Limited Packings in Graphs

Robert Gallant*
Georg Gunther*
Bert Hartnell†
Douglas Rall‡

October 2007

Abstract

We define a k-limited packing in a graph, which generalizes a 2-packing in a graph, and give several bounds on the size of a k-limited packing. One such bound involves the domination number of the graph, and here we show all trees attaining the bound can be built via a simple sequence of operations. We consider graphs where every maximal 2-limited packing is a maximum 2-limited packing, and characterize their structure in a number of cases.

1 Introduction

Consider the following scenarios:

Network Security: A set of sensors are to be deployed to covertly monitor a facility. Too many sensors close to any given location in the facility can be detected. Where should the sensors be placed so that the total number of sensors deployed is maximized?

NIMBY: A city requires a large number of obnoxious facilities (such as garbage dumps), but no neighborhood should be close to too many such facilities, nor should the facilities themselves be too close together. Where should the facilities be located?

Market Saturation: A fast food franchise is moving into a new city. Market analysis shows that each outlet draws customers from both its immediate city block

*Memorial University of Newfoundland, Sir Wilfred Grenfell College, Corner Brook, Newfoundland

†Saint Mary's University, Halifax, Nova Scotia

‡Furman University, Greenville, South Carolina

and from nearby city blocks. However it is also known that a given city block cannot support too many outlets nearby. Where should the outlets be placed?

A graph model of these scenarios might maximize the size of a vertex subset subject to the constraint that no vertex in the graph is near too many of the selected vertices. The well-known packing number of a graph is the maximum size of a set of vertices B such that for any vertex v the closed neighborhood of v, $N[v]$, satisfies $|N[v] \cap B| \leq 1$. In this paper we consider relaxing the constraint to $|N[v] \cap B| \leq k$, for some fixed integer k.

Our notation is standard. Specifically, given a graph G then $V(G)$ is the set of vertices of G, $\gamma(G)$ is the domination number of G, $\rho(G)$ is the 2-packing number, $\delta(G)$ is the minimum degree of a vertex in G, $\Delta(G)$ is the maximum degree of a vertex in G, and for a vertex $v \in V(G)$, $N[v]$ is the closed neighborhood of v, which is the set of vertices adjacent to v along with v itself. The girth of a graph is the length of the shortest cycle in the graph, which is said to be infinite if the graph is a forest. A vertex of degree one is a leaf, and a stem is a vertex that is adjacent to at least one leaf. The symbol P_t denotes the path with t vertices, and if a vertex v in a tree is adjacent to a stem of degree 2, we will say v has a P_2 attached.

Definition 1. *Let G be a graph, and let $k \in \mathbb{N}$. A set of vertices $B \subseteq V(G)$ is called a k-limited packing in G provided that for all $v \in V(G)$, we have $|N[v] \cap B| \leq k$.*

In [1], the author introduces a notation unifying the description of many graph theoretic parameters. Specifically, in the context of a given graph G, a set $B \subseteq V(G)$ is called a $[\rho_{\leq k}, \sigma_{\leq k-1}]$-set provided any vertex v in G has $|N[v] \cap B| \leq k$, which is what we are calling a k-limited packing. Similarly a 2-limited packing in a graph would be called a $[\rho_{\leq 2}, \sigma_{\leq 1}]$-set.

A k-limited packing B in a graph G is called *maximal* if there does not exist a k-limited packing B' in G such that $B \subsetneq B'$. A k-limited packing B in a graph G is called *maximum* if there does not exist a k-limited packing B' in G such that $|B| < |B'|$.

For example, on the path P_5 as indicated in Figure 1, the sets $\{1, 3, 5\}$ and

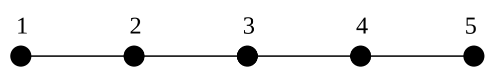

Figure 1: The path P_5

$\{1, 2, 4, 5\}$ are 2-limited packings in P_5. Both sets are maximal 2-limited packings in P_5, and the set $\{1, 2, 4, 5\}$ is a maximum 2-limited packing in P_5. We are interested in the maximum size of a k-limited packing in an arbitrary graph.

Definition 2. *Let G be a graph, and let $k \in \mathbb{N}$. The k-limited packing number of G, denoted $\mathrm{L}_k(G)$, is defined by*

$$\mathrm{L}_k(G) = \max\{|B| \mid B \text{ is a } k\text{-limited packing in } G\}.$$

If a subset of vertices B is a 2-packing then the distance between any pair of distinct vertices in B is at least 3, in which case $|N[v] \cap B| \leq 1$ for any vertex v in the graph, so B is also a 1-limited packing. But a 1-limited packing B has $|N[v] \cap B| \leq 1$ for any vertex v, and so the distance between any pair of distinct vertices in B is at least 3. Thus 1-limited packings and (distance) 2-packings are the same, and so $\mathrm{L}_1(G) = \rho(G)$.

Since a k-limited packing is also a $(k+1)$-limited packing we immediately obtain the following inequalities:

$$\rho(G) = \mathrm{L}_1(G) \leq \mathrm{L}_2(G) \leq \ldots \leq \mathrm{L}_{\Delta(G)+1}(G) = |V(G)|.$$

We collect some easily verified facts about the k-limited packing numbers of some familiar graphs in the following lemma.

Lemma 3. *Let $m, k, n \in \mathbb{N}$ with $m \geq 3$. Then,*

- $\mathrm{L}_1(P_m) = \lceil \frac{m}{3} \rceil$,
- $\mathrm{L}_2(P_m) = \lceil \frac{2m}{3} \rceil$
- $\mathrm{L}_1(C_m) = \lfloor \frac{m}{3} \rfloor$
- $\mathrm{L}_2(C_m) = \lfloor \frac{2m}{3} \rfloor$
- $\mathrm{L}_k(K_m) = \min\{k, m\}$
- $\mathrm{L}_k(K_{m,n}) = \begin{cases} 1 & \text{if } k = 1, \\ \min\{k-1, m\} + \min\{k-1, n\} & \text{if } k > 1. \end{cases}$

2 Bounds on k-limited packings

In this section we bound the k-limited packing number of a graph G. First we observe some connections to domination numbers of G.

For a positive integer $k \leq \delta(G) + 1$, a subset D of $V(G)$ is called a k-tuple dominating set in G if $|N[v] \cap D| \geq k$ for every vertex $v \in V(G)$. The minimum cardinality of a k-tuple dominating set in G is denoted by $\gamma_{\times k}(G)$. The familiar domination number is thus $\gamma(G) = \gamma_{\times 1}(G)$.

Lemma 4. *Let G be a graph with maximum degree Δ and minimum degree δ, and let $\{B, R\}$ be a partition of $V(G)$. Then:*

1. *If $k \leq \delta - 1$ and B is a $(\delta - k)$-limited packing in G, then R is a $(k+1)$-tuple dominating set in G.*

2. *If $k \leq \Delta - 1$ and R is a $(k+1)$-tuple dominating set in G, then B is a $(\Delta - k)$-limited packing in G.*

Proof. Let B be a $(\delta - k)$-limited packing in G. Then for any vertex v in G we have $|N[v] \cap B| \leq \delta - k$. Since $|N[v]| \geq \delta + 1$, we have $|N[v] \cap R| \geq (\delta + 1) - |N[v] \cap B| \geq (\delta + 1) - (\delta - k) = k + 1$. Thus R is a $(k+1)$-tuple dominating set in G.

This establishes (1). The proof of (2) is similar and is omitted. □

When the graph is regular even more can be said.

Lemma 5. *If G is an r-regular graph, and $k \leq r - 1$, then*

$$\mathrm{L}_{r-k}(G) + \gamma_{\times(k+1)}(G) = |V(G)|.$$

Proof. Lemma 4 implies if B is a maximum $(\delta - k)$-limited packing in G, then $R = V(G) - B$ is a $(k+1)$-tuple dominating set in G, and so $\mathrm{L}_{\delta-k}(G) = |B| = |V(G)| - |R| \leq |V(G)| - \gamma_{\times(k+1)}(G)$. Also, if R is a minimum $(k+1)$-tuple dominating set in G, then B is a $(\Delta - k)$-limited packing in G, and so $\mathrm{L}_{\Delta-k}(G) \geq |B| = |V(G)| - |R| = |V(G)| - \gamma_{\times(k+1)}(G)$. When G is r-regular, $r = \delta = \Delta$, which implies $\mathrm{L}_{r-k}(G) = |V(G)| - \gamma_{\times(k+1)}(G)$, from which the theorem assertion follows. □

The following bound also involves the domination number, and arises naturally when considering linear programs associated with k-limited packings.

Lemma 6. *If G is a graph, then $\mathrm{L}_k(G) \leq k\gamma(G)$. Furthermore, equality holds if and only if for any maximum k-limited packing B in G and any minimum dominating set D in G both the following hold:*

1. *For any $b \in B$ we have $|N[b] \cap D| = 1$.*

2. *For any $d \in D$ we have $|N[d] \cap B| = k$.*

Proof. Let B be any maximum k-limited packing in G, and let D be any minimum dominating set in G. Let $U = \{(b, d) | b \in B, d \in D,$ and $b \in N[d]\}$. For every $b \in B$, there is at least one $d \in D$ such that $b \in N[d]$ since D is a dominating set for G, and hence $|B| \leq |U|$. For each $d \in D$, we know $|N[d] \cap B| \leq k$, since B is a k-limited packing, and hence there are at most k vertices $b \in B$ with $(b, d) \in U$, and so $|U| \leq k|D|$. Thus $\mathrm{L}_k(G) = |B| \leq |U| \leq k|D| = k\gamma(G)$.

From these inequalities we see $\mathrm{L}_k(G) = k\gamma(G)$ holds if and only if $|N[b] \cap D| = 1$ for each $b \in B$, and $|N[d] \cap B| = k$ for each $d \in D$. As B is an arbitrary maximum k-limited packing in G, and D is an arbitrary minimum size dominating set in G, the result follows. □

One can bound the size of a k-limited packing solely in terms of the number of vertices in G.

Lemma 7. *If G is a connected graph with $|V(G)| \geq 3$, then $\mathrm{L}_2(G) \leq \frac{4}{5}|V(G)|$.*

Proof. Let B be a maximum 2-limited packing in G. We count the number, e, of edges with an endpoint in both B and $V(G) - B$.

Since B is a 2-limited packing, the induced subgraph $G[B]$ has maximum degree 1, and hence the components of $G[B]$ are either isolated vertices or P_2's. Since G is connected, and $|V(G)| \geq 3$, each component in $G[B]$ has an edge (in G) to some vertex in $V(G) - B$, and so e is at least as large as the number of components in $G[B]$, and so $|B|/2 \leq e$. Since B is a 2-limited packing each vertex in $V(G) - B$ has at most two neighbors in B. Hence $e \leq 2(|V(G)| - |B|)$. Together these inequalities imply $|B|/2 \leq 2(|V(G)| - |B|)$, which implies $|B| \leq \frac{4}{5}|V(G)|$. □

The upper bound $|B| = \frac{4}{5}|V(G)|$ is achieved only if both inequalities in the proof hold with equality. This means every vertex in $V(G) - B$ has two P_2's attached, and all the vertices in these P_2's are in B. Given any graph H we can attach two P_2's to every vertex in the graph to obtain a new graph containing H where this bound is met; in particular the newly added vertices are a 2-limited packing in G.

If we impose constraints on the minimum degree $\delta(G)$ of G, then similar reasoning gives the following.

Lemma 8. *If G is a connected graph, and $\delta(G) \geq k$, then $\mathrm{L}_k(G) \leq \frac{k}{k+1}|V(G)|$.*

This bound can always be achieved; let H be any connected graph, and to each vertex v in H attach a new K_k by making v adjacent to each vertex in the K_k. The resulting graph G has $\mathrm{L}_k(G) = k|V(H)| = \frac{k}{k+1}|V(G)|$. When $k = 2$ the cycles C_{3m} are another family of graphs which achieve this bound.

When the graph is regular stronger bounds are possible. The following is representative.

Lemma 9. *Let G be a cubic graph. Then $\frac{1}{4}|V(G)| \leq \mathrm{L}_2(G) \leq \frac{1}{2}|V(G)|$.*

3 Uniformly 2-limited graphs

A greedy algorithm will quickly find a maximal k-limited packing in a graph, but that set will not usually be a maximum k-limited packing. In this section we consider graphs G where every maximal 2-limited packing in G is a maximum 2-limited packing. This is the same as saying that every maximal 2-limited packing in G has the same cardinality.

Definition 10. *A graph G is said to be uniformly 2-limited if every maximal 2-limited packing in G has the same cardinality.*

For example P_3 is uniformly 2-limited, but P_4 and P_5 are not. The following gives a sufficient condition for a graph G to be uniformly 2-limited.

Lemma 11. *Let G be a graph, and let $\{s_1, s_2, \ldots, s_m\}$ be the set of stems in G. Suppose $\{N[s_i]|1 \leq i \leq m\}$ is a partition of $V(G)$, and if a stem s_i is adjacent to exactly one leaf, then all non-leaf neighbors of s_i have degree 2. Then G is uniformly 2-limited.*

Proof. Let B be a maximal 2-limited packing in G, and so $|N[s_i] \cap B| \leq 2$ for each stem s_i. We will first show $|N[s_i] \cap B| = 2$ for each s_i. On the contrary suppose some stem s_i has $|N[s_i] \cap B| < 2$.

One possibility is that the stem s_i is adjacent to at least two leaves. Since $|B \cap N[s_i]| < 2$, one of the leaves, say l_i, is not in B. But then $B \cup \{l_i\}$ is also a 2-limited packing, contradicting the maximality of B.

The other possibility is that the stem s_i is adjacent to exactly one leaf l_i. If $l_i \notin B$ then $B \cup \{l_i\}$ is a 2-limited packing, contradicting the maximality of B. So we must have $l_i \in B$, and also $s_i \notin B$ since $|N[s_i] \cap B| < 2$. The set $B' = B \cup \{s_i\}$ is also a 2-limited packing in G; in particular any non-leaf neighbor v of s_i must satisfy $|N[v] \cap B'| \leq 2$ since v has degree 2 and $v \notin B'$. So again we have contradicted the maximality of B.

Thus each stem s_i has $|N[s_i] \cap B| = 2$, and as the set $\{N[s_i]|1 \leq i \leq m\}$ is a partition of $V(G)$, $|B| = \sum_{1 \leq i \leq m} |N[s_i] \cap B| = 2m$. But as B was an arbitrary maximal 2-limited packing in G, every maximal 2-limited packing in G has the same size $2m$. □

The main result of this section is that the conditions of Lemma 11 are also necessary when a uniformly 2-limited graph G contains leaves and has girth at least 11. We will use the following notational convenience.

Definition 12. *Denote by $\mathcal{U}_2$ the set of uniformly 2-limited graphs;*

$$\mathcal{U}_2 = \{G|\ G \text{ is uniformly 2-limited graph }\}.$$

We first prove a series of conditions necessary for inclusion in $\mathcal{U}_2$.

Lemma 13. *If the graph G contains two adjacent stems, then $G \notin \mathcal{U}_2$.*

Proof. Suppose G contains adjacent stems s_1 and s_2, with adjacent leaves l_1, l_2 respectively. Extend $\{s_1, s_2\}$ to B, a maximal 2-limited packing in G. Since $\{s_1, s_2\} \subseteq B$, and s_1 and s_2 are adjacent, and B is a 2-limited packing in G, l_1 and l_2 are not in B. The set $B' = (B - \{s_2\}) \cup \{l_1, l_2\}$ is also a 2-limited packing in G, and $|B'| = |B| + 1$. Thus there are maximal 2-limited packings in G of different sizes, so $G \notin \mathcal{U}_2$. □

Lemma 14. *If G has two stems s_1, s_2 such that s_1 and s_2 are distance 2 apart, and either s_1 is adjacent to at least two leaves or s_1 has degree 2, then $G \notin \mathcal{U}_2$.*

Proof. Firstly suppose stems s_1, s_2 are distance 2 apart, each being adjacent to a vertex z, and s_2 is adjacent to leaf l_2 and s_1 is adjacent to two leaves l_1, l_1'. Extend $\{s_2, z, l_1\}$ to B, a maximal 2-limited packing in G. Because B is a 2-limited packing in G already containing $\{s_2, z, l_1\}$, it cannot also contain l_2 or l_1'. Then $B' = (B - \{z\}) \cup \{l_2, l_1'\}$ is a 2-limited packing in G, and $|B'| = |B| + 1$. Thus there are maximal 2-limited packings in G of different sizes, so $G \notin \mathcal{U}_2$.

Next suppose stems s_1, s_2 are distance 2 apart, each being adjacent to a vertex z, and s_1 has degree 2, and s_1 is adjacent to leaf l_1, and s_2 is adjacent to leaf l_2. In this case extend $\{s_2, z, l_1\}$ to B, a maximal 2-limited packing in G. In this case $B' = (B - \{z\}) \cup \{s_1, l_2\}$ is also a 2-limited packing in G, and $|B'| = |B| + 1$. Thus there are maximal 2-limited packings in G of different sizes, and so $G \notin \mathcal{U}_2$. □

Lemma 15. *If G is a graph with girth at least 5, with two stems at distance two apart, then $G \notin \mathcal{U}_2$.*

Proof. Let G be a graph with girth at least 5, and let s_1, s_2 be stems of G at distance two apart, with a common neighbor z. In light of Lemma 14, we may assume s_1 is adjacent to exactly one leaf l_1 and at least one non-leaf vertex $u_1 \neq z$, and similarly s_2 is adjacent to exactly one leaf l_2 and a non-leaf vertex $u_2 \neq z$. Since G has girth at least 5, the vertices $s_1, l_1, u_1, z, s_2, l_2, u_2$ are distinct and there does not exist a vertex w in G with $\{u_1, u_2, z\} \subseteq N[w]$. Hence the set $\{u_1, u_2, z\}$ is a 2-limited packing in G, and we can extend it to B, a maximal 2-limited packing in G. But the set $(B - \{z\}) \cup \{l_1, l_2\}$ is also a 2-limited packing in G with cardinality $|B| + 1$. Hence there are maximal 2-limited packings in G having different sizes, and so $G \notin \mathcal{U}_2$. □

Lemma 16. *Suppose G is a graph with girth at least 11, and $G \in \mathcal{U}_2$. Then any vertex v in G that is distance 2 from a stem is adjacent to exactly one stem.*

Proof. Suppose G is a graph with girth at least 11, and $G \in \mathcal{U}_2$, and s is a stem in G with an adjacent leaf l. Suppose vertex v is distance 2 from s, and vertex z is adjacent to both s and v. By Lemma 13, z is not a stem and so v is not a leaf, so let $C = \{c_1, c_2, \ldots, c_m\}$ be the (nonempty) set of neighbors of v other than z. By Lemma 15, v cannot be a stem, and so no c_i is a leaf. For each $c_i \in C$, let $D_i = \{d_{i1}, d_{i2}, \ldots, d_{ik_i}\}$ be the (nonempty) set of neighbors of c_i other than v.

To complete the proof, we must show that exactly one of the c_i's is a stem. By Lemma 15, we know that at most one of the c_i's is a stem, so by way of contradiction assume that none of the c_i's is a stem. We now build a maximal 2-limited packing B. Start by placing vertices s, z into B. To extend B we consider each vertex c_i and associated neighbors D_i of c_i in turn.

If D_i contains a stem, then without loss of generality, this stem is d_{i1}, with leaf e_{i1}. In this case place d_{i1} and e_{i1} into B. Any other vertex d_{ij} in D_i, with $j > 1$ (if it exists) has a neighbor e_{ij} other than c_i since c_i is not a stem. Since d_{i1} is a stem d_{ij} is not a stem (by Lemma 13) and so e_{ij} has a neighbor f_{ij} other than d_{ij}. Place e_{ij} and f_{ij} into B.

If D_i does not contain a stem, then since c_i is not a stem, d_{i1} has a neighbor e_{i1} other than c_i. Place d_{i1} and e_{i1} in B. Any vertex $d_{ij} \in D_i$ with $j > 1$ (if it exists) has a neighbor e_{ij} other than c_i since c_i is not a stem. Since D_i does not contain stems d_{ij} is not a stem and so e_{ij} has a neighbor f_{ij} other than d_{ij}. Place e_{ij} and f_{ij} into B.

For the resulting set B, there is no vertex $w \in G$ for which $|N[w] \cap B| > 2$ because any such vertex would lie on a cycle of length at most 10 and the girth of G is at least 11. Thus B is a 2-limited packing, and we can extend B to a maximal 2-limited packing B' in G. The way B is constructed ensures that each vertex c_i is adjacent to exactly one vertex in B'. Therefore, the set $B'' = (B' - \{z\}) \cup \{l, v\}$ is also a 2-limited packing in G, with cardinality $|B'| + 1$. Hence G has maximal 2-limited packings of different sizes, contradicting the fact that $G \in \mathcal{U}_2$, and hence exactly one of the c_i's is a stem.

□

Lemma 17. *Let G be a connected graph in $\mathcal{U}_2$ with girth at least 11, and suppose G has at least one stem. Then every vertex that is not a stem is adjacent to exactly one stem.*

Proof. In G, a vertex that is not a stem cannot be adjacent to two stems by Lemma 15, so suppose G contains a vertex v that is not a stem and that is not adjacent to a stem. Among all stems in G let s be one closest to v, and consider a shortest path $v, u_1, u_2, \ldots, u_t, s$ from v to s. If the path has length 2 then by Lemma 16 we know v must be adjacent to a stem which is a contradiction. If this path has length three or more (so $t \geq 2$) then Lemma 16 implies vertex u_{t-1} is adjacent to a stem s', in contradiction to s being the closest stem to v. □

Theorem 18. *Let G be a connected, uniformly 2-limited graph of girth at least 11. Suppose $\{s_1, s_2, \ldots, s_m\}$ is the set of stems in G, and $m \geq 1$. Then the set $\{N[s_i] | 1 \leq i \leq m\}$ is a partition of $V(G)$, and if a stem s_i is adjacent to exactly one leaf, then all non-leaf neighbors of s_i have degree 2.*

Proof. Suppose graph G satisfies the requirements of the theorem. Lemma 17 and Lemma 13 imply that the set $\{N[s_i] | 1 \leq i \leq m\}$ is a partition of $V(G)$, so the first assertion follows.

Now suppose some stem, say s_1, is adjacent to exactly one leaf l_1, and further has a non-leaf neighbor y_1 of degree at least 3. Let B_1 be a set consisting of each stem s_i and one leaf from those leaves adjacent to each s_i. Because the sets $N[s_i]$ partition $V(G)$, the set B_1 is a 2-limited packing. In fact, the set B_1 is a maximal 2-limited packing since it intersects each set $N[s_i]$ in two vertices, so adding another vertex to B_1 would cause some s_i to have $|N[s_i] \cap B_1| > 2$. We now build a second maximal 2-limited packing B_2 as follows. Choose two non-stem neighbors y_i and y_j of y_1 (elements of $N[s_i]$ and $N[s_j]$, respectively,) and put $\{y_i, s_i, y_j, s_j\}$ into B_2. For each of any remaining non-leaf neighbors z of s_1, choose a non-stem neighbor z_k (an element of $N[s_k]$) and put $\{z_k, s_k\}$ in B_2. The set B_2 is a 2-limited packing for G because any vertex v with $|N[v] \cap B_2| > 2$ would lie on a cycle of length at most 8, contradicting the fact that girth of G is 11 or more. So we can extend B_2 to a maximal 2-limited packing in G. The only vertex from $N[s_1]$ that could be in the resulting maximal 2-limited packing B_2 is the leaf l_1. Since this 2-limited packing in G contains at most 2 vertices from each other $N[s_i]$, we have $|B_2| < 2m$. But B_1 contains exactly two vertices from each $N[s_i]$, and so $|B_1| = 2m$. Thus $|B_2| < |B_1|$, contradicting the fact that $G \in \mathcal{U}_2$.

□

In light of Lemma 11, the conditions of Theorem 18 are in fact necessary and sufficient conditions for a graph of girth at least 11 that contains a stem to be uniformly 2-limited. In particular, these conditions are necessary and sufficient for a tree to be uniformly 2-limited.

In fact, it is possible to show that many graphs without stems are not uniformly 2-limited.

Lemma 19. *If G has girth at least 14 and has minimum degree at least 2, then $G \notin \mathcal{U}_2$.*

The proof is omitted; the basic idea is to take a P_4 in G, x, y, z, u, and extend $\{y, z\}$ to a maximal 2-limited packing B in such a way that $(B - \{y\}) \cup \{x, u\}$ is also a 2-limited packing with cardinality $|B| + 1$.

4 Trees T with $\mathrm{L}_2(T) = 2\gamma(T)$

By Lemma 6, all graphs G satisfy $\mathrm{L}_2(G) \leq 2\gamma(G)$. In this section we give a constructive characterization of those trees that attain this bound. First we note that the graphs considered in the last section are relevant here.

Lemma 20. *If T is a tree and T is uniformly 2-limited, then* $\mathrm{L}_2(T) = 2\gamma(T)$.

Proof. Let $\{s_1, s_2, \ldots, s_m\}$ be the set of stems in T. By Theorem 18 we know $\{N[s_i] | 1 \leq i \leq m\}$ is a partition of $V(T)$, and every maximal 2-limited packing B in T contains exactly two vertices from each set in the partition, so $\mathrm{L}_2(T) = 2m$. The set of stems of T are a dominating set of T, so $\gamma(T) \leq m \leq \mathrm{L}_2(T)/2$. Since $\mathrm{L}_2(T) \leq 2\gamma(T)$ by Lemma 6 we have $\mathrm{L}_2(T) = 2\gamma(T)$. □

For brevity we name the trees of interest in this section.

Definition 21. *Define*

$$\mathcal{L}_2 = \{T | T \textit{ is a tree and } \mathrm{L}_2(T) = 2\gamma(T)\}.$$

Lemma 20 says that every uniformly 2-limited tree is in $\mathcal{L}_2$. However, the tree shown in Figure 2 is in $\mathcal{L}_2$ but is not uniformly 2-limited. Hence the set

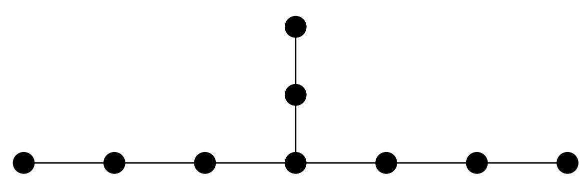

Figure 2: A tree T with $\mathrm{L}_2(T) = 2\gamma(T)$ that is not uniformly 2-limited.

of uniformly 2-limited trees are a strict subset of $\mathcal{L}_2$. Before giving an algorithmic description of the set $\mathcal{L}_2$, we state some necessary conditions for inclusion in $\mathcal{L}_2$. Although stated for trees, the following lemma holds for any graph G with $\mathrm{L}_2(G) = 2\gamma(G)$.

Lemma 22. *Let $T \in \mathcal{L}_2$. Then both of the following hold:*

- *T does not contain a stem that also has a P_2 attached.*
- *T does not contain a vertex that has three P_2's attached.*

Proof. Suppose T contains a stem s adjacent to a leaf l and with a P_2 a, b attached, as shown in Figure 3. Let B be a maximum 2-limited packing in T. By switching

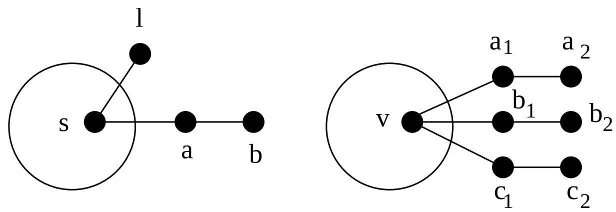

Figure 3: Subgraphs that can't occur in a tree in $\mathcal{L}_2$.

some vertices in B with others in $V(G) - B$ if necessary, we may assume that B contains the three vertices l, a, b. Let D be a minimum dominating set for T. By switching some vertices in D with others if necessary, we may assume that D contains the two stems s, a. But then T has a minimum dominating set D and a maximum 2-limited packing B where vertex $a \in B$ has $|N[a] \cap D| = 2 > 1$, and therefore by Lemma 6 the graph T cannot have $\mathrm{L}_2(T) = 2\gamma(T)$, and so $T \notin \mathcal{L}_2$.

Next suppose T contains a vertex v adjacent to three P_2's as illustrated in Figure 3. Let B be a maximum 2-limited packing in T. By switching some vertices in B with others if necessary, we may assume that B contains the five vertices a_2, a_1, b_2, b_1, c_2, and also that $v, c_1 \notin B$. Let D be a minimum dominating set for T. By switching some vertices in D with others if necessary, we may assume that D contains the three stems a_1, b_1, c_1. But then graph T has a minimum dominating set D and a maximum 2-limited packing B such that for vertex $c_1 \in D$ we have $|N[c_1] \cap B| = 1 < 2$, and therefore by Lemma 6 the graph T cannot have $\mathrm{L}_2(T) = 2\gamma(T)$, so $T \notin \mathcal{L}_2$.

□

Our aim is to show that $\mathcal{L}_2$ is precisely the set $\mathcal{C}$ defined next.

Definition 23. *Let $\mathcal{C}$ be the set of graphs consisting of P_2 together with any tree that can be obtained from P_2 by any finite sequence of the following operations.*

1. *Add a new leaf to any stem s already in the graph. We refer to this as a type-1 operation at s.*
2. *Add a new P_3 to the graph, making a leaf of the new P_3 adjacent to any vertex x already in the graph. We refer to this as a type-2 operation at x.*

3. *Add a new P_3 to the graph, making the central vertex of the P_3 adjacent to any vertex x already in the graph that is not in some maximum 2-limited packing in the graph. We refer to this as a type-3 operation at x.*

4. *Add a new P_5 to the graph, making the central vertex of the P_5 adjacent to any vertex x already in the graph that is not in some maximum 2-limited packing in the graph. We refer to this as a type-4 operation at x.*

Figure 4 illustrates the various operations.

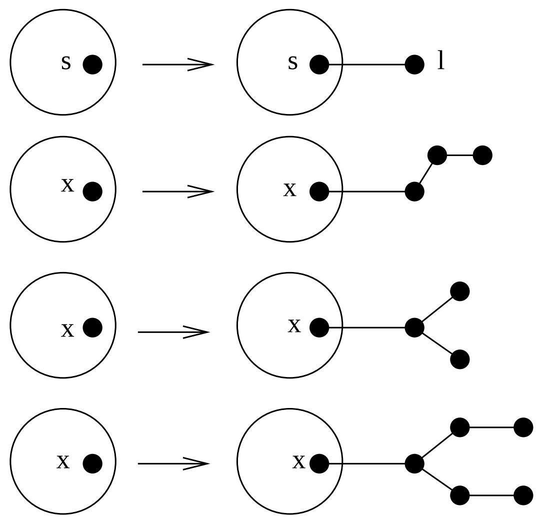

Figure 4: The effect of the operations defining $\mathcal{C}$.

Theorem 24. *Each tree T in $\mathcal{C}$ has $\mathrm{L}_2(T) = 2\gamma(T)$. (In brief: $\mathcal{C} \subseteq \mathcal{L}_2$.)*

Proof. The path $P_2 \in \mathcal{L}_2$. Suppose that a tree $T \in \mathcal{C}$ is in $\mathcal{L}_2$, so $\mathrm{L}_2(T) = 2\gamma(T)$. We show a new tree $T' \in \mathcal{C}$ constructed from T using any of the four operations defining $\mathcal{C}$ is in $\mathcal{L}_2$ also, and so $\mathcal{C} \subseteq \mathcal{L}_2$ will follow inductively. We consider the four operations in turn.

Let s be a stem in T; since $T \in \mathcal{L}_2$, we have $\mathrm{L}_2(T) = 2\gamma(T)$. Apply a type-1 operation to T at s to obtain tree T'. Because s is a stem in T we may assume a minimum dominating set of T contains s, and so dominates T', and so $\gamma(T') \leq \gamma(T)$. A 2-limited packing in T is also a 2-limited packing in T', so $\mathrm{L}_2(T') \geq \mathrm{L}_2(T) = 2\gamma(T) \geq 2\gamma(T')$. But since $\mathrm{L}_2(T') \leq 2\gamma(T')$ by Lemma 6 we have $\mathrm{L}_2(T') = 2\gamma(T')$ and hence $T' \in \mathcal{L}_2$.

Let x be any vertex in T. Apply a type-2 operation to T at x to obtain tree T'. A dominating set for T along with the new stem dominates T', so $\gamma(T') \leq \gamma(T) + 1$. A maximum 2-limited packing in T along with the new leaf and stem is 2-limiting for T', so $\mathrm{L}_2(T') \geq \mathrm{L}_2(T)+2$. Thus $\mathrm{L}_2(T') \geq \mathrm{L}_2(T)+2 = 2\gamma(T)+2 = 2(\gamma(T)+1) \geq 2\gamma(T')$. But since $\mathrm{L}_2(T') \leq 2\gamma(T')$ by Lemma 6 we have $\mathrm{L}_2(T') = 2\gamma(T')$ and hence $T' \in \mathcal{L}_2$.

Let x be a vertex of T that is not in some maximum 2-limited packing B in T. Apply a type-3 operation to T at x to obtain tree T' with new leaves u, v. A dominating set for T along with the new stem dominates T', so $\gamma(T') \leq \gamma(T) + 1$. Since $x \notin B$, the set $B \cup \{u, v\}$ is a 2-limited packing in T'. Thus $\mathrm{L}_2(T') \geq \mathrm{L}_2(T) + 2 = 2\gamma(T) + 2 = 2(\gamma(T) + 1) \geq 2\gamma(T')$. As above this implies $\mathrm{L}_2(T') = 2\gamma(T')$ and so $T' \in \mathcal{L}_2$.

Finally, let x be a vertex of T that is not in some maximum 2-limited packing B in T, and apply a type-4 operation to T at x to obtain tree T'. A dominating set for T, along with the two new stems will dominate T' so $\gamma(T') \leq \gamma(T) + 2$. Since $x \notin B$, the set B along with the new leaves and stems is 2-limiting for T', so $\mathrm{L}_2(T') \geq \mathrm{L}_2(T) + 4$. Thus $\mathrm{L}_2(T') \geq \mathrm{L}_2(T) + 4 = 2\gamma(T) + 4 = 2(\gamma(T) + 2) \geq 2\gamma(T')$, and again $T' \in \mathcal{L}_2$ follows.

□

Theorem 25. *If tree T has $\mathrm{L}_2(T) = 2\gamma(T)$, then $T \in \mathcal{C}$. (In brief: $\mathcal{L}_2 \subseteq \mathcal{C}$.)*

Proof. We proceed inductively on the number of vertices. No tree on one vertex is in $\mathcal{L}_2$, and the only tree on two vertices in $\mathcal{L}_2$ is the path P_2, and this tree is in $\mathcal{C}$ also. So inductively assume that for some positive integer $n \geq 2$, all trees T in $\mathcal{L}_2$ with $|V(T)| \leq n$ are contained in $\mathcal{C}$. Let tree T' have $n + 1$ vertices and assume $T' \in \mathcal{L}_2$. We will show $T' \in \mathcal{C}$.

First suppose T' has a stem s adjacent to at least three leaves l_1, l_2, l_3. Clearly there is a minimum dominating set D of T' not containing l_1 and similarly there is a maximum 2-limited packing B in T' not containing l_1. Let $T = T' - l_1$. Since $l_1 \notin D$, D dominates T, and since $l_1 \notin B$, B is a 2-limited packing for T, and so $\mathrm{L}_2(T) \geq |B| = 2|D| \geq 2\gamma(T)$ and hence $T \in \mathcal{L}_2$ by Lemma 6. Since T has n vertices by hypothesis $T \in \mathcal{C}$. As T' can be obtained by applying a type-1 operation to T at s we have $T' \in \mathcal{C}$ also.

Suppose tree T' on $n + 1$ vertices is in $\mathcal{L}_2$ and has no stem adjacent to three or more leaves. Let L be a longest path in tree T'. If T' has diameter 2 it is a star and is in $\mathcal{C}$, so we can assume the length of L is 3 or more, and that the sequence of vertices in the path L is $a, b, c, d, \ldots$ where a is a leaf. As L was a longest path, and the distance between a and c is 2, and T' has no stems with 3 or more leaves attached, the components of $T' - c$ not containing vertex d are either single vertices, or P_2's, P_3's. Furthermore in the last case it is the middle vertex in the P_3 that is adjacent to vertex c in T'. This is illustrated in Figure 5.

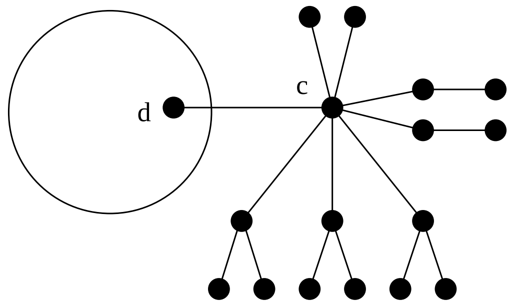

Figure 5: Structure of T' around c.

If some component of $T' - c$ is a single vertex, then c is a stem in T', and so by Lemma 22 the component of $T' - c$ containing vertices a, b is a P_3, containing exactly one further vertex x. Let $T = T' - \{a, b, x\}$. Since c is a stem in T', $\gamma(T) \leq \gamma(T') - 1$. Since we may assume some maximum 2-limited packing B in T' contains vertices a and x, $\mathrm{L}_2(T) \geq \mathrm{L}_2(T') - 2$. Thus $\mathrm{L}_2(T) \geq \mathrm{L}_2(T') - 2 = 2\gamma(T') - 2 = 2(\gamma(T') - 1) \geq 2\gamma(T)$ and so $\mathrm{L}_2(T) = 2\gamma(T)$ by Lemma 6. Thus T is also in $\mathcal{C}$ by hypothesis. Further, this tells us $B - \{a, x\}$ is a maximum 2-limiting set for T that does not contain c, so T' can be obtained by a applying a type-3 operation to T at c, and so $T' \in \mathcal{C}$ also.

If no component of $T' - c$ is a single vertex, then the components of $T' - c$ that do not contain vertex d are P_2's and P_3's; Assume there are α P_3's and β P_2's, so $\alpha + \beta \geq 1$, and by Lemma 22, $\beta \leq 2$. Let T be the tree obtained by removing these αP_3's, and these βP_2's, and the vertex c from T'. (In other words, T is the component of $T' - c$ containing the vertex d.) Because $T' - c$ has this structure we can assume T' has a minimum dominating set D such that $D - V(T)$ consists precisely of every neighbor of c (except d), of which there are $\alpha + \beta$ in number, and not vertex c, and so $\gamma(T) \leq \gamma(T') - (\alpha + \beta)$. Similarly this structure ensures T' has a maximum 2-limiting set B containing exactly two vertices from each of these $\alpha + \beta$ components, and $c \notin B$. So $\mathrm{L}_2(T) \geq \mathrm{L}_2(T') - 2(\alpha + \beta)$. Thus, as $\mathrm{L}_2(T') = 2\gamma(T')$, we have $\mathrm{L}_2(T) \geq \mathrm{L}_2(T') - 2(\alpha + \beta) = 2\gamma(T') - 2(\alpha + \beta) = 2(\gamma(T') - (\alpha + \beta)) \geq 2\gamma(T)$, and so as before Lemma 6 ensures $\mathrm{L}_2(T) = 2\gamma(T)$, so $T \in \mathcal{C}$ by hypothesis.

If we show T' can be obtained from T by an appropriate sequence of the operations that define $\mathcal{C}$, then we have $T' \in \mathcal{C}$ and our result will follow inductively. For this we consider the possibilities for β. If $\beta = 2$ then B cannot contain d, and so a type-4 operation to T at d followed by α applications of a type-3 operation at c gives T'. If $\beta = 1$ a type-2 operation to T at d, followed by α applications of a type-3 operation at c gives T'. If $\beta = 0$ then $\alpha \geq 1$ so a type-2 operation, followed by a type-1 operation, followed by $\alpha - 1$ type-3 operations will produce tree T' from tree T. $\square$

5 Summary

In this paper we introduce k-limited packings in a graph. It is natural to question whether the main result in section 3, the structural characterization of graphs of girth at least 11 that are uniformly two limited, in fact applies to graphs of some lower girth as well. Similarly one wonders if a characterization of the sort in section 4, for trees T with $\mathrm{L}_2(T) = 2\gamma(T)$, exists for non-trees.

The first and third authors acknowledge support from NSERC grants 327473-06 and 0004983, respectively.

References

[1] J. Arne Telle, Complexity of domination-type problems in graphs, Nordic Journal of Computing 1(1994), 157-171.

[2] John F. Fink and Michael S. Jacobson, On n-domination, n-dependence and forbidden subgraphs, Graph Theory and Its Applications to Algorithms and Computer Science, John Wiley & Sons, Inc. (1985), 301-312.

[3] A. Meir and J.W. Moon, Relations between packing and covering numbers of a tree, Pacific Journal of Mathematics 61 (1975), 225-233.