On graphs having one size of maximal open packings

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Abstract

A set P of vertices in a graph G is an open packing if no two distinct vertices in P have a common neighbor. Among all maximal open packings in G, the smallest cardinality is denoted $\rho_L^{\rm o}(G)$ and the largest cardinality is $\rho^{\rm o}(G)$. There exist graphs for which these two invariants are arbitrarily far apart. In this paper we begin the investigation of the class of graphs that have one size of maximal open packings. By presenting a method of constructing such graphs we show that every graph is the induced subgraph of a graph in this class. The main result of the paper is a structural characterization of those G that do not have a cycle of order less than 15 and for which $\rho_L^{\rm o}(G) = \rho^{\rm o}(G)$.

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1 Introduction

A subset P of the vertex set of a graph G is an *open packing* in G if the open neighborhoods of vertices in P are pairwise disjoint. That is, no pair of distinct vertices in P have a common neighbor. By a *maximal open packing* we mean an open packing that is maximal with respect to set containment. The cardinality of a largest open packing in G is called the *open packing number* of G and is denoted by $\rho^{\rm o}(G)$. The *lower open packing number* of G, denoted $\rho_L^{\rm o}(G)$, is the minimum cardinality of a maximal open packing in G. To see that these two numbers can differ by an arbitrary amount consider the tree T_n of order $4n+2$ obtained from the disjoint union of two stars $K_{1,n}$ by subdividing each edge once and then adding an edge to make their centers adjacent. For this tree, $\rho_L^{\rm o}(T_n) = 2$ while $\rho^{\rm o}(T_n) = 2n+2$.

The study of maximal open packings in graphs was initiated by Henning and Slater [7]. They determined the lower and upper open packing numbers for paths and cycles, and in a series of results they established bounds for $\rho^{\rm o}(G)$ and $\rho_L^{\rm o}(G)$.

In particular, they proved that for a connected graph G of order n with maximum degree Δ and minimum degree δ, we have $\frac{n}{\Delta(\Delta-1)+1} \le \rho_L^{\rm o}(G) \le \rho^{\rm o}(G) \le \frac{n}{\delta}$. Brešar, Kuenzel, and Rall [1] investigated graphs with a unique maximum open packing and showed that recognition of this class of graphs is polynomially equivalent to the recognition of the graphs with a unique maximum independent set. Furthermore, they gave a structural characterization of the class of trees T with a unique open packing of cardinality $\rho^{\rm o}(T)$. A number of other researchers have studied open packings. For example, see the following [2–6, 9, 10].

Open packings are related to total domination since every open neighborhood has a nonempty intersection with any total dominating set. (A set S is a total dominating set if every vertex in G is adjacent to at least one vertex in S.) This implies that the size of a smallest total dominating set in any graph is at least as large as its open packing number. In the class of trees these invariants are equal. In 2005, Rall [12] proved that if T is a nontrivial tree, then the cardinality of a smallest total dominating set is $\rho^{\rm o}(T)$. This result is "parallel" to the theorem of Meir and Moon [8] that showed the domination number of a tree is equal to the cardinality of a largest (closed) packing, which is a set of vertices whose closed neighborhoods are pairwise disjoint.

Henning and Slater [7] also considered the complexity of computing open packings. They proved that the following decision problem is NP-complete even when restricted to bipartite or to chordal graphs.

Open Packing	
Instance:	A graph G and a positive integer k.
Question:	Does G have an open packing of cardinality k?

Hamid and Saravanakumar [2] posed as an open problem the characterization of those graphs G such that $\rho_L^{\rm o}(G) = \rho^{\rm o}(G)$. The main goal of this paper is to give a structural characterization of the subclass of this class of graphs that have no cycles of length less than 15. For convenience we denote by $\mathcal{U}$ the set of all graphs G such that $\rho_L^{\rm o}(G) = \rho^{\rm o}(G)$. Equivalently, $G \in \mathcal{U}$ if and only if every maximal open packing in G has cardinality $\rho^{\rm o}(G)$. Note that Open Packing is solvable in linear time for the class $\mathcal{U}$ since a greedy algorithm will always produce an open packing of cardinality $\rho^{\rm o}(G)$ for every $G \in \mathcal{U}$.

Let $\mathcal{F}$ be the family of all finite simple graphs, G, such that there is a weak partition $L, S_1, S_2, D_{11}, D_{12}, D_2$ of $V(G)$ that satisfies the following properties.

1. L is the set of leaves in G, and $S_1 \cup S_2$ is the set of support vertices in G.

2. The sets S_1 and D_{11} are independent, and S_2 induces a matching in G.

3. $D_2 = \{x : d_G(x, S_1 \cup S_2) = 2\}$.

4. No vertex in $S_1 \cup D_{11}$ is adjacent to a vertex in $S_2 \cup D_{12}$.

5. Each vertex in D_{11} is adjacent to exactly one vertex in S_1, each vertex of D_{12} is adjacent to exactly one vertex in S_2, and each vertex in D_2 is adjacent to exactly one vertex in D_{11}.

Our main result is the following theorem which gives the characterization described above.

Theorem 1. *If G is a nontrivial graph having girth at least 15, then $G \in \mathcal{U}$ if and only if $G \in \mathcal{F}$.*

The remainder of the paper is organized as follows. In the next section we give the necessary definitions and notation used throughout the remainder of the paper. In Section 3, we present a construction to prove that every graph is an induced subgraph of some graph in $\mathcal{U}$. In addition, we establish a useful connection between $\mathcal{U}$ and the class of well-covered graphs. Section 4 is devoted to establishing some necessary conditions for any graph in $\mathcal{U}$ that has girth at least 15. The structural characterization (Theorem 1) is proved in Section 5, and we conclude with some open problems in Section 6.

2 Definitions and Notation

Let G be a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v)$ of all vertices in G that are adjacent to v. For a subset S of $V(G)$ the open neighborhood of S is denoted by $N_G(S)$ and is defined by $N_G(S) = \cup_{x \in S} N_G(x)$. The *closed neighborhood* of v is the set $N_G[v]$ defined by $N_G[v] = N_G(v) \cup \{v\}$. Whenever the graph G is understood from the context, it will be removed from the subscript. For $A \subseteq V(G)$, the subgraph of G induced by A will be denoted by $G[A]$. The *independence number* of G is the maximum cardinality, $\alpha(G)$, of a set of vertices that are pairwise non-adjacent, and the *independent domination number* of G is the smallest cardinality $i(G)$ of such an independent set that is maximal in the subset inclusion relation. A graph is *well-covered* (see Plummer [11]) if all of its maximal independent sets have the same cardinality. That is, a graph G is well-covered if $i(G) = \alpha(G)$.

A vertex x of G is a *leaf* if $\deg(x) = 1$, and a vertex is called a *support vertex* if it is adjacent to at least one leaf. A support vertex that has more than one leaf as a neighbor is called a *strong support* vertex. We denote the set of all leaves (support vertices) of G by L_G (respectively S_G). The set of leaves of G adjacent to a support vertex s is denoted $L_G(s)$. If u and v are vertices in G, then the distance between u and v is denoted $d_G(u, v)$ and is the length of a shortest uv-path in G. For $A \subseteq V(G)$ and a vertex u, we let $d_G(u, A)$ denote $\min\{d_G(u, v) : v \in A\}$. The *girth* of G, denoted $g(G)$, is the length of a shortest cycle in G. If G is acyclic, then we write $g(G) = \infty$. For a positive integer n, we let $[n]$ be the set of integers $\{1, 2, \ldots, n\}$. A *weak partition* of a nonempty set X is a collection of pairwise disjoint subsets of X whose union is X. Note that some of the subsets in a weak partition might be empty.

Suppose G is a graph with a support vertex s. Fix a leaf $x \in L_G(s)$. Let G' be the graph with $V(G') = V(G) \cup \{w\}$ and $E(G') = E(G) \cup \{sw\}$, where w is a new vertex. If A is a maximal open packing of G, then it follows that A is a maximal open packing of G'. Furthermore, if there exists $z \in A \cap L_G(s)$, then $(A - \{z\}) \cup \{w\}$ is a maximal open packing of G'. If B is a maximal open packing of G' and $w \notin B$, then B is a maximal open packing of G. On the other hand, if $w \in B$, then $(B - \{w\}) \cup \{x\}$ is a maximal open packing of G. This gives the following observation.

Observation 1. *If G is a graph with at least one support vertex and $G \in \mathcal{U}$, then any supergraph of G that is obtained from G by adding a new leaf adjacent to any support vertex of G is also in $\mathcal{U}$. If G is a graph with a strong support vertex s and $G \in \mathcal{U}$, then $G - w \in \mathcal{U}$ where $w \in L_G(s)$.*

3 Preliminary Results

For a given graph G, the *open neighborhood graph of G* is the graph, $N_\text{o}(G)$, whose vertex set is $V(G)$ such that distinct vertices u and v are adjacent in $N_\text{o}(G)$ if and only if $N_G(u) \cap N_G(v) \neq \emptyset$. It is clear that a subset $A \subseteq V(G)$ is a (maximal) open packing in G if and only if A is a (maximal) independent set in the graph $N_\text{o}(G)$. Consequently, $i(N_\text{o}(G)) = \rho_L^\text{o}(G) \leq \rho^\text{o}(G) = \alpha(N_\text{o}(G))$. Thus we have the following connection between well-covered graphs and the class $\mathcal{U}$.

Proposition 2. *A graph G is in $\mathcal{U}$ if and only if $N_\text{o}(G)$ is well-covered.*

A straightforward analysis shows that for $n \geq 3$ we have $N_\text{o}(C_{2n}) = 2C_n$ and $N_\text{o}(C_{2n-1}) = C_{2n-1}$, while $N_\text{o}(C_3) = C_3$ and $N_\text{o}(C_4) = 2P_2$. In addition, $N_\text{o}(P_{2n}) = 2P_n$ and $N_\text{o}(P_{2n+1}) = P_n \cup P_{n+1}$ for every positive integer n, while $N_\text{o}(P_1) = P_1$. The next result then follows by using Proposition 2 and what is known about well-covered cycles and paths.

Proposition 3. *If n is a positive integer, then*

(i) $P_n \in \mathcal{U}$ *if and only if* $n \in \{1, 2, 3, 4, 8\}$.

(ii) $C_n \in \mathcal{U}$ *if and only if* $n \in \{3, 4, 5, 6, 7, 8, 10, 14\}$.

The next proposition shows that, regardless of girth, there does not exist a forbidden subgraph characterization for the graphs in $\mathcal{U}$.

Proposition 4. *If H is any graph, then there exists a graph $G \in \mathcal{U}$ such that H is an induced subgraph of G.*

Proof. Suppose H has order n with vertex set $\{h_1, h_2, \ldots, h_n\}$. For each $i \in [n]$, let $a_ib_ic_i$ be a path of order 3. A graph G of order $4n$ is now constructed from the disjoint union of H and the n disjoint paths of order 3 by adding the set of edges, $\{h_ia_i : i \in [n]\}$. Suppose P is any maximal open packing in G. It is clear that $1 \le |P \cap \{h_i, a_i, b_i, c_i\}| \le 2$, for every $i \in [n]$. Suppose there exists $k \in [n]$ such that $|P \cap \{h_k, a_k, b_k, c_k\}| = 1$, say $P \cap \{h_k, a_k, b_k, c_k\} = \{x\}$. Each of the four possibilities for x leads to the contradiction that P is not maximal. If $x \in \{h_k, b_k\}$, then c_k can be added to P. On the other hand, if $x \in \{a_k, c_k\}$, then b_k can be added to P. This implies that $|P| = 2n$ and therefore $G \in \mathcal{U}$. □

4 Necessary Conditions

In this section we will derive a list of necessary conditions that are true about any graph $G \in \mathcal{U}$ that has girth at least 15. In Section 5 we use these conditions to prove the main characterization theorem. Note that some of these do not require such a restriction on the girth.

Lemma 5. *Let G be a triangle-free graph. If $G \in \mathcal{U}$, then no vertex of G is adjacent to more than one support vertex.*

Proof. Suppose that x is a vertex in a graph G such that x is adjacent to at least two support vertices, say s_1 and s_2. Let y_1 and y_2 be vertices of degree 1 adjacent to s_1 and s_2 respectively. Extend $\{x, s_1\}$ to a maximal open packing P. Note that $P \cap (N(\{x, s_1, s_2\}) - \{x, s_1\}) = \emptyset$. Since $s_1s_2 \notin E(G)$ (G is triangle-free), it follows that $(P - \{x\}) \cup \{y_1, y_2\}$ is an open packing, which implies that

$$\rho_L^{\mathrm{o}}(G) \le |P| < |(P - \{x\}) \cup \{y_1, y_2\}| \le \rho^{\mathrm{o}}(G)\,,$$

and therefore $G \notin \mathcal{U}$. □

Lemma 6. *Let G be a graph with girth at least 15. If $\delta(G) \ge 2$, then $G \notin \mathcal{U}$.*

Proof. Suppose G is a graph with girth at least 15 and $\delta(G) \ge 2$. If also $\Delta(G) \le 2$, then $G \notin \mathcal{U}$ by Proposition 3. Thus, we assume that $\Delta(G) \ge 3$. Let x be a vertex of degree at least 3; suppose $N(x) = \{y_1, y_2, \ldots, y_k\}$ for some $k \ge 3$. Consider a breadth-first search spanning tree T of G rooted at x. For each $i \in [7]$, let $L_i = \{u \in V(G) : d_G(x, u) = i\}$. Since G has girth at least 15, we see that L_k is an independent set for $1 \le k \le 6$. We now define a subset $A \subseteq V(G)$, beginning with $A = \emptyset$. There are a number of situations to consider, and we will add vertices to A as follows.

- For each $u \in L_3$ such that $d_G(y_1, u) = 2$, choose vertices $u_4 \in N(u) \cap L_4$ and $u_5 \in N(u_4) \cap L_5$. Add u_4 and u_5 to A.

- Select a single vertex $w \in N(y_2) \cap L_2$. For each vertex $p \in L_4$ such that $d_G(p, w) = 2$, choose vertices $p_5 \in N(p) \cap L_5$ and $p_6 \in N(p_5) \cap L_6$. Add p_5 and p_6 to A.

- For each $s \in (N(y_2) \cap L_2) - \{w\}$ and for each $s' \in N(s) \cap L_3$, select $s'_4 \in N(s') \cap L_4$ and $s'_5 \in N(s'_4) \cap L_5$. Add s'_4 and s'_5 to A.

- Select a single vertex $z \in N(y_3) \cap L_2$. For each vertex $q \in L_4$ such that $d_G(q, z) = 2$, choose vertices $q_5 \in N(q) \cap L_5$ and $q_6 \in N(q_5) \cap L_6$. Add q_5 and q_6 to A.

- For each $t \in (N(y_3) \cap L_2) - \{z\}$ and for each $t' \in N(t) \cap L_3$, select $t'_4 \in N(t') \cap L_4$ and $t'_5 \in N(t'_4) \cap L_5$. Add t'_4 and t'_5 to A.

- For each $j \in [k] - \{1, 2, 3\}$ and for each vertex $r \in N(y_j) \cap L_2$ select one vertex $r_3 \in N(r) \cap L_3$ and one vertex $r_4 \in N(r_3) \cap L_4$. Add r_3 and r_4 to A.

Note that the set A constructed above is an open packing in G. Let H be the subgraph of G induced by $S = \{x, y_1, y_2, y_3, w, z\}$. Extend A to a maximal open packing, B, of the induced subgraph $G - S$ of G. By the choice of the vertices placed into A, we see that if $g \in V(G - S)$ and g is within distance 2 of a vertex of S, then $g \notin B$. On the other hand, if a maximal open packing of H is added to B, then the resulting set is a maximal open packing of G. This implies that both $B \cup \{x, y_2\}$ and $B \cup \{y_1, w, z\}$ are maximal open packings of G. Therefore, $G \notin \mathcal{U}$. □

Lemma 7. *Let G be a graph with $\delta(G) = 1$ and with girth at least 11. If $G \in \mathcal{U}$, then every vertex of G is within distance 2 of a support vertex.*

Proof. Suppose, for the sake of contradiction, that there exists $G \in \mathcal{U}$ such that $\delta(G) = 1$ and $g(G) \geq 11$, but for some vertex w of G, the distance from w to the nearest support vertex is at least 3. Let s be a support vertex of G that is closest to w and let r be a vertex of degree 1 adjacent to s. Suppose v is a vertex on a shortest w,s-path such that $d_G(v, s) = 3$. Let $vxzs$ be a shortest v,s-path. For each $i \in [5]$, let $L_i = \{u \in V(G) : d_G(v, u) = i\}$. Since $g(G) \geq 11$, the set L_i is independent for each $i \in [4]$ and no vertex in L_5 has more than one neighbor in L_4. In addition, since no support vertex of G is within distance less than 3 of v, it follows that $N(u) \cap L_4 \neq \emptyset$, for each $u \in L_3$. Let $N(v) - \{x\} = \{x_1, \ldots, x_k\}$. Let $j \in [k]$. For each $a \in N(x_j) - \{v\}$, choose $a' \in L_3 \cap N(a)$ and $a'' \in L_4 \cap N(a')$. Let

$$D = \bigcup_{j=1}^{k} \left(\bigcup \{a', a'' : a \in N(x_j) - \{v\}\} \right).$$

Since $g(G) \geq 11$, it follows that $D \cup \{z, s\}$ is an open packing of G that can be extended to a maximal open packing, P, of G. Note that $P \cap [\{x_1, \ldots, x_k\}] = \emptyset$. However, now $(P - \{z\}) \cup \{r, v\}$ is a larger open packing of G, which is a contradiction. □

Lemma 8. *Let G be a graph with girth at least 7. If there exists a path $s_1u_1vu_2s_2$ in G such that s_1 and s_2 are support vertices and such that for each $i \in [2]$, the vertex u_i is the only neighbor of s_i that has degree at least 2, then $G \notin \mathcal{U}$.*

Proof. For $i \in [2]$, let x_i be a vertex of degree 1 such that $x_i \in N(s_i)$. In addition, define the following sets of vertices.

- $A = N(u_1) - \{v, s_1\}$, $B = N(v) - \{u_1, u_2\}$, $C = N(u_2) - \{v, s_2\}$,
- $A' = N(A) - \{u_1\}$, $B' = N(B) - \{v\}$, $C' = N(C) - \{u_2\}$.

Note that the sets A, B, C, A', B', C' are pairwise disjoint since $g(G) \geq 7$. Extend the open packing $\{x_1, v, u_2\}$ to a maximal open packing, P, of G. It follows that $P \cap (A \cup \{s_1, u_1\}) = \emptyset$, $P \cap (B \cup C \cup B' \cup C') = \emptyset$, and $P \cap (N[s_2] - \{u_2\}) = \emptyset$. The set $Q = (P - \{v, u_2\}) \cup \{s_1, s_2, x_2\}$ is an open packing, and $|Q| > |P|$. Therefore, $G \notin \mathcal{U}$. □

By Lemma 5 it follows that for a triangle-free graph G in $\mathcal{U}$, if G has vertices of degree 1, then the subgraph of G induced by S_G is a disjoint union of isolated vertices and edges. If s is such an isolated vertex in $G[S_G]$, then s is called a *single star support* vertex. If uv is an edge in $G[S_G]$, then u and v are called *double star supports.*

Lemma 9. *Let G be a connected graph such that $\delta(G) = 1$ and $g(G) \geq 15$. If $G \in \mathcal{U}$ and s is a single star support in G, then s has at most one neighbor that does not belong to L_G.*

Proof. Note that if u is a single star support vertex and $v \in S_G$ such that $u \neq v$, then $d_G(u, v) \geq 3$ since by Lemma 5 no vertex of G is adjacent to two support vertices. We will prove the lemma by establishing a sequence of claims.

Claim 1. *If s_1 and s_2 are both single star support vertices, then $d_G(s_1, s_2) \neq 3$.*

Proof. Suppose for the sake of contradiction that there exists a path s_1abs_2 in G. Let k_i be a leaf adjacent to s_i, for $i \in [2]$. For $i \in [4]$, let

$$L_i(s_1) = \{u : d_G(s_1, u) = i \text{ such that no shortest } us_1\text{-path contains } a\} - L_G\,,$$

and

$$L_i(s_2) = \{u : d_G(s_2, u) = i \text{ such that no shortest } us_2\text{-path contains } b\} - L_G\,.$$

For each $x \in L_1(s_1)$ and for each $x_2 \in N(x) \cap L_2(s_1)$ choose a vertex $x_3 \in N(x_2) \cap L_3(s_1)$ and a vertex $x_4 \in N(x_3) \cap L_4(s_1)$. Similarly, for each $x \in L_1(s_2)$ and for each $x_2 \in N(x) \cap L_2(s_2)$ choose a vertex $x_3 \in N(x_2) \cap L_3(s_2)$ and a vertex $x_4 \in N(x_3) \cap L_4(s_2)$. Let

$$P = \{x_3 : x \in L_1(s_1) \cup L_1(s_2)\} \cup \{x_4 : x \in L_1(s_1) \cup L_1(s_2)\} \cup \{a, b\}.$$

Since the girth of G is more than 13, the set P is an open packing. Extend P to a maximal open packing Q of G. Since $(Q - \{a, b\}) \cup \{s_1, k_1, s_2, k_2\}$ is also an open packing, we have reached a contradiction. This proves Claim 1.

Claim 2. *If s_1 is a single star support vertex, then there does not exist a double star support whose distance to s_1 is exactly* 3.

Proof. Suppose the claim is false. Let s_1 be a single star support and let s_2 and s_3 be adjacent double star supports such that $s_1abs_2s_3$ is a path in G. For each $i \in [3]$, let $k_i \in L_G \cap N(s_i)$. For each $i \in [4]$, let $L_i(s_1)$ be defined as in the proof of Claim 1 and let

$$L_i(s_2) = \{u : d_G(s_2, u) = i \text{ such that no shortest } us_2\text{-path contains } b \text{ or } s_3\} - L_G.$$

For each $x \in L_1(s_1) \cup L_1(s_2)$ choose x_3 and x_4 as in the proof of Claim 1. Let

$$P = \{x_3 : x \in L_1(s_1) \cup L_1(s_2)\} \cup \{x_4 : x \in L_1(s_1) \cup L_1(s_2)\} \cup \{a, b, k_3\}.$$

By the girth assumption on G, it follows that P is an open packing in G. Extend P to a maximal open packing Q of G. Since $(Q - \{a, b\}) \cup \{s_1, k_1, k_2\}$ is also an open packing, we have reached a contradiction. This proves Claim 2.

Claim 3. *No pair of single star support vertices are at distance exactly* 4.

Proof. Suppose the claim is not true. Let s_1 and s_2 be single star support vertices such that $d_G(s_1, s_2) = 4$. Let s_1abcs_2 be a path in G and let $k_i \in L_G \cap N(s_i)$ for $i \in [2]$. In a manner similar to the proofs of the above claims, for each $i \in [4]$, we let

$$L_i(s_1) = \{u : d_G(s_1, u) = i \text{ such that no shortest } us_1\text{-path contains } a\} - L_G,$$

and

$$L_i(s_2) = \{u : d_G(s_2, u) = i \text{ such that no shortest } us_2\text{-path contains } c\} - L_G.$$

Also, $P = \{x_3 : x \in L_1(s_1) \cup L_1(s_2)\} \cup \{x_4 : x \in L_1(s_1) \cup L_1(s_2)\} \cup \{a, b, k_2\}$. Since $g(G) \geq 15$, the set P is an open packing in G, and we extend it to a maximal open packing, Q, in G. However, $(Q - \{a, b\}) \cup \{k_1, s_1, s_2\}$ is a larger open packing. This contradiction proves the claim.

Now, let s be a single star support vertex that has distinct neighbors b and c that both have degree larger than 1. Since s is a single star support vertex, it follows by definition that neither b nor c is a support vertex. Fix $a \in N(b) - \{s\}$ and $d \in N(c) - \{s\}$. By Lemma 5, neither a nor d is a support vertex. For each $i \in [4]$, let

$$L_i(a) = \{u : d_G(a, u) = i \text{ such that no shortest } ua\text{-path contains } b\},$$

and

$$L_i(d) = \{u : d_G(d, u) = i \text{ such that no shortest } ud\text{-path contains } c\}.$$

By Claim 1 and Claim 2, $L_1(a) \cup L_1(d)$ does not contain a support vertex, which implies that $L_2(a) \cup L_2(d)$ does not contain any leaves. In addition, Claim 3 implies that $L_2(a) \cup L_2(d)$ does not contain any single star support vertices. Taken together this means that each vertex in $L_2(a)$ has a neighbor in $L_3(a)$ which in turn is adjacent to a vertex in $L_4(a)$. A similar conclusion holds for each vertex in $L_2(d)$. For each $x \in L_1(a)$ and for each $x_2 \in N(x) \cap L_2(a)$ choose a vertex $x_3 \in N(x_2) \cap L_3(a)$ that is adjacent to a vertex $x_4 \in L_4(a)$. In an analogous way for each $x \in L_1(d)$ and for each $x_2 \in N(x) \cap L_2(d)$ choose a vertex $x_3 \in N(x_2) \cap L_3(d)$ that is adjacent to a vertex $x_4 \in L_4(d)$. By the choice of these vertices and because the girth of G is at least 15, we see that

$$P = \{x_3 : x \in L_1(a) \cup L_1(d)\} \cup \{x_4 : x \in L_1(a) \cup L_1(d)\} \cup \{b, s\}$$

is an open packing in G. We extend P to a maximal open packing, Q, in G. However, $(Q - \{s\}) \cup \{a, d\}$ is a larger open packing. This final contradiction proves the lemma. □

Lemma 10. *Let G be a connected graph such that $\delta(G) = 1$ and $g(G) \geq 15$. If $G \in \mathcal{U}$, then every vertex at distance 2 from S_G has exactly one single star support vertex at distance exactly 2.*

Proof. Suppose G is a graph of girth at least 15 such that $G \in \mathcal{U}$ and $\delta(G) = 1$. Let v be a vertex such that $d_G(v, S_G) = 2$. By Claim 3 in the proof of Lemma 9, there cannot be more than one single star support vertex at distance exactly 2 from v. Suppose there are none. This means that there exist double star support vertices, say s_1 and s_2, and a shortest path $vas_2s_1k_1$, where $k_1 \in L_G \cap N(s_1)$. For each $i \in [4]$, let

$$L_i(v) = \{u : d_G(v, u) = i \text{ such that no shortest } uv\text{-path contains } a\}.$$

Let $x \in L_1(v)$ and let $x_2 \in N(x) \cap L_2(v)$. Since $x \notin S_G$, we get $\deg(x_2) \geq 2$. By our assumption that no single star support vertex has distance exactly 2 from v, we infer that there exist $x_3 \in N(x_2) \cap L_3(v)$ and $x_4 \in N(x_3) \cap L_4(v)$. Let

$$P = \{x_3 : x \in L_1(v)\} \cup \{x_4 : x \in L_1(v)\} \cup \{s_1, s_2\}.$$

Since $g(G) \geq 15$, the set P is an open packing. Extend P to a maximal open packing, Q, of G. This leads to a contradiction since $(Q - \{s_2\}) \cup \{v, k_1\}$ is an open packing of cardinality larger than $|Q|$. □

5 Proof of Theorem 1

In this section we prove Theorem 1. For this purpose we introduce the following notation. The set of single star support vertices in G will be denoted by $S_1(G)$, and $S_2(G)$ will denote the set of double star support vertices in G. By Lemma 7, every v in a graph of girth at least 15 that belongs to $\mathcal{U}$ is within distance 2 of a support vertex. Hence, we define $D_1(G) = \{x : d_G(x, S_G) = 1\} - L_G$ and $D_2(G) = \{x : d_G(x, S_G) = 2\}$. For simplification when the graph is clear from the context we simply write S_1 instead of $S_1(G)$, and so on. By Lemma 5, the set D_1 further partitions into D_{11} and D_{12} defined by $D_{11} = \{x \in D_1 : N(x) \cap S_1 \neq \emptyset\}$ and $D_{12} = \{x \in D_1 : N(x) \cap S_2 \neq \emptyset\}$.

We restate Theorem 1 for convenience.

Theorem 1 *If G is a nontrivial graph having girth at least 15, then $G \in \mathcal{U}$ if and only if $G \in \mathcal{F}$.*

Proof. Suppose G is in $\mathcal{U}$ and G has girth at least 15. It follows from Lemma 6 that $\delta(G) = 1$. Applying Lemmas 5, 7, 8, 9, and 10 we see that $G \in \mathcal{F}$.

Now suppose G belongs to the family $\mathcal{F}$ and let $L, S_1, S_2, D_{11}, D_{12}, D_2$ be a weak partition of $V(G)$ that satisfies conditions 1-5 in the definition of $\mathcal{F}$. Let P be any maximal open packing in G. We claim that $|P| = 2|S_1| + |S_2|$. Let $S_1 = \{a_1, \ldots, a_n\}$ and let $S_2 = \{b_1, \ldots, b_m, c_1, \ldots, c_m\}$ where $b_ic_i \in E(G)$ for each $i \in [m]$. For $i \in [n]$, let u_i be a leaf adjacent to a_i, let $X_i = \{x \in V(G) : d_G(x, a_i) \leq 2\}$, and let $A_i = X_i \cap D_2$. For $j \in [m]$, let $v_j \in L \cap N(b_j)$, let $w_j \in L \cap N(c_j)$, let $Y_j = \{x \in V(G) : d_G(x, \{b_j, c_j\}) \leq 1\}$, and let $B_j = Y_j \cap D_{12}$. It is clear that $|P \cap X_i| \leq 2$ and that $|P \cap Y_j| \leq 2$ for each $i \in [n]$ and each $j \in [m]$.

Suppose first that there exists $p \in [n]$ such that $|P \cap X_p| = 1$. For the sake of reference, let $\{x\} = P \cap X_p$. Since P is a maximal open packing, we have a contradiction. Indeed, if $x \in L \cap X_p$ or $x \in D_{11} \cap X_p$, then P can be expanded to include a_p. On the other hand, if $x = a_p$ or $x \in A_p$, then P can be expanded to include u_p. Now suppose there is $q \in [m]$ such that $|P \cap Y_q| = 1$, say $\{y\} = P \cap Y_q$. Again we arrive at a contradiction since P is a maximal open packing. In particular, if $y \in L \cap N(b_q)$, $y = c_q$, or $y \in D_{12} \cap N(b_q)$, then P can be expanded to include w_q. On the other hand if $y \in L \cap N(c_q)$, $y = b_q$, or $y \in D_{12} \cap N(c_q)$, then P can be expanded to include v_q. Therefore, $|P| = 2|S_1| + |S_2|$, and it follows that $G \in \mathcal{U}$. □

We note that the proof of Theorem 1 shows that regardless of girth, G belonging to $\mathcal{F}$ is sufficient to guarantee that $G \in \mathcal{U}$. See Figure 1 for an illustration of a graph belonging to $\mathcal{F}$. The vertices belonging to D_{11} are white squares, vertices in D_{12}

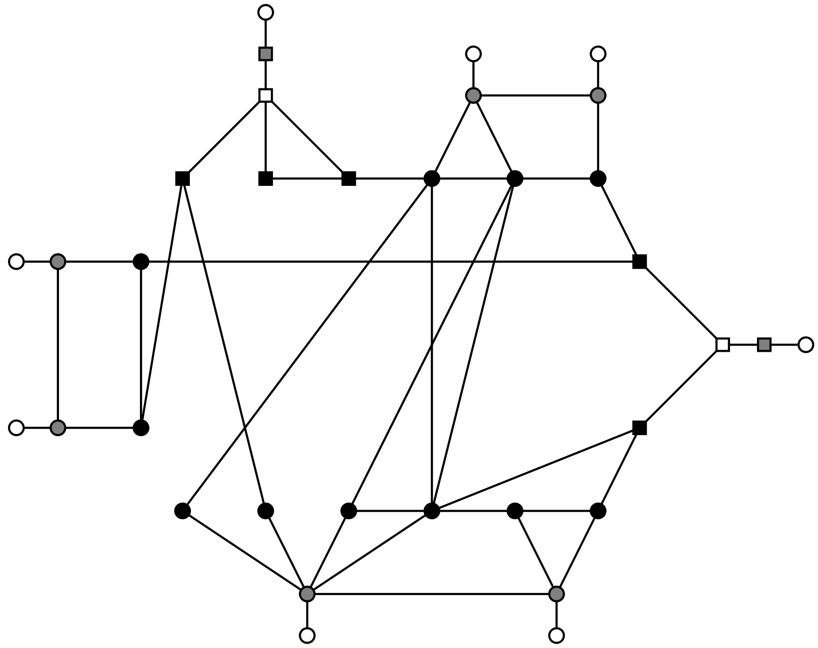

Figure 1: A graph in $\mathcal{F}$

are solid circles, vertices in D_2 are solid squares, vertices in S_1 are gray squares, and vertices in S_2 are gray circles.

Graphs of girth at least 15 that belong to the class $\mathcal{U}$ can be recognized in polynomial time. Specifically, suppose $g(G) \geq 15$. Identify the set L of vertices of degree 1 and let $S = N(L)$. Examine the subgraph of G induced by S to determine if its components have order at most 2. From this it is straightforward to determine if $G \in \mathcal{F}$, that is, if there is a weak partition of $V(G)$ that satisfies the five conditions in Section 1.

Using the above recognition algorithm applied to a tree T, one can of course decide whether $T \in \mathcal{U}$. It is also possible to give a different, but equivalent, description of the trees that belong to $\mathcal{U}$. With that in mind, recall that a double star is a tree of diameter 3 that has two support vertices. Suppose we have a finite collection $T_1, \ldots, T_n$ of double stars where the support vertices of T_i are u_i and v_i for $i \in [n]$. For each $k \in [n]$ we label some of the leaves in $L_{T_k}(u_k)$ and in $L_{T_k}(v_k)$ as *special* by applying exactly one of the following two rules to T_k.

(1) We label at least one (possibly all) of the leaves in $L_{T_k}(u_k)$ as special.

(2) We label at least one, but not all, of the leaves in $L_{T_k}(u_k)$ as special. Similarly, we label at least one, but not all, of the leaves in $L_{T_k}(v_k)$ as special.

Suppose A is the set of all vertices labeled as special. By adding edges between some pairs of vertices in A in such a way that every vertex of A is incident to at least one added edge and such that the resulting graph is connected without any cycles, we obtain a tree T that belongs to $\mathcal{U}$. To identify the partition $L, S_1, S_2, D_{11}, D_{12}, D_2$ of $V(T)$ that exists for any graph in $\mathcal{F}$, we note the following. The set L is the set of vertices of degree 1 in T. The set S_2 consists of the support vertices from those double stars T_k that were treated by rule (2) above or were processed by rule (1) but for which not all the leaves adjacent to u_k were labeled as special. The set S_1 consists of all v_k such that T_k was treated by rule (1) above and for which all the vertices in $L_{T_k}(u_k)$ were labeled as special. The sets D_{11} and D_{12} consist of non-leaf neighbors of vertices in S_1 and S_2, respectively. The set D_2 is made up of all those vertices that belong to $L_{T_k}(u_k)$, where T_k was treated by rule (1) such that all the vertices of $L_{T_k}(u_k)$ were labeled as special.

We conclude this section by noting that once the girth is lowered below 15, more complications occur. By Lemma 6, every graph of girth at least 15 that belongs to $\mathcal{U}$ contains a leaf. We now show that this girth restriction is in some sense best possible by exhibiting an infinite family of graphs with girth 14 and minimum degree 2 that belong to $\mathcal{U}$. Let k be a positive integer and for each $j \in [k]$, let $a_jb_jc_jd_je_jf_j$ be a path of order 6. To the disjoint union of these k paths and two new vertices x and y we add the set $\cup_{i=1}^{k}\{xa_i, yf_i\}$ of edges. The resulting graph is denoted G_k. See Figure 2.

We claim that $G_k \in \mathcal{U}$, for each $k \geq 2$. First, note that $G_2 = C_{14} \in \mathcal{U}$. Now suppose $k \geq 3$. To verify that $G_k \in \mathcal{U}$, we appeal to Proposition 2 and consider $N_o(G_k)$. Since G_k is connected and bipartite, it is easy to see that $N_o(G_k)$ consists of two isomorphic components. The component, say H, that contains b_1 has the following structure. The vertex set of H is $\{x\} \cup \left(\cup_{i=1}^{k}\{b_i, d_i, f_i\}\right)$. The set $\{f_1, f_2, \ldots, f_k\}$ induces a clique in H. The remaining edges of H are those in the set $\cup_{i=1}^{k}\{xb_i, b_id_i, d_if_i\}$. Let M be an arbitrary maximal independent set of H. If $x \in M$, then $H - N_H[x]$ is isomorphic to the corona of a clique of order k. Since this corona is well-covered with independence number k, it follows that $|M| = 1 + k$. On the other hand if $x \notin M$, then $M \cap \{b_1, b_2, \ldots, b_k\} \neq \emptyset$. Without loss of generality assume that $b_1 \in M$. Since either f_1 or a neighbor of f_1 is in M and since $\{f_1, f_2, \ldots, f_k\}$ induces a clique, it is clear that $|M \cap \{f_1, f_2, \ldots, f_k\}| = 1$. Furthermore, $|M \cap \{b_j, d_j\}| = 1$ for each j such that $2 \leq j \leq k$. Again we conclude that $|M| = k + 1$. Therefore, H is well-covered. By Proposition 2, we infer that $G_k \in \mathcal{U}$.

6 Open Problems

We conclude with the following open problem and question.

Problem 1. *Find a structural characterization of the class $\mathcal{U}$.*

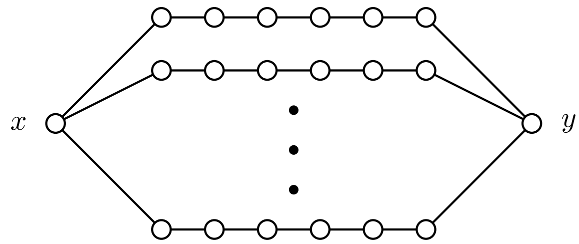

Figure 2: The graph G_k

Question 1. *Is there a polynomial time algorithm to recognize the class of graphs in which all the maximal open packings have the same cardinality?*

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