Packing and Domination Invariants on Cartesian Products and Direct Products

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Abstract

The dual notions of domination and packing in finite simple graphs were first extensively explored by Meir and Moon in [15]. Most of the lower bounds for the domination number of a nontrivial Cartesian product involve the 2-packing, or closed neighborhood packing, number of the factors. In addition, the domination number of any graph is at least as large as its 2-packing number, and the invariants have the same value for any tree. In this paper we survey what is known about the domination, total domination and paired-domination numbers of Cartesian products and direct products. In the process we highlight two other packing invariants that each play a role similar to that played by the 2-packing number in dominating Cartesian products.

Keywords: graph products, paired-domination, k-packing, Vizing's conjecture, open packing, total domination, direct product

AMS subject classification: 05C69, 05C70, 05C05

1 Introduction

One of the most well-known open problems in domination theory is the conjecture made by V. G. Vizing [19] almost forty years ago. The conjecture states that the domination number of the Cartesian product of any pair of graphs is no less than the product of their domination numbers. The closed neighborhood, or distance-two, packing number plays an important role in many of the attacks on this conjecture. In this paper we consider two other types of packing invariants and show how each plays a similar role in the study of a generalized domination number on a particular graph product.

1.1 Domination and Packing

For notation and graph theory terminology we generally follow [9]. Let G be a finite, simple, undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open* *neighborhood* of a vertex $v \in V$ is $N_G(v) = \{ u \in V \mid uv \in E \}$, while its *closed neighborhood* is $N_G[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, $N_G(S)$ denotes $\bigcup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$. If the graph is clear from the context, then we omit the subscript on these neighborhood names. The neighborhood of an edge $e = uv$ is $N(\{u, v\}) = N[\{u, v\}]$. For a set $S \subseteq V$, the subgraph of G induced by S is denoted by $\langle S \rangle_G$ or $\langle S \rangle$ and the complement of G by $\overline{G}$. A family $\{S_k\}_{k \in I}$ of subsets of vertices in G is a *cover* of (or *covers*) G if $V(G) = \cup_{k \in I} S_k$. The family is a *packing* if the subsets are pairwise disjoint. A *matching* in G is a collection of edges that, as sets of vertices, is a packing. A matching M is a *perfect* matching of G if it covers G when M is considered as a collection of 2-element subsets of vertices.

For a positive integer k, Meir and Moon [15] defined a *k-packing* in G to be a set $A \subseteq V$ such that for every pair of distinct vertices u and v in A, the distance between u and v in G is more than k. For $k = 2$ this is equivalent to requiring that $\{N[x]\}_{x \in A}$ is a packing in the sense defined in the previous paragraph. A vertex subset B is an *open packing* in G if the collection of open neighborhoods, $\{N(x)\}_{x \in B}$, is a packing in G. The *k-packing number* is the order of a largest k-packing of G and is denoted $\rho_k(G)$, while the *open packing number*, $\rho^{\mathrm{o}}(G)$, is the order of a largest open packing in G. Of course, 1-packings (independent sets) have played a prominent role in graph theory almost since its formal beginnings. We will use the more common notation $\alpha(G)$, instead of $\rho_1(G)$, for the *vertex independence number* of G, the cardinality of a largest independent set of vertices in G.

A subset D of vertices is a *dominating set* of G if every vertex x in V either belongs to D or is adjacent to a vertex in D. Equivalently, the family $\{N[x]\}_{x \in D}$ of closed neighborhoods covers G. Such a dominating set is *minimal* if no proper subset of D is a dominating set. The *domination number*, $\gamma(G)$, is the minimum cardinality of a dominating set of G. A set S of vertices is a *total dominating* set if the family $\{N(x)\}_{x \in S}$ of open neighborhoods covers G, and the *total domination number*, $\gamma_t(G)$, equals the minimum cardinality of a total dominating set of G. To guarantee that a graph has a total dominating set we must assume it has no isolated vertices. Hence we assume this to be the case for all graphs in the remainder of the paper. A dominating set D whose induced subgraph $\langle D \rangle$ contains a perfect matching is called a *paired-dominating* set, and the *paired-domination number* $\gamma_{\mathrm{pr}}(G)$ is the cardinality of a smallest such set. The vertices saturated by the edges of any maximal matching in G constitute a paired dominating set. Thus, $\gamma_{\mathrm{pr}}(G)$ is defined for all graphs under consideration and is no larger than twice the order of a smallest maximal matching. It follows from their definitions that $\gamma(G) \leq \gamma_t(G) \leq \gamma_{\mathrm{pr}}(G)$. The path of order five shows that all of the inequalities can be strict; all three values are equal on a path of order four. A dominating set D of cardinality $\gamma(G)$ is called a $\gamma(G)$-set or a γ-set if G is understood from context. We define γ_t-set, γ_{pr}-set, ρ_k-set and ρ^{o}-set similarly.

The fact that $\rho_2(G) \leq \gamma(G)$ for every G follows directly from the observations that a γ-set intersects every closed neighborhood and a ρ_2-set contains at most one vertex from each closed neighborhood. In an entirely similar way, and based only on the definitions, it is immediate that $\rho^{\mathrm{o}}(G) \leq \gamma_t(G)$ for every G that has no isolated vertices. In [3] Brešar, et al demonstrated a connection, although not so direct, between paired-domination and 3-packings. We summarize these relationships in the following.

Proposition 1.1 *For any graph G without isolated vertices,*

(i) $\gamma(G) \geq \rho_2(G)$*;*

(ii) $\gamma_t(G) \geq \rho^{\mathrm{o}}(G)$*; and*

(iii) ([3]) $\gamma_{\mathrm{pr}}(G) \geq 2\rho_3(G)$.

Proof. The first two inequalities follow as above. Let D be a $\gamma_{\mathrm{pr}}(G)$-set and let C be a $\rho_3(G)$-set. Suppose $|D| = 2k$. For each vertex $x \in C$, let $p(x)$ be a vertex such that $p(x) \in D \cap N[x]$ and let $D' = \cup_{x\in C}\{p(x)\}$. Since C is a 3-packing, $|C| = |D'|$ and D' is an independent set in $\langle D\rangle$. Now, $\langle D\rangle$ is covered by k edges and consequently

$$\gamma_{\mathrm{pr}}(G) = |D| \geq 2|D'| = 2\rho_3(G)\,.$$

□

1.2 Product Graphs

For product graphs we follow the book by Imrich and Klavžar [12]. Specifically, the *Cartesian product*, $G_1 \,\Box\, G_2$, of two graphs G_1 and G_2 has the ordinary (set) Cartesian product $V(G_1) \times V(G_2)$ as its vertex set. Two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \,\Box\, G_2$ if they are equal in one coordinate and adjacent in the other. (That is, for exactly one of $i = 1$ or $i = 2$, $u_i = v_i$ and u_{3-i} is adjacent to v_{3-i} in G_{3-i}.) The *direct product*, $G_1 \times G_2$, has the same vertex set as the Cartesian product, and u and v are adjacent in $G_1 \times G_2$ precisely when u_1v_1 is an edge in G_1 and u_2v_2 is an edge in G_2.[1]

For a vertex g in G, the subgraph of $G \,\Box\, H$ induced by the set $\{(g, h) \mid h \in V(H)\}$ is isomorphic to H and is denoted by ${}^gH$. For $h \in V(H)$, G^h is the subgraph, isomorphic to G and induced by $\{(g, h) \mid g \in V(G)\}$. Both of these subgraphs are called *fibers* of $G \,\Box\, H$. In a direct product $G \times H$ we also refer to these as fibers, but in this case they are independent sets in $G \times H$.

Depending on the particular graph product and the particular property of vertex subsets, it may be easy to establish that the (set) Cartesian product of two sets with the property retains that property in the product graph. For example, if I_j is an independent set in the graph G_j for $j = 1, 2$, then the set $I_1 \times I_2$ is independent in the graph $G_1 \,\Box\, G_2$. Because α measures the largest order of an independent set, it follows that $\alpha(G_1 \,\Box\, G_2) \geq \alpha(G_1)\alpha(G_2)$. On the other hand, if D_i is a dominating set of G_i for $i = 1, 2$, then $D_1 \times D_2$ dominates $G_1 \,\Box\, G_2$ only if $D_i = V(G_i)$ for at least one of the values of i.

For a given graphical invariant σ and given graph product $\otimes$ it is thus natural to investigate the behavior of σ on $\otimes$. It is sometimes the case that the value $\sigma(G \otimes H)$ depends directly on the two values $\sigma(G)$ and $\sigma(H)$ for all pairs of graphs G and H. We say that σ is

[1] The direct product has gone by a number of different names in the literature including categorical, tensor, cardinal, etc.

supermultiplicative (respectively, *submultiplicative*) on $\otimes$ if $\sigma(G \otimes H) \geq \sigma(G)\sigma(H)$ (respectively, $\sigma(G \otimes H) \leq \sigma(G)\sigma(H)$) for all pairs G and H. A class of graphs $\mathcal{C}$ is called a *universal multiplicative class for* σ *on* $\otimes$ if for every graph H, it follows that $\sigma(G \otimes H) = \sigma(G)\sigma(H)$ whenever G is from the class $\mathcal{C}$. (We only require the given inequality or equality to hold for those graphs for which the invariant is defined.)

Possibly the most well-known problem in this part of domination theory is the conjecture of V. G. Vizing [19] that dates from 1968.

Vizing's Conjecture *The domination invariant γ is supermultiplicative on the Cartesian product $\Box$.*

We will say that Vizing's conjecture holds for (or is satisfied by) the graph G if $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$ for every H.

In the remainder of the paper we summarize what is known on questions related to whether various domination invariants are sub- or super- multiplicative on the graph products $\Box$ and $\times$. In the process we illustrate the importance of a packing invariant in three of these cases and show similarities between these packing-domination pairs.

2 Domination

2.1 Cartesian Product

For nearly a dozen years no positive results were published on Vizing's conjecture. In 1979 Barcalkin and German [2] showed that the conjecture holds for a large class of graphs that includes some well-known families of graphs.

Theorem 2.1 ([2]) *If G is the spanning subgraph of a graph G' such that $\gamma(G) = \gamma(G')$ and G' has $\gamma(G')$ complete subgraphs that cover G', then Vizing's conjecture holds for G.*

Let us say that a graph G is a BG-graph[2] if it satisfies the hypothesis of Theorem 2.1. Suppose $A = \{v_1, \ldots, v_k\}$ is a $\rho_2(G)$-set and that $k = \gamma(G)$. The set A may not dominate G; in any case, let $B = V(G) - N[A]$. Add edges to G so that each closed neighborhood $N_G[v_i]$ becomes a complete subgraph. If B is not empty, add edges to make $N[v_1] \cup B$ a complete subgraph. Call this graph with added edges G'. It is clear that $\rho_2(G') = k$ and thus G is shown to be a BG-graph. The following result of Meir and Moon [15] then shows that every tree is a BG-graph.

Theorem 2.2 ([15]) *If T is any tree, then $\gamma(T) = \rho_2(T)$.*

[2]It should be noted that Barcalkin and German called this collection of graphs the A-class. BG-graphs is our term to give them credit for this significant result.

The proof given in [15] is by induction on the order of the tree. One can also prove this theorem by an argument similar to that given in [17] for Theorem 3.5 by considering the intersection graph on the closed neighborhoods of T. In [2] it is also shown that graphs with domination number two are BG-graphs. One can readily prove that every cycle is a BG-graph. It is not always an easy matter to determine if a given graph is a BG-graph since there are potentially many ways to add edges to form the required number of complete subgraphs that cover the graph while at the same time not reducing the domination number. An example of a small graph that is not a BG-graph and yet satisfies Vizing's conjecture is the cubic graph F of order eight that is formed from $K_{3,3}$ by "expanding" one vertex to a K_3. For every pair u, v of non-adjacent vertices in F, $\gamma(F + uv) = 2 < 3 = \gamma(F)$, and clearly F is not covered by three complete subgraphs proving that F is not a BG-graph. Part (i) of the next result shows that F satisfies the conjecture.

Theorem 2.3 *Vizing's conjecture holds for any graph G satisfying one of the following conditions:*

(i) ([18]) $\gamma(G) = 3$;

(ii) ([6]) $\gamma(G) = \rho_2(G) + 1$;

(iii) ([1]) *G is chordal.*

Another approach—instead of showing that the conjecture holds for a new class of graphs—was taken by Clark and Suen. They were the first to show that the domination number of the Cartesian product of two graphs is always bounded below by a positive constant multiple of the product of their domination numbers. Their result has also proven to be useful in studying other domination parameters on Cartesian products.

Theorem 2.4 ([5]) *For any graphs G and H, $\gamma(G \,\Box\, H) \geq \frac{1}{2}\gamma(G)\gamma(H)$.*

Attempts to increase the constant in the above have so far proven to be futile. A proof of Vizing's conjecture would, of course, solve the following problem. But, short of that, finding a solution would represent significant progress towards settling the conjecture.

Problem 1 *Find a constant $c > \frac{1}{2}$ such that, for all graphs G and H,*

$$c \cdot \gamma(G)\gamma(H) \leq \gamma(G \,\Box\, H)\,.$$

In contrast to what we will show in the case of total domination of direct products, the following result of Hartnell and Rall implies there is no nontrivial universal multiplicative class for γ on $\Box$.

Theorem 2.5 ([8]) *If T is a tree that has a vertex adjacent to at least two leaves, then $\gamma(G \,\Box\, T) > \gamma(G)\gamma(T)$ for every connected graph G of order at least two.*

Jacobson and Kinch demonstrated the importance of 2-packings in the study of dominating sets in Cartesian products. They used the fact that the closed neighborhood of a vertex (u, v) in $G \,\square\, H$ is contained in $N_G[u] \times V(H)$ and that any dominating set of $G \,\square\, H$ must contain at least $\gamma(H)$ vertices from $N_G[u] \times V(H)$ to prove the following result. Note that the bound might provide little information about the domination number of the Cartesian product (e.g., if both graphs have a small 2-packing number).

Theorem 2.6 ([13]) *For any pair of graphs G and H,*

$$\gamma(G \,\square\, H) \geq \max\{\gamma(G)\rho_2(H), \gamma(H)\rho_2(G)\}\,.$$

We summarize three important connections between packings and domination, at least as they relate to dominating Cartesian products.

Theorem 2.7 *Let G and H be any graphs and let T be any tree. Then*

(i) $\gamma(G) \geq \rho_2(G)$;

(ii) $\gamma(G \,\square\, H) \geq \max\{\gamma(G)\rho_2(H), \gamma(H)\rho_2(G)\}$;

(iii) $\gamma(T) = \rho_2(T)$.

2.2 Direct Product

The domination invariant γ is not supermultiplicitive on direct products. For example, as shown in [16], if G is a complete graph of even order at least six with a perfect matching removed, then $\gamma(G \times G) = 3 < \gamma(G)\gamma(G)$. In [14] Klavžar and Zmazek exhibited an infinite collection of graphs, $\{G_n\}$, such that $\gamma(G_n \times G_n) \leq \frac{7}{9}\gamma(G_n)\gamma(G_n)$. Also, γ is not submultiplicative on $\times$ as illustrated by $\gamma(K_3 \times K_3) = 3 > \gamma(K_3)\gamma(K_3)$. However, using Theorem 3.3 and the easily established fact that $\gamma_t(G) \leq 2\gamma(G)$ it follows immediately that

$$\gamma(G \times H) \leq 4\gamma(G)\gamma(H). \tag{1}$$

It thus seems reasonable to ask for the largest value of the constant c and the smallest value of C such that for every pair of graphs G and H,

$$c \cdot \gamma(G)\gamma(H) \leq \gamma(G \times H) \leq C \cdot \gamma(G)\gamma(H). \tag{2}$$

The above examples show that c can be at most 3/4 while C must be at least 3. At the present time we do not know of a constant $0 < c < 1$ such that the first inequality in (2) holds for all graphs G and H. A recent paper by Brešar, Klavžar and Rall established $C = 3$ as the correct value in the second inequality.

Theorem 2.8 ([4]) *For any graphs G and H, $\gamma(G \times H) \leq 3\gamma(G)\gamma(H)$.*

The authors of [4] show that if G is a graph such that $\gamma(G) > \rho_2(G)$, then for every H, $\gamma(G \times H) < 3\gamma(G)\gamma(H)$. However, choose each of G and H to be a graph having a ρ_2-set that is also a dominating set and then add two pendant vertices to each vertex in each of these γ-sets. Denote the resulting graphs G^+ and H^+, respectively. They prove that $\gamma(G^+ \times H^+) = 3\gamma(G^+)\gamma(H^+)$, thus proving that the bound in Theorem 2.8 is sharp.

Nowakowski and Rall [16] proved that if G and H have no isolated vertices, then

$$\gamma(G \times H) \geq \max\{\gamma_t(H)\rho_2(G), \gamma_t(G)\rho_2(H)\}. \tag{3}$$

For any tree T, it then follows from Theorem 2.2 that

$$\gamma(T \times H) \geq \rho_2(T)\gamma_t(H) = \gamma(T)\gamma_t(H) \geq \gamma(T)\gamma(H)\,.$$

Hence, with appropriate restrictions on one of the factors, c can be chosen to be 1 in (2). In [4] it is shown that if $\gamma(G) = \gamma_t(G)$ and $\gamma(H) = \gamma_t(H)$, then $\gamma(G \times H) \leq \gamma(G)\gamma(H)$. If, in addition, either $\rho_2(G) = \gamma(G)$ or $\rho_2(H) = \gamma(H)$, then by appealing to (3) we see that $\gamma(G \times H) = \gamma(G)\gamma(H)$. That is, if additional conditions are imposed on the factors, then we are able to say more about the constants in (2). For the general case we pose the following question.

Question 1 *What is the largest constant c such that, for all graphs G and H,*

$$c \cdot \gamma(G)\gamma(H) \leq \gamma(G \times H)\,?$$

3 Total Domination

3.1 Cartesian Product

The total domination invariant, γ_t, is neither submultiplicative nor supermultiplicative on Cartesian products as $\gamma_t(K_n \,\Box\, K_n)$ shows for $n \geq 5$ and $n = 3$, respectively. In fact, by letting n be an arbitrary integer larger than five in the above, we see that the ratio of $\gamma_t(G \,\Box\, H)$ to $\gamma_t(G)\gamma_t(H)$ can be made arbitrarily large. That is, there does not exist a constant C such that $\gamma_t(G \,\Box\, H) \leq C \cdot \gamma_t(G)\gamma_t(H)$. In this section we show—in the other direction—that for any pair of graphs the above ratio is at least 1/6 and that if one of the graphs is, for example a tree, then it is at least 1/2.

If D is any $\gamma(G)$-set, then let D' be the set obtained by enlarging D to include a vertex v' from the open neighborhood of each vertex v that is isolated in the subgraph $\langle D \rangle$. It is clear that D' is a total dominating set of G and that $|D'| \leq 2|D|$. Thus, $\gamma_t(G) \leq 2\gamma(G)$, and by combining this with the bound of Clark and Suen from Section 2.1 we get the following chain of inequalities

$$\gamma_t(G)\gamma_t(H) \leq 4\gamma(G)\gamma(H) \leq 8\gamma(G \,\Box\, H) \leq 8\gamma_t(G \,\Box\, H)\,,$$

proving the ratio is always at least 1/8. This was first observed by Henning and Rall in [11] in the only study to date of the multiplicative nature of γ_t on $\Box$. By employing a technical counting argument based around a partition of $V(G)$ that is formed via a minimum paired-dominating set of G, the authors were able to establish the following result that provided the first nontrivial lower bound for $\gamma_t(G \,\square\, H)$ as a multiple of $\gamma_t(G)\gamma_t(H)$.

Theorem 3.1 ([11]) *For any graphs G and H without isolated vertices,*

$$\gamma_t(G)\gamma_t(H) \leq 6\gamma_t(G \,\square\, H).$$

We do not know of any pair of graphs for which the six in the established bound cannot be replaced by two. As is true in many of these situations, if one factor is appropriately restricted then the bound can be improved. We give the proof of this result to illustrate the interplay between several of the domination and packing invariants.

Theorem 3.2 ([11]) *Let G and H be graphs without isolated vertices and assume that $\rho_2(G) = \gamma(G)$. Then*

$$\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G \,\square\, H)\,.$$

Proof. Let D be a minimum total dominating set of the product $G \,\square\, H$, and let $S = \{g_1, \ldots, g_{\gamma(G)}\}$ be a $\rho_2(G)$-set. Choose a partition $\{V_1, \ldots, V_{\gamma(G)}\}$ of $V(G)$ such that for $i = 1, \ldots, \gamma(G)$, $N_G[g_i] \subseteq V_i$. For $i = 1, \ldots, \gamma(G)$, let $D_i = D \cap (V_i \times V(H))$. Further, let S_i be a minimum set of vertices in $G \,\square\, H$ that every vertex of the fiber ${}^{g_i}H$ has a neighbor in S_i and such that S_i contains as many vertices in ${}^{g_i}H$ as possible. Then, $S_i \subseteq V_i \times V(H)$. If S_i contains a vertex x not in ${}^{g_i}H$, then x has a unique neighbor $x' \in {}^{g_i}H$. Replacing x in the set S_i with any neighbor of x' in ${}^{g_i}H$ (notice that such a vertex exists since H has no isolated vertices) produces a new minimum set S_i' of vertices in $G \,\square\, H$ such that ${}^{g_i}H \subseteq N(S_i')$ and S_i' contains more vertices in ${}^{g_i}H$ than does S_i, a contradiction. Hence, $S_i \subseteq H_i$, and so S_i is a total dominating set of the fiber ${}^{g_i}H$. Since ${}^{g_i}H$ is isomorphic to H, it follows that $|S_i| \geq \gamma_t(H)$. But ${}^{g_i}H \subseteq N(D_i)$ as well, and hence $|D_i| \geq |S_i|$. Thus,

$$\gamma_t(G \,\square\, H) = |D| = \sum_{i=1}^{\gamma(G)} |D_i| \geq \sum_{i=1}^{\gamma(G)} |S_i| \geq \sum_{i=1}^{\gamma(G)} \gamma_t(H) = \gamma(G)\gamma_t(H) \geq \frac{1}{2}\gamma_t(G)\gamma_t(H). \qquad (4)$$

$\square$

In particular, by Theorem 2.2 it follows that if T is any tree of order at least two, then

$$\gamma_t(T)\gamma_t(H) \leq 2\gamma_t(T \,\square\, H). \qquad (5)$$

In [11] Henning and Rall showed the bound of Theorem 3.2 is sharp and, in fact, showed that for connected graphs, equality holds in (5) if and only if $\gamma_t(T) = 2\gamma(T)$ and $H = K_2$. This is further evidence for an affirmative answer to the following.

Question 2 *Is $\frac{1}{2}\cdot\gamma_t(G)\gamma_t(H) \leq \gamma_t(G \,\square\, H)$ for all graphs G and H without isolated vertices?*

3.2 Direct Product

In this section we will show that open packings assume a role for total domination in direct products that is strikingly similar to that played by 2-packings relative to ordinary domination in Cartesian products. In particular, we prove a result that corresponds to Theorem 2.7.

If R is a total dominating set of G and S is a total dominating set of H, it follows immediately from the definition of the direct product that every vertex (g, h) of $G \times H$ is adjacent to a vertex in $R \times S$. An immediate consequence of this is that γ_t is submultiplicative on the direct product.

Theorem 3.3 ([16]) *If G and H have no isolated vertices, then $\gamma_t(G \times H) \leq \gamma_t(G)\gamma_t(H)$.*

A lower bound for the total domination number of a direct product can be expressed in terms of the open packing numbers and total domination numbers of the factor graphs in a manner analogous to Theorem 2.6 in the case of Cartesian products.

Theorem 3.4 ([17]) *For any graphs G and H with no isolated vertices,*

$$\gamma_t(G \times H) \geq \max\{\gamma_t(G)\rho^{\mathrm{o}}(H), \gamma_t(H)\rho^{\mathrm{o}}(G)\}.$$

Proof. It suffices to show that $\gamma_t(G \times H) \geq \gamma_t(G)\rho^{\mathrm{o}}(H)$. Fix a vertex h in H, let B be a $\rho^{\mathrm{o}}(H)$-set and let S be a $\gamma_t(G \times H)$-set. For $g \in V(G)$ the open neighborhood of (g, h) is given by $N((g, h)) = N_G(g) \times N_H(h)$. Then, S contains a vertex (x, y) adjacent to (g, h) such that $(x, y) \in \cup_{w \in N(h)} G^w$. Let $S_h = \{\, x \mid (x, y) \in S \cap (\cup_{w \in N(h)} G^w) \,\}$. It follows that S_h is a total dominating set of G. But now

$$S \supseteq \bigcup_{h \in B} [S \cap (\bigcup_{w \in N(h)} G^w)],$$

and thus

$$\gamma_t(G \times H) = |S| \geq \sum_{h \in B} |S \cap (\bigcup_{w \in N(h)} G^w)| \geq \sum_{h \in B} |S_h| \geq \rho^{\mathrm{o}}(H)\gamma_t(G)\,.$$

□

Putting Theorem 3.4 together with the fact that γ_t is submultiplicative on $\times$ we can now show that there is a nontrivial universal multiplicative class $\mathcal{U}$, for γ_t on $\times$. Any graph G such that $\gamma_t(G) = \rho^{\mathrm{o}}(G)$ belongs to $\mathcal{U}$, for then

$$\gamma_t(G)\gamma_t(H) = \rho^{\mathrm{o}}(G)\gamma_t(H) \leq \gamma_t(G \times H) \leq \gamma_t(G)\gamma_t(H)\,.$$

As noted in [17], the complete bipartite graphs, cycles of order divisible by four as well as any graph G constructed by adding at least one endvertex adjacent to each vertex of any connected graph F belong to $\mathcal{U}$. Again, in a result reminiscent of Theorem 2.2 for 2-packings and domination in trees, Rall proved the following result for open packings and total domination in trees. As a consequence, every nontrivial tree belongs to $\mathcal{U}$.

Theorem 3.5 ([17]) *If T is any tree of order at least two, then $\gamma_t(T) = \rho^{\mathrm{o}}(T)$. Thus, for any graph G with no isolated vertices, $\gamma_t(T \times G) = \gamma_t(T)\gamma_t(G)$.*

Combining Proposition 1.1(ii), Theorem 3.4 and Theorem 3.5 we observe that there is an analogous relationship between open packings and total domination in direct products as there is between 2-packings and ordinary domination in Cartesian products. See Theorem 2.7.

Theorem 3.6 *Let G and H be any graphs without isolated vertices,and let T be any tree of order at least two. Then*

(i) $\gamma_t(G) \geq \rho^{\mathrm{o}}(G)$;

(ii) $\gamma_t(G \times H) \geq \max\{\gamma_t(G)\rho^{\mathrm{o}}(H), \gamma_t(H)\rho^{\mathrm{o}}(G)\}$;

(iii) $\gamma_t(T) = \rho^{\mathrm{o}}(T)$.

We close this section with a question about which essentially nothing is known, except that if such a constant exists then it can be at most 3/4.

Question 3 *Is there a positive constant c (and if so, what is the largest) such that*

$$c \cdot \gamma_t(G)\gamma_t(H) \leq \gamma_t(G \times H)$$

hold for all graphs G and H without isolated vertices?

4 Paired-domination

4.1 Cartesian Product

The invariant γ_{pr} is neither supermultiplicative nor submultiplicative on $\square$. For example,

$$\gamma_{\mathrm{pr}}(P_2 \,\square\, P_2) = 2 < \gamma_{\mathrm{pr}}(P_2)\gamma_{\mathrm{pr}}(P_2) \quad \text{and} \quad \gamma_{\mathrm{pr}}(K_{2n} \,\square\, K_{2n}) = 2n > 4 = \gamma_{\mathrm{pr}}(K_{2n})\gamma_{\mathrm{pr}}(K_{2n})\,.$$

The latter example shows that the ratio of $\gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(H)$ to $\gamma_{\mathrm{pr}}(G \,\square\, H)$ can be arbitrarily close to zero. On the other hand, this ratio cannot be larger than 8 as is shown by the following inequality chain,

$$\gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(H) \leq 4\gamma(G)\gamma(H) \leq 8\gamma(G \,\square\, H) \leq 8\gamma_{\mathrm{pr}}(G \,\square\, H). \tag{6}$$

The second inequality above follows from Theorem 2.4 while the first is a consequence of the easily established fact that the paired-domination number of a graph is at most twice its domination number. It is not known whether a smaller constant can replace the 8 in the above inequality. However, Brešar, Henning and Rall showed that 2 will suffice with an appropriate restriction placed on one of the factor graphs. An infinite family of graphs was described to show this bound is sharp.

Theorem 4.1 ([3]) *If $\gamma_{\rm pr}(G) = 2\rho_3(G)$, then for every H with no isolated vertices,*

$$\gamma_{\rm pr}(G)\gamma_{\rm pr}(H) \le 2\gamma_{\rm pr}(G \,\Box\, H)\,.$$

In particular, they showed that the hypothesis holds if G is any nontrivial tree, thus establishing a 3-packing, paired-domination result for trees that is similar to Theorem 2.2 and Theorem 3.5. We omit the proof.

Theorem 4.2 ([3]) *For every nontrivial tree T, $\gamma_{\rm pr}(T) = 2\rho_3(T)$.*

From Inequality (6) and Proposition 1.1 it follows that $\gamma_{\rm pr}(G \,\Box\, H) \ge \frac{1}{4}\rho_3(G)\gamma_{\rm pr}(H)$. A stronger lower bound for the paired-domination number of the Cartesian product in terms of the 3-packing and paired-domination numbers of the factors was established in [3]. Note the similarity of this result to those given earlier in Theorem 2.6 and Theorem 3.4.

Theorem 4.3 ([3]) *For any graphs G and H without isolated vertices,*

$$\gamma_{\rm pr}(G \,\Box\, H) \ge \max\{\gamma_{\rm pr}(G)\rho_3(H), \gamma_{\rm pr}(H)\rho_3(G)\}.$$

Proof. Let $S = \{v_1, \ldots, v_{\rho_3(G)}\}$ be a $\rho_3(G)$-set. For $i = 1, \ldots, \rho_3(G)$, let $V_i = N_G[v_i]$, and so the sets V_i are pairwise disjoint. Let D be a $\gamma_{\rm pr}(G \,\Box\, H)$-set. Fix a perfect matching M of $\langle D\rangle_{G \,\Box\, H}$. For $u \in D$ the unique vertex $v \in D$ such that $uv \in M$ is called the partner of u with respect to M. For $i = 1, \ldots, \rho_3(G)$, let D_i consist of all vertices of D in $V_i \times V(H)$ together with their partners (some of which may possibly not belong to $V_i \times V(H)$), and let $H_i = V({}^{v_i}H) = \{v_i\} \times V(H)$. Since S is a 3-packing in G, and since every vertex of a paired-dominating set has a unique partner, we note that the sets D_i are pairwise disjoint. Hence

$$|D| \ge \sum_{i=1}^{\rho_3(G)} |D_i|\,.$$

For $i = 1, \ldots, \rho_3(G)$, let S_i be a minimum set of vertices in $G \,\Box\, H$ such that (i) S_i dominates H_i and (ii) the subgraph induced by S_i in $G \,\Box\, H$ contains a perfect matching M_i. Among all such sets S_i, we choose S_i to contain as many vertices in H_i as possible. Since the set D_i satisfies conditions (i) and (ii), we have that

$$|D_i| \ge |S_i|\,.$$

If S_i contains a vertex x not in $V_i \times V(H)$, then x is the partner of a vertex x' (with respect to the perfect matching M_i) not in H_i that uniquely dominates a vertex $y \in H_i$. Replacing x in the set S_i with the vertex y produces a new set of vertices in $G \,\Box\, H$ that satisfies conditions (i) and (ii) but that contains more vertices in H_i than does S_i, a contradiction. Hence, $S_i \subseteq V_i \times V(H)$.

We show next that $S_i \subseteq H_i$. Suppose S_i contains a vertex u not in H_i. Let u' be the partner of u (with respect to M_i). If $u' \notin H_i$, then let w (respectively, w') be the neighbor of u (respectively, u') in $G \,\Box\, H$ that belongs to H_i. We distinguish two cases with respect to w and w' being distinct.

Case 1. $w \neq w'$. Then neither w nor w' is in S_i (or else neither u nor u' would have been needed in S_i) and at least one of w and w' is not dominated by $S_i - \{u, u'\}$. So we can replace the pair u and u' in the set S_i with the pair w and w' to produce a new set of vertices in $G \,\Box\, H$ that satisfies conditions (i) and (ii) but that contains more vertices in H_i than does S_i, contradicting our choice of S_i.

Case 2. $w = w'$. Then no vertex from $N[w] \cap H_i$ belongs to S_i (or again S_i would not have contained either u or u'). So by replacing u' by w in S_i we produce a new set that satisfies conditions (i) and (ii) but that contains more vertices in H_i than does S_i, a contradiction.

Both cases imply that $u' \in H_i$. If S_i contains every neighbor of u' in H_i, then we can simply delete u and u' from the set S_i to contradict our choice of S_i. Hence at least one neighbor u'' of u' in H_i does not belong to S_i. Replacing u in the set S_i with u'' produces a new set of vertices in $G \,\Box\, H$ that satisfies conditions (i) and (ii) (with u' paired with u'' in the new set) but that contains more vertices in H_i than does S_i, a contradiction. Hence, $S_i \subseteq H_i$, and so S_i is a paired-dominating set of the subgraph ${}^{v_i}H$. Since ${}^{v_i}H$ is isomorphic to H it follows that

$$|S_i| \geq \gamma_{\rm pr}(H)\,.$$

Thus,

$$\gamma_{\rm pr}(G \,\Box\, H) = |D| \geq \sum_{i=1}^{\rho_3(G)} |D_i| \geq \sum_{i=1}^{\rho_3(G)} |S_i| \geq \sum_{i=1}^{\rho_3(G)} \gamma_{\rm pr}(H) = \rho_3(G)\gamma_{\rm pr}(H)\,,$$

which establishes the desired bound. □

Again, as in the case of domination in Cartesian products (Theorem 2.7) and in total domination of direct products (Theorem 3.6) we summarize the relationships involving paired-domination and 3-packings. This result follows from Proposition 1.1(iii), Theorem 4.3 and Theorem 4.2.

Theorem 4.4 *Let G and H be any graphs and let T be any tree. Then*

(i) $\gamma_{\rm pr}(G) \geq 2\rho_3(G)$;

(ii) $\gamma_{\rm pr}(G \,\Box\, H) \geq \max\{\gamma_{\rm pr}(G)\rho_3(H), \gamma_{\rm pr}(H)\rho_3(G)\}$;

(iii) $\gamma_{\rm pr}(T) = 2\rho_3(T)$.

Rewriting inequality (6) we see that for every pair of graphs

$$\frac{1}{8} \cdot \gamma_{\rm pr}(G)\gamma_{\rm pr}(H) \leq \gamma_{\rm pr}(G \,\Box\, H)\,.$$

We close with the following question related to improving this bound.

Question 4 *What is the largest constant $c \geq \frac{1}{8}$ such that, for all graphs G and H,*

$$c \cdot \gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(H) \leq \gamma_{\mathrm{pr}}(G \,\Box\, H)\,?$$

4.2 Direct Product

If A and B are paired-dominating sets of G and H respectively, then clearly $A \times B$ dominates $G \times H$ and one easily shows that $\langle A \times B \rangle$ contains a perfect matching. Thus, γ_{pr} is submultiplicative on $\times$. That is, for every pair of graphs G and H with no isolates, $\gamma_{\mathrm{pr}}(G \times H) \leq \gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(H)$. In [4] it is shown that there are infinite families for which equality is actually achieved, although no universal multiplicative class is shown to exist. On the other hand, if S_n denotes the subdivided star with n pendant vertices, then it is shown that

$$\gamma_{\mathrm{pr}}(S_n \times S_n) \leq 2n^2 + 2n < 4n^2 = \gamma_{\mathrm{pr}}(S_n)\gamma_{\mathrm{pr}}(S_n)\,.$$

Question 5 *Is there a positive constant c (and if so, what is the largest) such that*

$$c \cdot \gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(H) \leq \gamma_{\mathrm{pr}}(G \times H)$$

hold for all graphs G and H without isolated vertices?

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