 1$ for every $v \in V_1 \cup (V_2 - W)$. Since W contains at least two vertices, namely s and t, it is straightforward to show that f' is a 2-rainbow independent dominating function of G such that $w(f') \leq n - 2$. It follows that $\gamma_{\text{ri2}}(G) \leq n-2$ and as a result $\gamma_{\text{ri2}}(G) + \gamma_{\text{ri2}}(\overline{G}) \leq n+3$, which is the final contradiction.

We have shown that the assumption on equality in (1) leads to a contradiction, thus

$$\gamma_{\text{ri2}}(G) + \gamma_{\text{ri2}}(\overline{G}) \leq n + 3. \tag{2}$$

Note that if G is a cycle on 5 vertices, the equality in (2) is attained. Any graph of order 3 attains the lower bound. ■

The only graphs that attain the lower bound in Theorem 4.3 have order 3. Indeed, suppose that G has order n where $n > 3$ and that $\gamma_{\rm ri2}(G) + \gamma_{\rm ri2}(\overline{G}) = 5$. We may assume without loss of generality that $\gamma_{\rm ri2}(G) = 2$. Let $(\{x_3, \ldots, x_n\}, \{x_1\}, \{x_2\})$ be a 2RiDF-partition of G. The vertices x_1 and x_2 are either isolated or induce a component of order 2 in $\overline{G}$, and it follows that $\gamma_{\rm ri2}(\overline{G}) = 2 + \gamma_{\rm ri2}(\overline{G}\langle\{x_3, \ldots, x_n\}\rangle) \geq 4$ since $n \geq 4$.

5 Concluding remarks

In this paper we have presented a new domination concept and explored some of its basic properties. A number of natural questions remain unanswered. One of these is whether the 5-cycle is the only graph for which the upper bound is attained in the Nordhaus-Gaddum type inequality. Further, we observed (in Proposition 2.5) that $i(G) \leq \gamma_{{\rm ri}k}(G)$ for any graph G and positive integer k, and presented some families of graphs for which the equality holds. In addition, we found a property that must hold if $i(G) = \gamma_{{\rm ri}k}(G)$ (see Corollary 2.6). A characterization of graphs for which the latter equality holds remains open. It would also be interesting to explore algorithmic aspects of computing the k-rainbow independent domination number. It is quite likely that the problem of deciding if a graph has a k-rainbow independent dominating function of a given weight is NP-complete. However, it would be interesting to consider this question for specific families of graphs as well.

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