On graphs having maximal independent sets of exactly t distinct cardinalities

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Abstract

For a given positive integer t we consider graphs having maximal independent sets of precisely t distinct cardinalities and restrict our attention to those that have no vertices of degree one. In the situation when t is four or larger and the length of the shortest cycle is at least $6t-6$, we completely characterize such graphs.

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1 Introduction

A well-covered graph (Plummer [6]) is one in which every maximal independent set of vertices is of one cardinality and is hence a maximum independent set. Finbow, Hartnell and Whitehead [5] defined the class $\mathcal{M}_t$ to consist of those graphs which have exactly t different sizes of maximal independent sets. Finbow, Hartnell and Nowakowski [4] proved that the well-covered graphs (the $\mathcal{M}_1$ collection) of girth (the length of a shortest cycle) 6 or more, with the exceptions of K_1 and C_7, have the property that every vertex has degree one or has exactly one vertex of degree one in its neighborhood. Thus, C_7 is the unique graph in $\mathcal{M}_1$ with girth at least 6

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that has minimum degree at least two. The graphs in $\mathcal{M}_2$ of girth 8 or more have also been characterized ([5]). There are precisely five graphs in $\mathcal{M}_2$ of girth at least 8 and minimum degree 2 or more, namely the cycles C_8, C_9, C_{10}, C_{11} and C_{13}. This implies there are no $\mathcal{M}_1$ graphs of girth at least 8 with minimum degree 2 or more and no $\mathcal{M}_2$ graphs of girth 14 or more and having minimum degree at least 2. For related work on the class $\mathcal{M}_t$ see [1] and [2].

In this paper we investigate the graphs in $\mathcal{M}_t$ that have minimum degree at least 2 and higher girth and establish that the characterization of these in $\mathcal{M}_1$ and $\mathcal{M}_2$ is part of a general pattern. In particular, for $t \geq 3$ we show that among graphs with minimum degree at least 2, $\mathcal{M}_t$ does not contain a graph of girth at least $6t+2$ and that $C_{6t-4}, C_{6t-3}, C_{6t-2}, C_{6t-1}$ and C_{6t+1} are the only exceptions for girth at least $6t-4$. Furthermore, if $t \geq 4$, then these cycles along with C_{6t-6} are the only graphs in $\mathcal{M}_t$ that have minimum degree at least 2 and girth at least $6t-6$.

Let G be a finite simple graph. A vertex of degree 1 is called a *leaf* and any vertex that is adjacent to a leaf is called a *support vertex*. If C is a cycle in a graph G and u and v belong to C, we let uCv denote the shorter of the two u,v-paths that are part of C. For $A \subseteq V(G)$ and u a vertex in G, $d(u, A)$ will denote the length of a shortest path in G from u to a vertex of A. We will use $\mathcal{M}(G)$ to denote the collection of all maximal independent sets of G and we define the *independence spectrum* (*spectrum* for short) of G to be the set $\mathcal{S}(G) = \{\,|I| : I \in \mathcal{M}(G)\}$. The class $\mathcal{M}_t$ consists of those graphs G for which $|\mathcal{S}(G)| = t$. The spectrum is not necessarily a set of consecutive positive integers (e.g., $\mathcal{S}(K_{2,4,5}) = \{2, 4, 5\}$), but for paths and cycles it is. We denote the set of positive integers between p and q inclusive by $[p, q]$. The following proposition is easy to establish.

Proposition 1 *For each positive integer n at least 3,*

$$\mathcal{S}(C_n) = [\lceil n/3\rceil, \lfloor n/2\rfloor] \quad \text{and} \quad \mathcal{S}(P_n) = [\lceil n/3\rceil, \lceil n/2\rceil]\,.$$

Hence, $C_n \in \mathcal{M}_t$ and $P_n \in \mathcal{M}_s$ where $t = \lfloor n/2\rfloor - \lceil n/3\rceil + 1$ and $s = \lceil n/2\rceil - \lceil n/3\rceil + 1$.

The following lemma from [5] will be used throughout—often without mention.

Lemma 2 [5] *If the graph G belongs to $\mathcal{M}_t$ and I is an independent set of G, then for every component C of $G - N[I]$ there exists $k \leq t$ such that $C \in \mathcal{M}_k$. In addition, $G - N[I] \in \mathcal{M}_r$ for some $r \leq t$.*

Lemma 2 will most often be used in the following way. We will find an independent set I in a graph G and demonstrate that $G - N[I]$ has a component that is in the class $\mathcal{M}_s$ for some $s > t$ and conclude that $G \notin \mathcal{M}_t$. The following lemma will be used in that context with Lemma 2.

Lemma 3 *If a cycle C is in $\mathcal{M}_t$ and a new vertex is added as a leaf adjacent to a single vertex of C, then the resulting graph belongs to $\mathcal{M}_{t+1}$.*

Proof. Assume $\mathcal{S}(C) = [k, k+t-1]$. Let H be the graph formed by adding a leaf x adjacent to y. Let u and v be the neighbors of y on C. Note that $\{I \in \mathcal{M}(H) : y \in I\} = \{J \in \mathcal{M}(C) : y \in J\}$, and because of the symmetry of the cycle, $\mathcal{S}(C) = \{|J| : J \in \mathcal{M}(C), y \in J\}$. Also, $\{I \in \mathcal{M}(H) : u \in I\} = \{J \cup \{x\} : J \in \mathcal{M}(C), u \in J\}$. This shows that $[k, k+t] \subseteq \mathcal{S}(H)$. If H has a maximal independent set A of size less than k, then $x \in A$ and neither u nor v is in A, for otherwise $A \cap C$ is a maximal independent set in C of cardinality less than k. But now $A' = (A - \{x\}) \cup \{y\} \in \mathcal{M}(C)$ and $|A'| < k$, a contradiction. Therefore, $\mathcal{S}(H) = [k, k+t]$. We conclude that $H \in \mathcal{M}_{t+1}$.

□

In the class of graphs with leaves there is no connection between girth and the size of the spectrum. This can be seen by the following general construction. Let $t \geq 2$ and $g \geq 3$ be integers. Let H be the graph formed by adding a single leaf adjacent to each vertex of a cycle of order g. For a single vertex x on the cycle attach a path $v_1, v_2, \ldots, v_{2t-3}$ to H by making x and v_1 adjacent. Then add two leaves adjacent to v_i if i is odd, and add one leaf adjacent to v_j if j is even. The resulting graph of order $2g + 5t - 7$ has girth g and belongs to the class $\mathcal{M}_t$. (The spectrum of this graph is $[g + 2t - 3, g + 3t - 4]$.) For this reason we will henceforth consider only graphs having minimum degree at least 2. For ease of reference we denote the class of graphs that are in $\mathcal{M}_t$ and have no leaves (i.e., minimum degree at least 2) by $\mathcal{M}_t^2$. Note that $\mathcal{M}_t^2 \subseteq \mathcal{M}_t$. In the course of several of our proofs we will show that some given graph is not in $\mathcal{M}_t^2$ by demonstrating it does not belong to $\mathcal{M}_t$.

The remainder of this paper is devoted to verifying the entries in the following table.

	girth								
	$6t-6$	$6t-5$	$6t-4$	$6t-3$	$6t-2$	$6t-1$	$6t$	$6t+1$	$\geq 6t+2$
$t=1$				Δ	Δ	Δ	$\emptyset$	C_7	$\emptyset$
$t=2$	Δ	Δ	C_8	C_9	C_{10}	C_{11}	$\emptyset$	C_{13}	$\emptyset$
$t=3$	C_{12}	Δ	C_{14}	C_{15}	C_{16}	C_{17}	$\emptyset$	C_{19}	$\emptyset$
$t=4$	C_{18}	$\emptyset$	C_{20}	C_{21}	C_{22}	C_{23}	$\emptyset$	C_{25}	$\emptyset$
$t \geq 5$	C_{6t-6}	$\emptyset$	C_{6t-4}	C_{6t-3}	C_{6t-2}	C_{6t-1}	$\emptyset$	C_{6t+1}	$\emptyset$

Table 1: Graphs of given girth in $\mathcal{M}_t^2$

The entry for a given girth (written as a function of t) and a given value of t should be interpreted as follows. If a specific graph is given, then this is the unique graph of that girth that belongs to $\mathcal{M}_t^2$. For example, C_{15} is the only graph of girth 15 in $\mathcal{M}_3^2$. If $\emptyset$ appears, then there are no graphs of that girth in $\mathcal{M}_t^2$. When the entry is Δ, then it is known that $\mathcal{M}_t^2$ contains at least one graph of that girth (and it is not just a cycle). Some of these type of entries have been verified in previous papers. For example, see [4] and [5] for $\mathcal{M}_1^2$ and $\mathcal{M}_2^2$, respectively.

2 Establishing Table Entries

We begin by showing that for a given positive integer t the only graphs in $\mathcal{M}_t$ with large enough girth must have leaves. The next result was proved for well-covered graphs ($t = 1$) in [3]. Proposition 1 shows it is sharp in terms of girth.

Theorem 4 *Let t be a positive integer. If $g(G) \geq 6t + 2$ and $\delta(G) \geq 2$, then $G \in \mathcal{M}_r(G)$ for some $r > t$.*

Proof. Assume $t \geq 2$. Let G have girth at least $6t+2$ and minimum degree at least two. We will show that G has maximal independent sets of at least $t + 1$ different sizes. Choose a cycle $C = v_1, v_2, \ldots, v_s$ of minimum length in G.

Assume first that $s \geq 6t + 4$ and let P denote the path $v_3, v_4, \ldots, v_{6t+1}$. Since $\delta(G) \geq 2$ and $g(G) = s$, each vertex $u \notin C$ that is adjacent to a vertex of P has another neighbor u' that does not belong to P and is not adjacent to any vertex of P. Choose one such neighbor u' for each u and let J denote the set of these neighbors. By the girth restriction it follows that the set $I = J \cup \{v_1, v_{6t+3}\}$ is independent. (If $s = 6t + 2$, then proceed as above except let $I = J \cup \{v_1\}$.) However, P is a component of $G - N[I]$ and by Proposition 1, $P \in \mathcal{M}_{t+1}$. Similar to the proof of Lemma 2 this implies that G has maximal independent sets of at least $t+1$ different sizes.

If $s = 6t+3$, let P be the path $v_3, v_4, \ldots, v_{6t+2}$. The set J is chosen as before, and now $G - N[J \cup \{v_1\}]$ has the path P of order $6t$ as a component. By Proposition 1 it once again follows that G has at least $t+1$ distinct sizes of maximal independent sets.

□

For any positive integer t it follows from Proposition 1 that $C_{6t+1} \in \mathcal{M}_t$. In [4] it was shown that C_7 is the only well-covered graph of girth 7 and minimum degree 2 or more. The following theorem shows the similar result is true for larger values of t.

Theorem 5 *Let $t \geq 2$ be an integer. The cycle C_{6t+1} is the only graph of girth $6t + 1$ in $\mathcal{M}_t^2$, and $\mathcal{M}_t^2$ contains no graphs of girth $6t$.*

Proof. By Proposition 1 the cycle of order $6t + 1$ belongs to $\mathcal{M}_t^2$. Suppose G is a graph not isomorphic to C_{6t+1} such that $g(G) = 6t + 1$ and $\delta(G) \geq 2$. Then G

has an induced cycle C of order $6t+1$, and C has a vertex w of degree at least 3. Since $g(G)=6t+1$ and $\delta(G)\geq 2$ we can find an induced path w,a,b,c, such that none of a,b or c belongs to C. Let $X=\{u\in V(G): d(u,C)=2\}-N(a)$ and let $Y=\{u\in V(G): d(u,a)=2, d(u,w)=3\}$. For any two vertices on C there is a path using part of C of length at most $3t$ joining them. Since $g(G)\geq 13$ it follows that Y is independent. Suppose two vertices $x_1,x_2\in X$ are adjacent. Let x_1,v_1,w_1 and x_2,v_2,w_2 be paths in G with w_1 and w_2 on the cycle C. Then the cycle $x_1,v_1,w_1Cw_2,v_2,x_2,x_1$ has length at most $3t+5$. But then $3t+5\geq 6t+1$, which implies that $t=1$, a contradiction. Finally, if a vertex in X is adjacent to a vertex in Y, then a similar argument shows that G has a cycle of length at most $3t+6$ which also leads to a contradiction.

Therefore, $X\cup Y$ is an independent set. One of the components of the graph $G-N[X\cup Y]$ is the cycle C with a single leaf a attached at the support vertex w. By Lemma 3 this component is in $\mathcal{M}_{t+1}$. An application of Lemma 2 then shows that $G\notin\mathcal{M}_t^2$.

Now let G be a graph of girth $6t$, and as above find an induced cycle C of length $6t$. This time let $X=\{u\in V(G): d(u,C)=2\}$. This set is independent unless there is a cycle of the form $x_1,v_1,w_1Cw_2,v_2,x_2,x_1$ that has length at most $3t+5$. But this means $3t+5\geq 6t$ contradicting our assumption that $t\geq 2$. Hence X is independent. The cycle C is one of the components of $G-N[X]$. Since $C_{6t}\in\mathcal{M}_{t+1}$, Lemma 2 implies that $G\notin\mathcal{M}_t^2$.

□

By following a line of reasoning similar to the first part of the proof of Theorem 5 one can prove the following result. The proof is omitted. As noted earlier, Theorem 6 also holds for $t=2$. See [5].

Theorem 6 *Let $t\geq 3$ be a positive integer. For each integer n such that $6t-4\leq n\leq 6t-1$, the cycle C_n is the unique graph of girth n that belongs to $\mathcal{M}_t^2$.*

We now establish the uniqueness (for $t\geq 3$) of the table entry corresponding to those graphs with no leaves whose shortest cycle has length $6t-6$ and which have maximal independent sets of exactly t distinct cardinalities.

Theorem 7 *For each integer $t\geq 3$, the cycle C_{6t-6} is the only graph of girth $6t-6$ that belongs to $\mathcal{M}_t^2$.*

Proof. The cycle of order $6t-6$ is in $\mathcal{M}_t^2$ by Proposition 1. Suppose that G is a graph of girth $6t-6$ with no leaves. If G is not C_{6t-6}, then we can find an induced cycle C of length $6t-6$ in G with w,a,b,c, X and Y defined as in the proof of Theorem 5. The set Y is independent because $g(G)\geq 12$, and X is independent since $t\geq 3$. If some vertex of X is adjacent to a vertex of Y, then G contains a cycle of length at most $3t - 3 + 6$. It follows that $3t + 3 \geq g(G) = 6t - 6$, or equivalently $t \leq 3$.

If the set $X \cup Y$ is independent, then $G - N[X \cup Y]$ has a component isomorphic to a cycle of length $6t - 6$ with a single leaf attached at w. By Lemma 3 this component is in $\mathcal{M}_{t+1}$ and so it follows from Lemma 2 that $G \notin \mathcal{M}_t$.

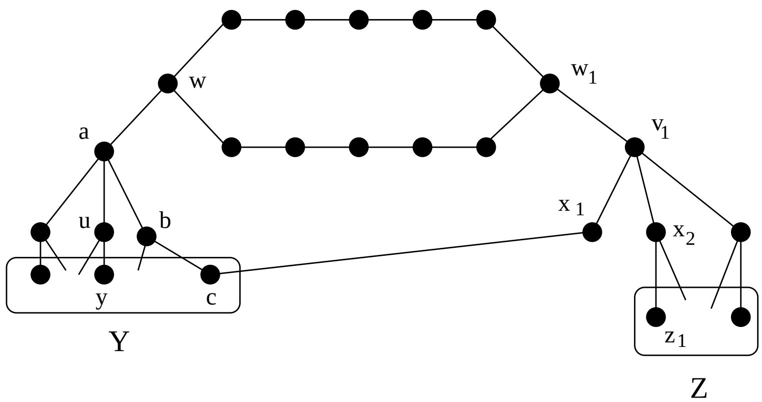

Figure 1: Part of G

Thus we may assume that $t = 3$ and that $X \cup Y$ is not independent. Without loss of generality we may assume that c from Y is adjacent to x_1 such that $x_1 \in X$ and x_1, v_1, w_1 is a path where w_1 is on the cycle C. See Figure 1. By using the fact that C has length 12 and $g(G) = 12$ we infer that the length of wCw_1 is 6. Let $X' = X - N(v_1)$ and let $Z = \{u : d(u, v_1) = 2, d(u, w_1) = 3, ux_1 \notin E(G)\}$. It is clear that Z is independent.

As above, if a vertex of Z is adjacent to a vertex h of X', then if $d(h, w) > 2$ a cycle of length at most 11 is present and if $d(h, w) = 2$ then G contains a cycle of length 10, contradicting $g(G) = 12$. Suppose $z_1 \in Y \cap Z$, say $z_1 = y$ as in Figure 1. Then $z_1 \neq c$, and $a, b, c, x_1, v_1, x_2, z_1, u, a$ is a cycle, contradicting the girth assumption. Similarly, since G has no cycles of length 9, it follows that $Z \cup Y$ is independent.

The set $X' \cup Y \cup Z$ is independent, and one of the components of the graph $G - N[X' \cup Y \cup Z]$ is the cycle C with a single leaf attached at vertices w and w_1. But this component has spectrum $\{4, 5, 6, 7, 8\}$ from which it follows that $G \notin \mathcal{M}_3$.

□

We now show that when $t \geq 4$ there is a "gap" at girth $6t - 5$ among the leafless graphs. That is, if G has minimum degree at least 2 and the shortest cycle of G has order $6t - 5$, then G does not belong to $\mathcal{M}_t$.

Theorem 8 *For each integer t at least 4, the class $\mathcal{M}_t^2$ contains no graphs of girth $6t-5$.*

Proof. First observe that $C_{6t-5} \in \mathcal{M}_{t-1}$. Our approach will be similar as that pursued in earlier proofs, except that we will be attempting to isolate a cycle of length $6t-5$ with a path of order 5 attached as in Figure 2. It is easy to check, using either $\{a, c, e\}$ or $\{a, d\}$ together with all possible maximal independent sets of a path of order $6t-6$, that this component has spectrum $[2t, 3t]$ and hence belongs to $\mathcal{M}_{t+1}$. This in turn implies via Lemma 2 that $G \notin \mathcal{M}_t^2$.

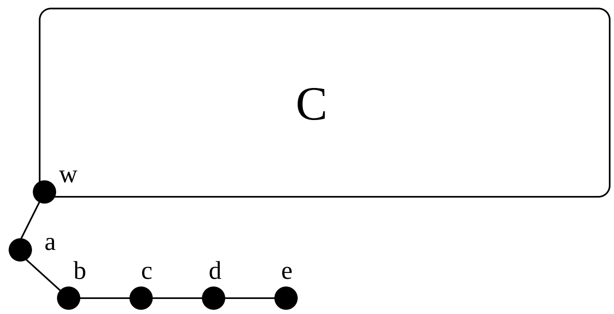

Figure 2: The cycle C with attachments

Suppose that G has girth $6t-5$ and has minimum degree at least 2. Let C be an induced cycle of length $6t-5$ in G. There must exist a vertex w on C having degree at least 3. For any two vertices on C there is a path on C joining them whose length is at most $3t-3$. Because of the girth and minimum degree assumptions on G we can find a path w, a, b, c, d, e as in Figure 2. Let $A = \{a, b, c, d, e\}$. Let $X = \{u : d(u, C) = 2\} - N(a)$ and let $Y = \{u : u \notin C, d(u, A) = 2, d(u, w) \geq 2\}$.

As in previous proofs it is straightforward to show that X is independent. Since $g(G) = 6t - 5 \geq 19$ no pair of vertices in Y can be adjacent. Suppose first that $X \cup Y$ is independent. The graph in Figure 2 is a component of $G - N[X \cup Y]$. As remarked at the outset, this shows that $G \notin \mathcal{M}_t^2$. We note that for $t \geq 5$, the girth restriction ensures that $X \cup Y$ is independent.

Now consider $t = 4$. Thus C is of length 19. Let s_1 and s_2 be the adjacent vertices on C that are at distance 9 from w. If both s_1 and s_2 are of degree two, then $X \cup Y$ is independent or else a cycle of length 18 would exist in G. Assume then without loss of generality that s_1 has a neighbor r that is not on C. Let $U = N(r) - \{s_1\}$. For each $u_i \in U$ choose a vertex $v_i \in N(u_i) - \{r\}$, and set $V = \{v_i : u_i \in U\}$. Similarly, let $B = N(a) - \{w\}$. For each $b_i \in B$ choose a vertex $c_i \in N(b_i) - \{a\}$, and set $D = \{c_i : b_i \in B\}$. Since $g(G) = 19$ the set $V \cup D \cup (X - U)$ is independent, and one of the components of $G - N[V \cup D \cup (X - U)]$ is a cycle of order 19 with a single leaf a adjacent to w and a single leaf r adjacent to s_1. This component belongs to $\mathcal{M}_5$ which proves that $G \notin \mathcal{M}_4^2$ and establishes the theorem.

$\square$

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