On minimum identifying codes in some Cartesian product graphs

Douglas F. Rall
Furman University
Greenville, SC, USA
doug.rall@furman.edu

Kirsti Wash
Trinity College
Hartford, CT, USA
kirsti.wash@trincoll.edu

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Abstract

An identifying code in a graph is a dominating set that also has the property that the closed neighborhood of each vertex in the graph has a distinct intersection with the set. The minimum cardinality of an identifying code, or ID code, in a graph G is called the ID code number of G and is denoted $\gamma^{\rm ID}(G)$. In this paper, we give upper and lower bounds for the ID code number of the prism of a graph, or $G\Box K_2$. In particular, we show that $\gamma^{\rm ID}(G\Box K_2) \geq \gamma^{\rm ID}(G)$ and we show that this bound is sharp. We also give upper and lower bounds for the ID code number of grid graphs and a general upper bound for $\gamma^{\rm ID}(G\Box K_2)$.

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1 Introduction

An identifying code, or ID code, in a graph is a dominating set that also has the property that the closed neighborhood of each vertex in the graph has a distinct intersection with the set. Thus every vertex of the graph can be uniquely located by using this intersection. Analogous to the domination number, the ID code number of a graph G is the minimum cardinality of an ID code of G and is denoted $\gamma^{\rm ID}(G)$. ID codes were first introduced in 1998 by Karpovsky, Chakrabarty and Levitin [15] who used them to analyze fault-detection problems in multi-processor systems. Since 1998 ID codes have been studied in many classes of graphs and an excellent, detailed list of references on ID codes can be found on Antoine Lobstein's webpage [17].

We shall focus on ID codes in a specific graph product, the Cartesian product. The *Cartesian product* of graphs G and H, denoted $G\Box H$, is the graph whose vertex set is $V(G)\times V(H)$. Two vertices (u_1, u_2) and (v_1, v_2) in $G\Box H$ are adjacent if either $u_1v_1 \in E(G)$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2v_2 \in E(H)$. When $H = K_2$, we refer to $G\Box K_2$ as the *prism* *of G*. Cartesian products have been studied for some time, and extensive information on their structural properties can be found in [8] and [13].

With respect to graph products, ID codes have been studied in the direct product of cliques [18], hypercubes [2, 12, 14, 16, 19], and infinite grids [1, 3, 11]. As we will be focusing on Cartesian products, we wish to point out that some of the more recent results regarding ID codes have been in the study of the Cartesian product of cliques [5, 7], and the Cartesian product of a path and a clique [10]. In light of these results, we first focus on the prism of a graph. When studying any parameter in a Cartesian product, an important question is whether there exists some formula relating the value of the parameter in the product to the value of the parameter in the underlying factor graphs. In [9] the authors prove the following result that relates the domination number of the prism of a graph G to the domination number of G.

Theorem 1 ([9])**.** *If G is any graph, then $\gamma(G) \leq \gamma(G \Box K_2) \leq 2\gamma(G)$.*

Since identifying codes are in the first place dominating sets, it seems natural to suspect that if G has an identifying code then a similar relationship would hold between $\gamma^{\mathrm{ID}}(G)$ and $\gamma^{\mathrm{ID}}(G \Box K_2)$. Namely, it would be natural to suspect that $\gamma^{\mathrm{ID}}(G) \leq \gamma^{\mathrm{ID}}(G \Box K_2) \leq 2\gamma^{\mathrm{ID}}(G)$. Indeed, we will prove that the lower bound in this inequality is correct and will show that the upper bound need not be true unless we make some additional assumptions on the minimum ID codes of G. It is known that for any graph G of order n, $\gamma^{\mathrm{ID}}(G) \leq n-1$. In [4] Foucaud et al. identify the class of all graphs which attain this bound, and interestingly enough, a subset of this class achieves the lower bound $\gamma^{\mathrm{ID}}(G \Box K_2) = \gamma^{\mathrm{ID}}(G)$. We also demonstrate an infinite family of graphs with identifying codes that show the upper bound is sharp.

Finally, we concentrate on the ID code number of grid graphs, i.e. the Cartesian product of two paths. The problem of finding the exact value for the domination number of grid graphs was quite difficult and finally settled in [6]. We expect finding the exact value for the ID code number of grid graphs to be just as difficult. In this paper, we give both upper and lower bounds for the ID code number of grid graphs, and we also give a general upper bound for the ID code number of the Cartesian product of a graph G and a path.

The remainder of the paper is organized as follows. In Section 2 we give some useful definitions and terminology as well as prove some basic facts about minimum ID codes. In Section 3 we prove the natural upper bound for the ID code number of the prism of a graph G when an additional assumption is imposed on G and show this bound is sharp. Section 4 is devoted to giving a lower bound for $\gamma^{\mathrm{ID}}(G \Box K_2)$ for any graph G. We also prove that the bound is sharp in this section. In Section 5, we give upper and lower bounds for $\gamma^{\mathrm{ID}}(P_m \Box P_n)$ for any positive integers $2 \leq m \leq n$ and we give a general upper bound for $\gamma^{\mathrm{ID}}(G \Box P_m)$.

2 Definitions and Preliminary Results

Given a simple undirected graph G and a vertex x of G, we let $N(x)$ denote the *open neighborhood* of x, that is, the set of vertices adjacent to x. The *closed neighborhood* of x

is $N[x] = N(x) \cup \{x\}$. By a *code* in G we mean any nonempty subset of vertices in G. The vertices in a code are called *codewords*. A code D in G is a *dominating set* of G if D has a nonempty intersection with the closed neighborhood of every vertex of G. The *domination number* of G is the cardinality of a smallest dominating set of G; it is denoted by $\gamma(G)$. A code having the property that the distance between any two codewords is at least 3 is called a *2-packing* of G, and $\rho_2(G)$ is the largest cardinality of a 2-packing in G. For compact writing we denote $N[x] \cap D$ by $I_D(x)$. A code D *separates* two distinct vertices x and y if $I_D(x) \neq I_D(y)$. When $D = \{u\}$ we say that u separates x and y. As mentioned above, an *identifying code* (*ID code* for short) of G is a code C that is a dominating set of G with the additional property that C separates every pair of distinct vertices of G. The minimum cardinality of an ID code of G is denoted $\gamma^{\rm ID}(G)$. Note that any graph having two vertices with the same closed neighborhood (so-called *twins*) does not have an ID code. If a graph has no twins, then we say it is *twin-free*.

If $h \in V(H)$, then the subgraph of $G \Box H$ induced by $V(G) \times \{h\}$ is called a *G-fiber* and is denoted by G^h. In the special case of the prism of G we will assume that $\{1, 2\}$ is the vertex set of K_2, and these two G-fibers are then G^1 and G^2. When dealing with the prism we will simplify the notation and denote the vertex (g, i) by g^i for $i \in [2]$. Here $[n]$ denotes the set of positive integers less than or equal to n. The map $p_G : V(G \Box H) \to V(G)$ defined by $p_G(a, b) = a$ is the *projection onto G*.

While our main emphasis is on minimum ID codes in prisms of graphs, we will also need some basic facts about ID codes in more general Cartesian products. The proof of the following is straightforward and is omitted.

Proposition 2. *If G and H both have minimum degree at least 1, then $G \Box H$ is twin-free.*

If C is any ID code in a twin-free graph G of order n, then $\{I_C(x)\}_{x \in V(G)}$ is a collection of n, pairwise distinct, nonempty subsets of C. This fact immediately implies the following result, which was first given in [15].

Proposition 3 ([15])**.** *Let G be any twin-free graph of order n. If $\gamma^{\rm ID}(G) = k$, then $n \leq 2^k - 1$. Equivalently, $\gamma^{\rm ID}(G) \geq \lceil \log_2(n + 1) \rceil$.*

In particular, an easy application of Proposition 3 to prisms yields the following corollary.

Corollary 4. *If H is any graph of order m with no isolated vertices, then*

$$\gamma^{\rm ID}(H \Box K_2) \geq \lceil \log_2(2m + 1) \rceil .$$

It also follows directly from Corollary 4 that if the prism of a graph G has ID code number 3, then G has order at most 3. Thus, we have the following result.

Corollary 5. *If G is a twin-free graph with no isolated vertices such that $\gamma^{\rm ID}(G) = 3$, then $\gamma^{\rm ID}(G \Box K_2) > 3$.*

By more closely analyzing how an identifying code separates vertices in a prism we can deduce some restrictions on ID codes in prisms.

Lemma 6. *Let G be a nontrivial, connected graph of order n. If C is an identifying code of $G \Box K_2$ that has m_i codewords in the G-layer G^i, for $i \in [2]$, then*

$$n \leq \min\{2^{m_1} - 1 + m_2, 2^{m_2} - 1 + m_1\}.$$

Proof. Let C be any ID code of $G \Box K_2$ and for $i \in [2]$ and let $m_i = |C_i|$ where $C_i = C \cap V(G^i)$. Note that $\{a^1 : a^2 \notin C_2\}$, $\{a^1 : a^2 \in C_2\}$ is a partition of $V(G^1)$. Any two vertices in the former subset are separated by C_1, and it follows that $|\{a^1 : a^2 \notin C_2\}| \leq 2^{m_1} - 1$. Clearly the second of these parts of the partition has cardinality m_2. Combining these we get that $n = |V(G^1)| \leq 2^{m_1} - 1 + m_2$. The result follows by applying a similar argument to G^2. $\square$

Proposition 7. *If the graph G has no isolated vertices, then $\gamma^{\mathrm{ID}}(G \Box K_2) > \gamma(G)$.*

Proof. Suppose to the contrary that $G \Box K_2$ has a minimum ID code C such that $|C| \leq \gamma(G)$. Since C dominates $G \Box K_2$, it follows from [9] that $\gamma(G) \geq |C| \geq \gamma(G \Box K_2) \geq \gamma(G)$, and hence $|C| = \gamma(G)$. As shown in [9], it follows that $C = (D_1 \times \{1\}) \cup (D_2 \times \{2\})$ where $D = D_1 \cup D_2$ and D is a minimum dominating set of G with the property that $V(G) - N[D_1] = D_2$ and $V(G) - N[D_2] = D_1$. Let $X = V(G) - D$. If some vertex $x \in X$ is adjacent to distinct vertices a and b in D_1 or in D_2, then by removing a and b from D and adding in x a dominating set smaller than D is constructed, which is a contradiction. Thus, every vertex of X has exactly one neighbor in D_1 and exactly one neighbor in D_2. Let $x \in X$ and suppose $\{d\} = N(x) \cap D_1$. It now follows that $I_C(d, 2) = \{(d, 1)\} = I_C(x, 1)$, which contradicts the assumption that C is an ID code for $G \Box K_2$. $\square$

3 Upper Bound

In this section we prove that under a certain condition on the minimum ID codes of a graph the natural upper bound holds for the ID code number of its prism.

Proposition 8. *If G has a minimum ID code I such that $G[I]$ has no isolated vertices, then*
$\gamma^{\mathrm{ID}}(G \Box K_2) \leq 2\gamma^{\mathrm{ID}}(G)$.

Proof. Let $D = I \times \{1, 2\}$, let $D_1 = I \times \{1\}$, and let $D_2 = I \times \{2\}$. It is clear that D dominates $G \Box K_2$ since I dominates G. Let x and y be distinct vertices of $G \Box K_2$. We show that D separates x and y. Suppose first that at least one of x and y belongs to D. Without loss of generality we assume that $x \in D_1$. If $y \in D_1$, then x and y have distinct neighbors in D_2. If $y \in G^1 - D_1$, then x has a neighbor in D_2 but y does not. If $y \in G^2 - D_2$, then since $G[I]$ has no isolated vertices it follows that x has a neighbor in D_1, but y does not. Finally, suppose that $y \in D_2$. If $p_G(x) = p_G(y)$, then $(N[x] \cap D_1) - N[y] \neq \emptyset$ since $G[I]$ has no isolated vertices. If $p_G(x) \neq p_G(y)$, then $x \in N[x] - N[y]$. Thus D separates x and y if at least one of them belongs to D. Now suppose that $x \in V(G^1) - D_1$. If y also belongs to $V(G^1) - D_1$, then D separates x and y because I separates $p_G(x)$ and $p_G(y)$. On the other hand, if $y \in V(G^2) - D_2$, then $N[y] \cap D \subseteq D_2$ while $N[x] \cap D \subseteq D_1$ and thus D separates x and y. $\square$

If we do not require that the subgraph of G induced by a minimum ID code has no isolated vertices, then the conclusion may not hold. As an example, let $X = \{1, 2, 3, 4\}$ and let $Y = \{A : A \subset \{1, 2, 3, 4\} \text{ and } |A| \geq 2\}$. Construct a bipartite graph G where $V(G) = X \cup Y$. In G the vertex $j \in X$ is adjacent to the vertex $A \in Y$ exactly when $j \in A$. It is clear that X is an identifying code in G and it then follows by Proposition 3 that $\gamma^{\mathrm{ID}}(G) \geq \log_2(|V(G)| + 1) = 4$. It can be easily verified that $\gamma^{\mathrm{ID}}(G \Box K_2) = 9$, which shows that the conclusion of Proposition 8 does not hold for this graph.

The upper bound given in Proposition 8 is sharp. To see this we consider the infinite class of so-called corona graphs. For a given graph H the *corona* of H is the graph constructed from H by adding a single (new) vertex of degree 1 adjacent to each vertex of H. The corona of H is denoted by $H \circ K_1$. Suppose that H is twin-free and connected. The set of vertices in the original graph H is a minimum dominating set of $H \circ K_1$ and also separates all pairs of vertices in this corona since H is twin-free. Consequently, $\gamma^{\mathrm{ID}}(H \circ K_1) = |V(H)|$. As the following proposition shows, we can also determine the identifying code number of the prisms of a more general class of graphs that includes these coronas. This result will also then yield an infinite family of graphs that achieve the upper bound given in Proposition 8.

Let n be any positive integer larger than 1. The class of graphs $\mathcal{H}_n$ consists of all the finite graphs that can be obtained from any connected graph of order n by adding at least one new vertex of degree 1 adjacent to each of these n vertices. (Note that $\mathcal{H}_n$ contains the corona of each connected graph of order n.)

Proposition 9. *If $H \in \mathcal{H}_n$, then $\gamma^{\mathrm{ID}}(H \Box K_2) = |V(H)|$.*

Proof. Suppose $H \in \mathcal{H}_n$, let $u_1, \ldots, u_n$ represent the vertices of the underlying graph of order n, and for each $i \in [n]$ let $x_{i,1}, \ldots, x_{i,k_i}$ represent the vertices of degree 1 adjacent to u_i. One can easily verify that $V(H^1)$ is an ID code for $H \Box K_2$. Hence $\gamma^{\mathrm{ID}}(H \Box K_2) \leq |V(H)|$. Suppose that C is an ID code for $H \Box K_2$. For each $i \in [n]$, let

$$A_i = \left(\bigcup_{j=1}^{k_i} \{(x_{i,j}, 1), (x_{i,j}, 2)\} \right) \cup \{(u_i, 1), (u_i, 2)\}.$$

We claim that $|A_i \cap C| \geq k_i + 1$ for each $i \in [n]$. Note first that if $\{(x_{i,j}, 1), (x_{i,j}, 2)\} \cap C = \emptyset$ for some $1 \leq j \leq k_i$, then $\{(u_i, 1), (u_i, 2)\} \subseteq C$ since C dominates $H \Box K_2$. If $k_i = 1$, then we are done. So assume that $k_i > 1$. If there exists $\ell \neq j$ such that

$$\{(x_{i,j}, 1), (x_{i,j}, 2), (x_{i,\ell}, 1), (x_{i,\ell}, 2)\} \cap C = \emptyset,$$

then C does not separate $(x_{i,j}, 1)$ and $(x_{i,\ell}, 1)$. So in this case, $|A_i \cap C| \geq k_i + 1$.

Next, suppose $|\{(x_{i,j}, 1), (x_{i,j}, 2)\} \cap C| \geq 1$ for each $1 \leq j \leq k_i$. If some j satisfies $\{(x_{i,j}, 1), (x_{i,j}, 2)\} \subseteq C$, then we are done. So we may assume $|\{(x_{i,j}, 1), (x_{i,j}, 2)\} \cap C| = 1$ for each $1 \leq j \leq k_i$. However, in this case one of $(u_i, 1)$ or $(u_i, 2)$ is in C for otherwise $(x_{i,j}, 1)$ and $(x_{i,j}, 2)$ are not separated. Thus, $|A_i \cap C| \geq k_i + 1$ in each case. This shows that $|C| \geq \sum_{i=1}^{n} |A_i \cap C| \geq \sum_{i=1}^{n} (k_i + 1) = |V(H)|$. □

If H is connected and twin-free, then by Proposition 9 we see that the corona $H \circ K_1$ is a graph that achieves the upper bound in Proposition 8. Hence this bound is achieved for infinitely many graphs.

4 Lower Bound

As mentioned in Section 1, Hartnell and Rall show in [9] that $\gamma(G\Box K_2) \geq \gamma(G)$ and we would naturally expect that $\gamma^{\mathrm{ID}}(G\Box K_2) \geq \gamma^{\mathrm{ID}}(G)$ to be true as well. However, the same projection argument that was used in [9] creates complications when applied to an ID code. In particular, given an ID code C of $G\Box K_2$, $p_G(C)$ need not be an ID code of G since $p_G(C)$ may induce isolated edges. However, we show in the following result that we can construct an ID code of G from $p_G(C)$.

Theorem 10. *For any twin-free graph G, $\gamma^{\mathrm{ID}}(G\Box H) \geq \gamma^{\mathrm{ID}}(G)\rho_2(H)$.*

Proof. Let C be a minimum ID code of $G\Box H$ and let A be a maximum 2-packing of the graph H. We will show that $|C \cap (V(G) \times N_H[h])| \geq \gamma^{\mathrm{ID}}(G)$ for every $h \in A$. Since the closed neighborhoods of vertices in A are pairwise disjoint, this will prove that $\gamma^{\mathrm{ID}}(G\Box H) = |C| \geq \gamma^{\mathrm{ID}}(G)\rho_2(H)$.

Fix a vertex h in H and let $C' = p_G(C \cap V(G^h))$. If C' is an ID code of G, then $|C \cap (V(G) \times N_H[h])| \geq |C \cap V(G^h)| \geq |C'| \geq \gamma^{\mathrm{ID}}(G)$. Thus, assume there exists at least one pair of vertices $g_1, g_2 \in V(G)$ such that $I_{C'}(g_1) = I_{C'}(g_2)$. Note that we are allowing the possibility here that these sets are empty; that is, that C' does not dominate G. We define a binary relation on $V(G)$ as follows: for $x, y \in V(G)$ we shall say that x and y are *restricted twins with respect to* C' if $I_{C'}(x) = I_{C'}(y)$. In this case we write $x \sim y$. It is clear that $\sim$ is an equivalence relation on $V(G)$. For a vertex x in G we let $R(x)$ represent the equivalence class of x under the relation $\sim$. If $V(G) - N[C'] \neq \emptyset$, then every pair of vertices in $V(G) - N[C']$ belong to the same equivalence class.

Let $R(a_1), \ldots, R(a_m)$ be a complete set of distinct equivalence classes of $\sim$ restricted to $N[C']$, and let $R(a_0) = V(G) - N[C']$. Note that $N[C'] = \cup_{i=1}^{m} R(a_i)$. If $R(a_0) \neq \emptyset$, then we assume $a_0 \in R(a_0)$. Furthermore, by reindexing if necessary we may assume that there exists $m_1 \in [m]$ such that $|R(a_i)| > 1$ for each $i \in [m_1]$ and $|R(a_i)| = 1$ for all $i > m_1$.

<u>Claim 1</u> If $R(a_0) \neq \emptyset$, then we can choose a set of $|R(a_0)|$ vertices from $V(G) - C'$ that dominates and separates each pair of vertices of $R(a_0)$.

<u>Proof</u> We proceed by induction on the cardinality of $R(a_0)$. Suppose first that $R(a_0) = \{a_0\}$. It is clear that $\{a_0\}$ dominates and separates $R(a_0)$. Next, assume that $R(a_0) = \{a_0, v\}$. If a_0 is not adjacent to v, then $\{a_0, v\}$ dominates and separates $R(a_0)$. So assume that a_0 is adjacent to v. If $(V(G) - C') \cap N[a_0] = (V(G) - C') \cap N[v]$, then a_0 and v are twins in G since $I_{C'}(a_0) = I_{C'}(v)$. Since G is twin-free, either there exists $w \in (V(G) - C') \cap (N[a_0] - N[v])$ or there exists $w \in (V(G) - C') \cap (N[v] - N[a_0])$. In either case, $\{a_0, w\}$ dominates and separates $R(a_0)$.

Assume that when $|R(a_0)| = k$, we can choose a set of k vertices from $V(G) - C'$ to dominate and separate each pair of vertices of $R(a_0)$. Suppose that $R(a_0) = \{u_1, \ldots, u_k, u_{k+1}\}$. By the induction hypothesis, there exists a set $W \subseteq V(G) - C'$ that dominates and separates each pair of vertices of $R(a_0) - \{u_{k+1}\}$ and $|W| = k$. If W dominates and separates each pair of vertices in $R(a_0)$, then we are done. So first assume that W does not dominate u_{k+1}. Note that since W dominates $R(a_0) - \{u_{k+1}\}$, then $W \cap N[u_{k+1}] \neq W \cap N[u_j]$ for all

$1 \leq j \leq k$. Thus, $W' = W \cup \{u_{k+1}\}$ is a set of $k+1$ vertices from $V(G) - C'$ that dominates and separates each pair of vertices of $R(a_0)$.

Next, suppose that W dominates u_{k+1} but there exists some $j \in [k]$ such that $W \cap N[u_{k+1}] = W \cap N[u_j]$. Note that if there exists $i \notin \{j, k+1\}$ such that $W \cap [u_i] = W \cap N[u_{k+1}]$, then W does not separate u_i and u_j, which is a contradiction. Thus, u_j is the only vertex of $R(a_0) - \{u_{k+1}\}$ that satisfies $W \cap N[u_{k+1}] = W \cap N[u_j]$. There exists a vertex in $V(G) - (W \cup C')$ that is adjacent to exactly one of u_{k+1} or u_j for otherwise u_{k+1} and u_j are twins in G. Assume first that there exists $z \in (N[u_j] - N[u_{k+1}]) - (W \cup C')$. It follows that $W \cup \{z\}$ separates every pair of vertices of $R(a_0)$. Otherwise, there exists $z \in (N[u_{k+1}] - N[u_j]) - (W \cup C')$ and $W \cup \{z\}$ separates every pair of vertices in $R(a_0)$. In either case, we have found a set of $|R(a_0)|$ vertices in $V(G) - C'$ that dominates and separates each pair of vertices in $R(a_0)$.($\square$)

<u>Claim 2</u> For each $i \in [m]$, we can choose a set of $|R(a_i)| - 1$ vertices from $V(G) - C'$ that separates each pair of vertices of $R(a_i)$.

<u>Proof</u> First, let $m_1 < i \leq m$. Note that $R(a_i) = \{a_i\}$ in which case there is no need to choose any vertices to separate a_i from itself. Now suppose $i \in [m_1]$. As in the proof of Claim 1, we proceed by induction on the cardinality of $R(a_i)$. Suppose first that $R(a_i) = \{a_i, v\}$. If a_i is not adjacent to v, then it follows that a_i and v are not vertices of C'. Moreover, a_i separates a_i and v. On the other hand, if a_i is adjacent to v, then either there exists $w \in (V(G) - C') \cap (N[a_i] - N[v])$ or there exists $w \in (V(G) - C') \cap (N[v] - N[a_i])$ for otherwise a_i and v are twins in G. In either case, w separates a_i and v. So we shall assume that when $|R(a_i)| = k$, there exists a set of $k-1$ vertices in $V(G) - C'$ that separates each pair of vertices of $R(a_i)$.

Suppose that $R(a_i) = \{u_1, \ldots, u_{k+1}\}$. By the induction hypothesis, there exists a set $W \subseteq V(G) - C'$ of cardinality $k-1$ that separates each pair of vertices in $R(a_i) - \{u_{k+1}\}$. If W separates u_{k+1} and u_j for all $j \in [k]$, then we are done. So assume that for some $j \in [k]$ that $W \cap N[u_j] = W \cap N[u_{k+1}]$. If there exists $1 \leq i \leq k, i \neq j$, such that $W \cap N[u_i] = W \cap N[u_{k+1}]$, then W does not separate u_i and u_j, which is a contradiction. Therefore, u_j is the only vertex of $R(a_i) - \{u_{k+1}\}$ that satisfies $W \cap N[u_j] = W \cap N[u_{k+1}]$. Since u_j and u_{k+1} are not twins in G, there exists $z \in V(G) - (W \cup C')$ that is adjacent to exactly one of u_j or u_{k+1}. Thus, $W \cup \{z\}$ separates every pair of vertices of $R(a_i)$ and $|W \cup \{z\}| = k$. ($\square$)

Finally, choose a minimal set W of vertices from $V(G) - C'$ that dominates $R(a_0)$ and that separates every pair of vertices from $R(a_i)$, for $0 \leq i \leq m$. By Claim 1 and Claim 2 such a set W exists. Note that $W \cup C'$ dominates every vertex of $V(G)$ since every vertex not dominated by C' belongs to $R(a_0)$ and W dominates $R(a_0)$. Next, note that if C' does not separate a pair of vertices, say $x, y \in V(G)$, then there exists a_i such that $\{x, y\} \subseteq R(a_i)$ for some $0 \leq i \leq m$. In this case, some vertex of W separates x and y. Thus, $W \cup C'$ is an ID code of G and $\gamma^{\rm ID}(G) \leq |W \cup C'|$. We claim that $|W \cup C'| \leq |C \cap (V(G) \times N_H[h])|$. Indeed, if $(u, h) \in V(G^h)$ where $u \in R(a_0)$, then there exists $h' \in V(H)$ such that $hh' \in E(H)$ and $(u, h') \in C$ since C is an ID code of $G \Box H$. Moreover, for each $1 \leq i \leq m$, consider the set $S_i = \{(u, h) \in V(G^h) : u \in R(a_i)\}$. Since C is an ID code of $G \Box H$, $|C \cap (V(G) \times N_H(h))| \geq$

$|S_i| - 1$. Thus, $|W| \leq |C \cap (V(G) \times N_H(h))|$, which implies that

$$\begin{aligned}\gamma^{\mathrm{ID}}(G) \leq |W \cup C'| &= |W| + |C'| \\ &\leq |C \cap (V(G) \times N_H(h))| + |C \cap V(G^h)| \\ &= |C \cap (V(G) \times N_H[h])|.\end{aligned}$$

Since A is a maximum 2-packing of H, we get

$$\rho_2(H)\gamma^{\mathrm{ID}}(G) = |A|\gamma^{\mathrm{ID}}(G) \leq \sum_{h \in A} |C \cap (V(G) \times N_H[h])| \leq |C| = \gamma^{\mathrm{ID}}(G \Box H)\,.$$

□

We call the reader's attention to the fact that Theorem 10 does not require that H be twin-free. Thus, an immediate consequence of Theorem 10 is the following.

Corollary 11. *For any twin-free graphs G and H,*

$$\gamma^{\mathrm{ID}}(G \Box H) \geq \max\{\gamma^{\mathrm{ID}}(G)\rho_2(H), \rho_2(G)\gamma^{\mathrm{ID}}(H)\}.$$

Next, we show that the bound given in Theorem 10 is indeed sharp. For the remainder of this section, we consider only Cartesian products of the form $G \Box K_2$. Note that by Corollary 4, $\gamma^{\mathrm{ID}}(G \Box K_2) > \gamma^{\mathrm{ID}}(G)$ when $\gamma^{\mathrm{ID}}(G) \leq 3$ as $\gamma^{\mathrm{ID}}(G) \leq |V(G)| - 1$ for all graphs. So the first case we consider is when $\gamma^{\mathrm{ID}}(G) = 4$.

Surprisingly, the class of graphs for which $\gamma^{\mathrm{ID}}(G \Box K_2) = \gamma^{\mathrm{ID}}(G) = 4$ is a subclass of the graphs which satisfy $\gamma^{\mathrm{ID}}(G) = |V(G)| - 1$. Foucaud et al. classified all such graphs that satisfy $\gamma^{\mathrm{ID}}(G) = |V(G)| - 1$ in [4]. For ease of reference, we include the description of this class of graphs here along with their result.

For any integer $k \geq 1$, let $A_k = (V_k, E_k)$ be the graph with vertex set $V_k = \{x_1, \ldots, x_{2k}\}$ and edge set $E_k = \{x_i x_j : |i - j| \leq k - 1\}$. So for $k \geq 2$, $A_k = P_{2k}^{k-1}$ and $A_1 = \overline{K_2}$. Let $\mathcal{A}$ be the closure of $\{A_i : i \in \mathbb{N}\}$ with respect to the join operation $\bowtie$. Figure 1 depicts several graphs in $\mathcal{A}$.

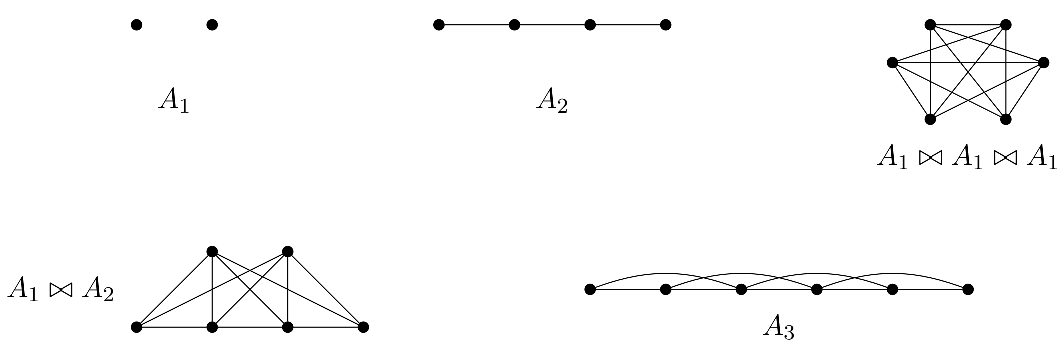


Figure 1: Examples of graphs in $\mathcal{A}$

Theorem 12 ([4])**.** *Given a connected graph G, we have $\gamma^{\mathrm{ID}}(G) = |V(G)| - 1$ if and only if $G \in \{K_{1,t} \mid t \geq 2\} \cup \mathcal{A} \cup (\mathcal{A} \bowtie K_1)$ and $G \not\cong A_1$.*

We now show that a subclass of $\mathcal{A}$ contains precisely those graphs for which $\gamma^{\mathrm{ID}}(G\Box K_2) = \gamma^{\mathrm{ID}}(G) = 4$.

Theorem 13. *For any connected twin-free graph G such that $\gamma^{\mathrm{ID}}(G) = 4$, $\gamma^{\mathrm{ID}}(G\Box K_2) = \gamma^{\mathrm{ID}}(G)$ if and only if $G \in \mathcal{A} \bowtie K_1$.*

Proof. Notice that if $G \in \mathcal{A} \bowtie K_1$ with $\gamma^{\mathrm{ID}}(G) = 4$, then $G = A_1 \bowtie A_1 \bowtie K_1$ or $G = A_2 \bowtie K_1$. In either case, we represent the vertices of $A_1 \bowtie A_1$ or A_2 by x_1, x_2, x_3, x_4. If $G = A_1 \bowtie A_1 \bowtie K_1$, then $C = \{x_1^1, x_2^1, x_3^2, x_4^2\}$ is an ID code of $G\Box K_2$. If $G = A_2 \bowtie K_2$, then $C = \{x_1^1, x_3^1, x_2^2, x_4^2\}$ is an ID code of $G\Box K_2$. Thus, $\gamma^{\mathrm{ID}}(G\Box K_2) \leq 4$. An application of Theorem 10 yields $\gamma^{\mathrm{ID}}(G\Box K_2) \geq \gamma^{\mathrm{ID}}(G) = 4$. Therefore, $\gamma^{\mathrm{ID}}(G\Box K_2) = 4$.

We now show the other direction. That is, let G be a connected twin-free graph such that $\gamma^{\mathrm{ID}}(G) = 4 = \gamma^{\mathrm{ID}}(G\Box K_2)$. Let C be a minimum ID code of $G\Box K_2$ and partition the projection of C onto $V(G)$, $p_G(C)$, as

$$\begin{aligned} C_1 &= \{v \in V(G) : v^1 \in C \text{ and } v^2 \notin C\} \\ C_2 &= \{v \in V(G) : v^1 \notin C \text{ and } v^2 \in C\} \\ D &= \{v \in V(G) : v^1 \in C \text{ and } v^2 \in C\}. \end{aligned}$$

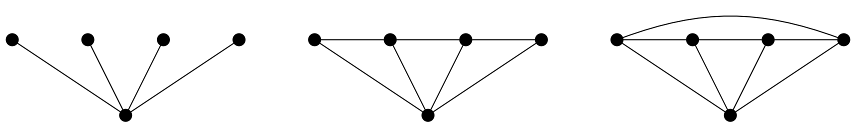
Suppose first that $|C_1| = 1$, $|D| = 0$, and let $C_1 = \{v\}$. Thus, $I_C(v^1) = \{v^1\}$, which implies for every $u \in V(G) - \{v\}$, $u^2 \in C$. It follows that $|V(G)| = 4$, which contradicts the assumption that $\gamma^{\mathrm{ID}}(G) = 4$. On the other hand, suppose $|C_1| = 0$, $|D| = 1$ and $D = \{v\}$. There exist precisely two vertices, say x and y in G such that $x^2 \in I_C(x^1)$ and $y^2 \in I_C(y^1)$ as $|C_2| = 2$. Every $w^1 \in V(G^1) - \{v^1, x^1, y^1\}$ is dominated only by v^1 and this implies that $|V(G)| = 4$, which is another contradiction. Thus, $|C_1 \cup D| = 2$ and similarly $|C_2 \cup D| = 2$.

(1) Suppose that $|D| = 2$ and let $D = \{u, v\}$. It follows that the order of G is at most 5, and since $\gamma^{\mathrm{ID}}(G) = 4$, we have $|V(G)| = 5$. Theorem 12 guarantees that $G \in \{K_{1,4}, A_1 \bowtie A_1 \bowtie K_1, A_2 \bowtie K_1\}$, pictured below in Figure 2. Note that $uv \in E(G)$ since the subgraph induced by C contains no isolated edge. Furthermore, since $|V(G)| = 5$, there exists $w \in V(G)$ such that w is adjacent to both u and v. Therefore, G contains a triangle and it follows that $G \in \{A_1 \bowtie A_1 \bowtie K_1, A_2 \bowtie K_1\}$.

(2) Suppose that $|D| = 1$, meaning $|C_1| = 1 = |C_2|$, and let $C_1 = \{u\}$, $D = \{v\}$, and $C_2 = \{w\}$. Since the subgraph induced by C contains no isolated edges, we may assume without loss of generality that $uv \in E(G)$. This immediately implies that $|V(G)| = 5$ and there exist vertices x and y in G such that $I_C(x^1) = \{u^1\}$ and $I_C(y^1) = \{v^1\}$. Therefore, $G \in \{A_1 \bowtie A_1 \bowtie K_1, A_2 \bowtie K_1\}$ since the subgraph induced by x, y, u, and v is a path.

(3) Suppose that $|D| = 0$, $|C_1| = 2 = |C_2|$, and let $C_1 = \{u, v\}$ and $C_2 = \{x, y\}$. Note that $uv \notin E(G)$ and $xy \notin E(G)$ since the subgraph induced by C contains no isolated edge. Thus, for any $w \in V(G) - (C_1 \cup C_2)$, $I_C(w^1) = \{u^1, v^1\}$ and $I_C(w^2) = \{x^2, y^2\}$. So $|V(G)| = 5$, $N[w] = V(G)$, and $G \in \{K_{1,4}, A_1 \bowtie A_1 \bowtie K_1, A_2 \bowtie K_1\}$. Also, $I_C(x^1) \neq \{x^2\}$ so x has a neighbor in C_1. Therefore, we may conclude that $G \in \{A_1 \bowtie A_1 \bowtie K_1, A_2 \bowtie K_1\}$.

□



$K_{1,4}$ $A_2 \bowtie K_1$ $A_1 \bowtie A_1 \bowtie K_1$

Figure 2: Graphs of order 5 with ID code number 4

Based on the above result, we next show that for any integer $k \geq 4$, there exists a graph G such that $\gamma^{\mathrm{ID}}(G \square K_2) = \gamma^{\mathrm{ID}}(G) = k$.

Theorem 14. *If $G \in \mathcal{A} \cup (\mathcal{A} \bowtie K_1)$ has order at least 5, then $\gamma^{\mathrm{ID}}(G \square K_2) = \gamma^{\mathrm{ID}}(G)$. Moreover, if $G = G_1 \bowtie G_2$ where $G_1, G_2 \in \mathcal{A} \cup (\mathcal{A} \bowtie K_1) - \{A_1, A_2\}$, then $\gamma^{\mathrm{ID}}((G_1 \bowtie G_2) \square K_2) = \gamma^{\mathrm{ID}}(G_1 \square K_2) + \gamma^{\mathrm{ID}}(G_2 \square K_2) + 1$.*

Proof. For the time being, assume that $G \in \mathcal{A}$. We proceed by induction. Write $G = G_1 \bowtie \cdots \bowtie G_m$ where each $G_i \in \{A_j : j \in \mathbb{N}\}$. Suppose first that $G = A_k$ where $k > 2$. That is, $G = P_{2k}^{k-1}$ for some $k \geq 3$. We show that

$$C = \{x_1^1, \ldots, x_{k-1}^1, x_{k+1}^1, \ldots, x_{2k-2}^1, x_k^2, x_{2k}^2\}$$

is an ID code for $G \square K_2$ of order $2k-1$. Figure 3 (a) depicts C for $A_5 \square K_2$. Let u and v be any pair of vertices in $G \square K_2$. One can easily verify that C is a dominating set for $G \square K_2$ and if $u \in V(G^1)$ and $v \in V(G^2)$, then C separates u and v. We check all remaining cases.

(a) ID code for $A_5 \square K_2$ (b) ID code for $(A_1 \bowtie A_4) \square K_2$

Figure 3: Examples of ID codes of $A_k \square K_2$ or $(A_k \bowtie A_\ell) \square K_2$

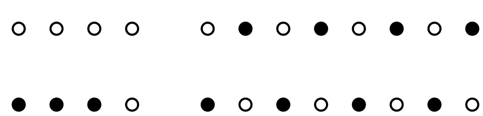


Figure 4: Example of ID code of $(A_2 \bowtie A_4) \square K_2$

Suppose first that $u = x_i^1$ and $v = x_j^1$ where $1 \leq i < j \leq 2k$. If $1 \leq i < j \leq k-1$, then $x_{j+(k-1)}^1$ separates u and v. If $1 \leq i \leq k-1$ and $j = k$, then x_k^2 separates u and v. If $1 \leq i \leq k$ and $k+1 \leq j \leq 2k$, then x_1^1 separates u and v. If $k \leq i < j \leq 2k-1$, then $x_{i-(k-1)}^1$ separates u and v. If $k \leq i \leq 2k-1$ and $j = 2k$, then x_{2k}^2 separates u and v.

Next, suppose that $u = x_i^2$ and $v = x_j^2$ where $1 \leq i < j \leq 2k$. If $i \notin \{k, 2k-1\}$, then x_i^1 separates u and v. If $i = k$ and $j = 2k-1$, then x_{2k}^2 separates u and v. If $i = k$ and

$j = 2k$, then u separates u and v. Finally, if $i = 2k-1$ and $j = 2k$, then x_k^2 separates u and v. Thus, C is an ID code of G, and we have shown that $\gamma^{\mathrm{ID}}(G\Box K_2) \le 2k-1$. On the other hand, $G \in \mathcal{A}$ so $\gamma^{\mathrm{ID}}(G) = 2k-1$ by Theorem 12. Thus, by Theorem 10 $\gamma^{\mathrm{ID}}(G\Box K_2) \ge \gamma^{\mathrm{ID}}(G) = 2k-1$, which implies that $\gamma^{\mathrm{ID}}(G\Box K_2) = \gamma^{\mathrm{ID}}(G)$.

Next, suppose $G = A_k \bowtie A_\ell$ where $k \in [\ell]$. Since G has order at least 5, $\ell \ge 2$. Let $x_1, \ldots, x_{2k}$ represent the vertices of A_k and $y_1, \ldots, y_{2\ell}$ represent the vertices of A_ℓ. We construct an ID code of $G\Box K_2$ based on the following three cases, where in each case $A = \{y_i^1 : i = 2j+1 \text{ for } 0 \le j \le \ell-1\}$ and $B = \{y_i^2 : i = 2j \text{ for } 1 \le j \le \ell\}$.

1. Suppose $k = 1$. Note that $\ell \ge 2$ since the order of G is at least 6. We show that $C = A \cup B \cup \{x_1^1\}$ is an ID code for $G\Box K_2$. Figure 3 (b) depicts C for $(A_1 \bowtie A_4)\Box K_2$. For $i \in [2]$ and each $v^i \in V(G^i)$, $|N_{G^i}[v^i] \cap C| \ge 2$, and it follows that C separates any vertex in G^1 from any vertex in G^2. Note that x_1^1 separates x_1^1 and x_2^1, and x_1^1 separates x_1^2 from any other vertex of G^2. Next, for $j \in [\ell]$, y_{2j-1}^1 separates y_{2j-1}^2 from every other vertex of G^2. By definition of A_ℓ, $N[y_{2i}^2] \cap B \ne N[y_{2j}^2] \cap B$ for $1 \le i < j \le \ell$. Since $N[x_2^2] \cap B = B$, C separates x_2^2 from y_{2i}^2 for $i \in [\ell]$. Similarly, C separates any two vertices in G^1. Therefore, C is an ID code of $G\Box K_2$, which implies $\gamma^{\mathrm{ID}}(G\Box K_2) \le \gamma^{\mathrm{ID}}(G)$.

2. Suppose $k \ge 2$. Let $T = \{x_i^1 : 1 \le i \le 2k-1\}$. We show that $C = A \cup B \cup T$ is an ID code of $G\Box K_2$. Figure 4 depicts C for $(A_2 \bowtie A_4)\Box K_2$. As in Case 1, C separates any vertex in G^1 from any vertex in G^2. Note that for $i \in [2k-1]$, x_i^1 separates x_i^2 from any other vertex of G^2. Next, for $j \in [\ell]$, y_{2j-1}^1 separates y_{2j-1}^2 from every other vertex of G^2. By definition of A_ℓ, $N[y_{2i}^2] \cap B \ne N[y_{2j}^2] \cap B$ for $1 \le i < j \le \ell$. Since $N[x_{2k}^2] \cap B = B$, C separates x_{2k}^2 from y_{2i}^2 for $i \in [\ell]$. Similarly, C separates any two vertices. Moreover, C is an ID code of the subgraph induced by $\{x_i^1 : 1 \le i \le 2k\}$. For each $j \in [\ell]$, y_{2j}^2 separates y_{2j}^1 from every other vertex in G^1. By definition A_ℓ, $N[y_{2i-1}^1] \cap A \ne N[y_{2j-1}^1] \cap A$ for $1 \le i < j \le \ell$. Furthermore, this shows that C separates y_{2i-1}^1 from x_j^1 where $i \in [\ell]$ and $j \in [2k]$ since $N[x_j^1] \cap A = A$. Therefore, C is an ID code of $G\Box K_2$, which implies $\gamma^{\mathrm{ID}}(G\Box K_2) \le \gamma^{\mathrm{ID}}(G)$.

Finally, note that by Theorem 10, we know $\gamma^{\mathrm{ID}}(G\Box K_2) \ge \gamma^{\mathrm{ID}}(G)$, which implies $\gamma^{\mathrm{ID}}(G\Box K_2) = \gamma^{\mathrm{ID}}(G)$. This concludes the base cases.

Suppose now that $r \ge 2$ and that if $G = G_1 \bowtie \cdots \bowtie G_r$ where each $G_j \in \{A_i : i \in \mathbb{N}\}$, then $\gamma^{\mathrm{ID}}(G\Box K_2) = \gamma^{\mathrm{ID}}(G)$. Now consider $H = A_s \bowtie G_1 \bowtie \cdots \bowtie G_r$ where $s \ge 1$. Let $G = G_1 \bowtie \cdots \bowtie G_r$. We can assume with no loss of generality that in the expansion $G = G_1 \bowtie \cdots \bowtie G_r$ the following inequality holds $|V(G_a)| \le |V(G_b)|$ when $a < b$. Thus, if $|V(A_s)| > |V(G_1)|$, we can let $H = G_1 \bowtie (A_s \bowtie G_2 \bowtie \cdots \bowtie G_s)$.

Suppose first that $H = A_1 \bowtie G$. Let C be a minimum ID code for $G\Box K_2$ and let x_1, x_2 represent the vertices of A_1. We claim that $C' = C \cup \{x_1^1, x_2^1\}$ is an ID code for $H\Box K_2$. Clearly C' dominates $H\Box K_2$, and any pair of vertices in $V(G\Box K_2)$ are separated by C, and therefore by C'. Suppose that $u, v \in \{x_1^1, x_2^1, x_1^2, x_2^2\}$. Note that if $u \in \{x_1^1, x_2^1\}$, then $I_{C'}(u) \cap G^1 \ne \emptyset$. Similarly, if $u \in \{x_1^2, x_2^2\}$, then $I_{C'}(u) \cap G^2 \ne \emptyset$. Thus, if $u \in \{x_1^1, x_2^1\}$ and $v \in \{x_1^2, x_2^2\}$, then $I_{C'}(u) \ne I_{C'}(v)$. If $u = x_1^1$ and $v = x_2^1$, then $u \in I_{C'}(u)$ but

$u \notin I_{C'}(v)$. Similarly, if $u = x_1^2$ and $v = x_2^2$, then $x_1^1 \in I_{C'}(u)$ but $x_1^1 \notin I_{C'}(v)$. Finally, if $u \in \{x_1^1, x_2^1, x_1^2, x_2^2\}$ and $v \notin \{x_1^1, x_2^1, x_1^2, x_2^2\}$, then one of x_1^1 or x_2^1 separates u and v. Thus, C' is an ID code of $H \Box K_2$ and by the inductive assumption and Theorem 12

$$|C'| = 2 + |C| = 2 + \gamma^{\text{ID}}(G \Box K_2) = 2 + \gamma^{\text{ID}}(G) = |V(H)| - 1 = \gamma^{\text{ID}}(H)\,.$$

This implies that $\gamma^{\text{ID}}(H \Box K_2) \leq \gamma^{\text{ID}}(H)$. An application of Theorem 10 gives $\gamma^{\text{ID}}(H \Box K_2) = \gamma^{\text{ID}}(H)$.

Next, suppose that $H = A_i \bowtie G$ where $i \geq 2$. Let C be a minimum ID code of $G \Box K_2$. We claim that $C' = C \cup A_i^1$ is an ID code of $H \Box K_2$. Clearly C' dominates $H \Box K_2$, and any pair of vertices in $V(G \Box K_2)$ are separated by C, and therefore by C'. Next, note that A_i^1 is an ID code of $A_i \Box K_2$. Thus, C' separates every pair of vertices in $A_i^1 \cup A_i^2$. Finally, suppose that $u \in A_i^1 \cup A_i^2$ and $v \in G^1 \cup G^2$. If $u \in A_i^1$ and $v \in G^2$, then u separates u and v. Similarly, if $u \in A_i^2$ and $v \in G^2$, then some vertex of A_i^1 separates u and v. So assume that $v \in G^1$. No vertex of $A_i^1 \cup A_i^2$ is adjacent to every vertex of A_i^1, but $A_i^1 \subset I_{C'}(v)$. Hence C' separates every pair of vertices in $H \Box K_2$, and consequently C' is an ID code of $H \Box K_2$. In a manner similar to that in the previous case, by using our induction assumption together with Theorems 10 and 12, we get that $\gamma^{\text{ID}}(H \Box K_2) = \gamma^{\text{ID}}(H)$.

Next, suppose that $G \in \mathcal{A} \bowtie K_1$. As above, we proceed by induction with base case $G = A_k \bowtie K_1$ where $k \in \mathbb{N}$. Note that $k \geq 2$ since the order of G is at least 5. If $k = 2$, then we are done by Theorem 13. If $k > 2$, then one can easily verify that $C = A \cup B$ where $A = \{x_{2j-1}^1 : j \in [k]\}$ and $B = \{x_{2j}^2 : j \in [k]\}$ is an ID code for $G \Box K_2$. Thus, $\gamma^{\text{ID}}(G \Box K_2) \leq |V(G)| - 1$ and by Theorem 10, we have $\gamma^{\text{ID}}(G \Box K_2) = \gamma^{\text{ID}}(G)$. We now assume that for some $m \geq 2$, $\gamma^{\text{ID}}(G \Box K_2) = \gamma^{\text{ID}}(G)$ if $G = G_1 \bowtie G_2 \bowtie \cdots \bowtie G_{m-1} \bowtie K_1$, where each $G_j \in \{A_i : i \in \mathbb{N}\}$.

Suppose $H = G_1 \bowtie \cdots \bowtie G_m \bowtie K_1$ where $G_j \in \{A_i : i \in \mathbb{N}\}$. Label the vertices of $G_j = A_{t_j}$, $t_j \geq 1$, as $x_{j,1}, \ldots, x_{j,2t_j}$ and let y be the vertex of K_1. For each j where $G_j = A_1$, let $C_j = \{x_{j,1}^1, x_{j,2}^1\}$ if j is odd and let $C_j = \{x_{j,1}^2, x_{j,2}^2\}$ if j is even. For each $G_j = A_{t_j}$ where $t_j > 1$, let

$$C_{j,1} = \{x_{j,2k-1}^1 : k \in [t_j]\},$$

and

$$C_{j,2} = \{x_{j,2k}^2 : k \in [t_j]\}.$$

Finally, let $C_j = C_{j,1} \cup C_{j,2}$. We show that $C = \cup_{j=1}^m C_j$ is an ID code for $G \Box K_2$. Let u and v be any two vertices in $V(G \Box K_2)$. If $u \in G^1$ and $v \in G^2$, then $|I_C(u) \cap V(G^1)| \geq 2$ and $|I_C(v) \cap V(G^2)| \geq 2$, which implies that C separates u and v. Now suppose that u and v belong to G^1. If $u = y^1$, then $I_C(u) = C \cap V(G^1) \neq I_C(v)$, which shows that C separates u and v. If $u \notin C$, then u is adjacent to a codeword in G^2, and this implies that C separates u and v. If $u \in \{x_{j,1}^1, x_{j,2}^1\}$ for some j such that $G_j = A_1$, say $u = x_{j,1}^1$, then $x_{j,2}^1$ separates u and v. If $u = x_{j,2k-1}^1$ for $k \in [t_j]$, then there exists a codeword d such that $d \in G_j^1$ but $d \notin I_C(u)$. If v does not belong to G_j^1, then d separates u and v. If v is in G_j^1, the structure of A_{t_j} shows that C separates u and v. A similar argument shows that C separates u and v when both belong to G^2. Hence C is an ID code for $H \Box K_2$, and it follows that $\gamma^{\text{ID}}(H \Box K_2) \leq |V(H)| - 1 = \gamma^{\text{ID}}(H)$. By Theorem 10 we now conclude that

$\gamma^{\rm ID}(H\Box K_2) = \gamma^{\rm ID}(H)$. By induction we have shown that if $G \in \mathcal{A} \cup (\mathcal{A} \bowtie K_1)$ has order at least 5, then $\gamma^{\rm ID}(G\Box K_2) = \gamma^{\rm ID}(G)$.

Finally, notice that if we have $G \in \mathcal{A} \cup (\mathcal{A} \bowtie K_1)$ where $G = G_1 \bowtie G_2$ and $G_1, G_2 \notin \{A_1, A_2\}$, then

$$\gamma^{\rm ID}((G_1 \bowtie G_2)\Box K_2) = |V(G_1)| + |V(G_2)| - 1 = \gamma^{\rm ID}(G_1\Box K_2) + \gamma^{\rm ID}(G_2\Box K_2) + 1.$$

□

The next immediate question is whether or not the graphs given in the statement of Theorem 14 are the only graphs which satisfy $\gamma^{\rm ID}(G\Box K_2) = \gamma^{\rm ID}(G)$. Unfortunately, there are an infinite number of graphs that are not contained in the class $\mathcal{A} \cup (\mathcal{A} \bowtie K_1)$ which satisfy $\gamma^{\rm ID}(G\Box K_2) = \gamma^{\rm ID}(G)$. For example, consider the graph G obtained from $A_2 \bowtie A_2 \bowtie A_2$ as follows. Label the vertices of $A_2 = P_4$ as u, v, x, y and let u_i, v_i, x_i, y_i represent the vertices of the i^{th} copy of A_2 for $i \in [3]$. To obtain G, let w represent an additional vertex and add an edge between w and x_3 and an edge between w and y_3. Figure 5 depicts the graph G without the edges between vertices of A_i and A_j when $i \neq j$, $\{i, j\} \subset \{1, 2, 3\}$.

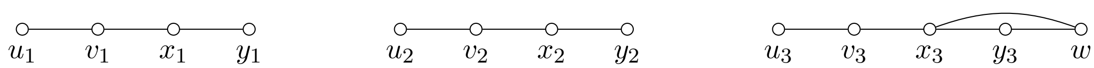


Figure 5: G obtained from $A_2 \bowtie A_2 \bowtie A_2$

We claim that $\gamma^{\rm ID}(G\Box K_2) = 11 = \gamma^{\rm ID}(G)$. First, note that $V(G) - \{u_3, w\}$ is an ID code of G. Next, we show that $\gamma^{\rm ID}(G) \geq 11$. Let C be a minimum ID code of G. If $w \notin C$, then it is clear that $|C| \geq 11$ since $G[\cup_{i=1}^{3}\{u_i, v_i, x_i, y_i\}]$ is isomorphic to $A_2 \bowtie A_2 \bowtie A_2$. So assume that $w \in C$. For each $i \in [3]$, $x_i \in C$ in order to separate u_i and v_i. Similarly, $v_i \in C$ in order to separate x_i and y_i. For $i \in [2]$, either $u_i \in C$ or $y_i \in C$ in order to separate v_i and x_i and, with no loss of generality, we may assume $u_i \in C$ for $i \in [2]$. Finally, notice that in order to separate v_1, v_2, and v_3, at least two vertices of $\{y_1, y_2, y_3\}$ are in C. In any case, we have shown, $\gamma^{\rm ID}(G) \geq 11$. Furthermore, Theorem 10 guarantees that $\gamma^{\rm ID}(G\Box K_2) \geq 11$. On the other hand, notice that $G\Box K_2$ is illustrated in Figure 6 and that the black vertices form an ID code of $G\Box K_2$. Thus, we have constructed a graph $G \notin \mathcal{A} \cup (\mathcal{A} \bowtie K_1)$ where $\gamma^{\rm ID}(G\Box K_2) = \gamma^{\rm ID}(G)$. Moreover, any graph G obtained from the join of k copies of A_2 by appending an additional vertex w in the same way as above will satisfy $\gamma^{\rm ID}(G\Box K_2) = \gamma^{\rm ID}(G)$.

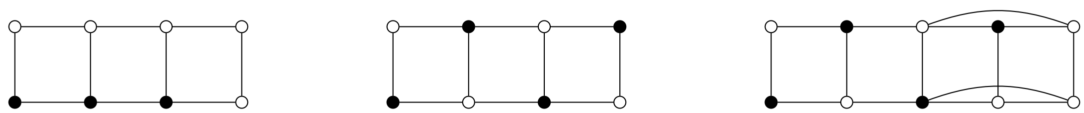


Figure 6: ID code of $G\Box K_2$

5 Grid graphs and general upper bounds

We now give upper bounds for the ID code number of $G\Box P_m$ where G is any graph and $m \geq 2$. First, we consider when G is a path.

Theorem 15. *For any positive integers m and k where $m \leq 3k$,*

$$\gamma^{\text{ID}}(P_m \Box P_{3k}) \leq mk + k\left\lceil \frac{m}{3} \right\rceil,$$

$$\gamma^{\text{ID}}(P_m \Box P_{3k+1}) \leq mk + k\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{m}{2} \right\rceil,$$

$$\gamma^{\text{ID}}(P_m \Box P_{3k+2}) \leq m(k+1) + (k-1)\left\lceil \frac{m}{3} \right\rceil.$$

Proof. First, suppose $m \not\equiv 1 \pmod 3$. We construct ID codes for each of the above cases. Let $\{0, 1, \ldots, m-1\}$ represent the vertices of P_m and let $\{0, 1, \ldots, y\}$ represent the vertices of P_{3k+a} for $a \in \{0, 1, 2\}$. Define

$$A = \{(i,j) : 0 \leq i \leq m-1, j \equiv 1 \pmod 3\}$$

and

$$B = \{(i,j) : i \equiv 1 \pmod 3, j \equiv 2 \pmod 3\}.$$

Figure 7 (a) depicts the set $A \cup B$ for $k = 2$ and $a = 0$. One can easily verify that $A \cup B$ is an ID code for $P_m \Box P_{3k}$. Next, consider $P_m \Box P_{3k+1}$. Let

$$C = \{(i,j) : i \equiv 1 \pmod 2, j = y\}.$$

Figure 7 (b) depicts the set $A \cup B \cup C$ for $k = 2$ and $a = 1$. Again, it is straightforward to verify that $A \cup B \cup C$ is an ID code for $P_m \Box P_{3k+1}$. Finally, consider $P_m \Box P_{3k+2}$. Define

$$X = \{(i,j) : 0 \leq i \leq m-1, j \equiv 1 \pmod 3, j \neq y\},$$

$$Y = \{(i,j) : i \equiv 1 \pmod 3, j \equiv 2 \pmod 3, j \neq y-2\},$$

and

$$Z = \{(i,j) : 0 \leq i \leq m-1, j = y-1\}.$$

The set $X \cup Y \cup Z$ is an ID code for $P_m \Box P_{3k+2}$.

Now, suppose that $m \equiv 1 \pmod 3$ and let

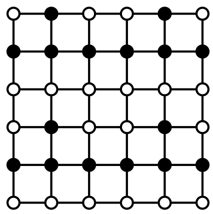
$$B' = B \cup \{(i,j) : i = m-1, j \equiv 2 \pmod 3\}$$

and

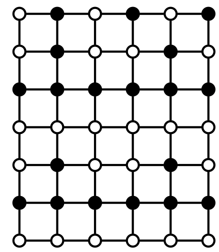
$$Y' = Y \cup \{(i,j) : i = m-1, j \equiv 2 \pmod 3, j \neq y-2\}.$$

One can easily verify that $A \cup B'$ is an ID code for $P_m \Box P_{3k}$, $A \cup B' \cup C$ is an ID code of $P_m \Box P_{3k+1}$, and $X \cup Y' \cup Z$ is an ID code of $P_m \Box P_{3k+2}$. $\Box$

Theorem 16. *For positive integers m and n where $2 \leq m \leq n$, $\gamma^{\text{ID}}(P_m \Box P_n) \geq mn/3$.*



(a) ID code for $P_6 \square P_6$



(b) ID code for $P_6 \square P_7$

Figure 7: Examples of ID codes when $m \not\equiv 1 \pmod 3$

Proof. Let C be a minimum ID code of $G = P_m \square P_n$. Partition $V(G)$ as follows. Let

$$C_1 = \{v \in V(G) : v \in C, v \text{ isolated in } G[C]\},$$
$$C_2 = C - C_1,$$

and for each $i \in [4]$,

$$N_i = \{v \in V(G) - C : v \text{ is adjacent to } i \text{ vertices in } C\}.$$

We further partition C_1 and C_2 as follows. For $i \in [3]$, let $A_i = \{v \in C_1 : \deg(v) = i + 1\}$ and let $B_i = \{v \in C_2 : \deg(v) = i + 1\}$. Note that the number of edges between C and $V(G) - C$ is at most $2|A_1| + 3|A_2| + 4|A_3| + |B_1| + 2|B_2| + 3|B_3|$. On the other hand, the number of edges between C and $V(G) - C$ is precisely $|N_1| + 2|N_2| + 3|N_3| + 4|N_4|$. Thus,

$$\begin{aligned} |C| + |A_1| + 2|A_2| + 3|A_3| + |B_2| + 2|B_3| &= 2|A_1| + 3|A_2| + 4|A_3| + |B_1| + 2|B_2| + 3|B_3| \\ &\geq |N_1| + 2|N_2| + 3|N_3| + 4|N_4| \\ &= mn - |C| + |N_2| + 2|N_3| + 3|N_4|. \end{aligned}$$

Therefore,

$$2|C| + |A_1| + 2|A_2| + 3|A_3| + |B_2| + 2|B_3| \geq mn + |N_2| + 2|N_3| + 3|N_4|.$$

Next, notice that

$$|A_1| + 2|A_2| + 3|A_3| + |B_2| + 2|B_3| \leq 3|C_1| + 2|C_2| = 3|C| - |C_2|,$$

which implies that

$$2|C| + 3|C| - |C_2| \geq mn + |N_2| + 2|N_3| + 3|N_4|.$$

On the other hand, no vertex of N_1 is adjacent to a vertex of C_1 for otherwise C would not separate such a pair of vertices. Thus, each vertex of N_1 is adjacent to precisely one vertex of C_2. Moreover, there can exist no more than $|C_2|$ vertices in N_1. Therefore, we may conclude that

$$\begin{aligned} 5|C| &\geq mn + |N_2| + 2|N_3| + 3|N_4| + |C_2| \\ &\geq mn + |N_2| + 2|N_3| + 3|N_4| + |N_1| \\ &\geq mn + mn - |C| + |N_3| + 2|N_4| \\ &\geq 2mn - |C|. \end{aligned}$$

It follows that $|C| \geq mn/3$. $\square$

Note that when $n = 3k$ for some $k \in \mathbb{N}$, it follows from Theorem 16 that $\gamma^{\mathrm{ID}}(P_m \square P_{3k}) \geq mk$. Thus the gap between Theorem 15 and Theorem 16 is $mk/3$. Now we provide a general upper bound for $\gamma^{\mathrm{ID}}(G \square P_m)$ whenever $m \geq 3$ and G is twin-free.

Theorem 17. *For any twin-free graph G of order n and any positive integer $m \geq 3$,*

$$\gamma^{\mathrm{ID}}(G \square P_m) \leq \min\{m\gamma^{\mathrm{ID}}(G), m\gamma(G) + \left\lceil \frac{m}{3} \right\rceil (n - \gamma(G))\}.$$

Proof. Let D be an ID code of G. Certainly $C = \{(u, v) \mid u \in D, v \in P_m\}$ is an ID code of $G \square P_m$. Next, let A be a minimum dominating set of G. Let $\{0, 1, \ldots, m-1\}$ represent the vertices of P_m. Let $X = \{(u, v) \mid u \in A, v \in P_m\}$ and $Y = \{(u, v) \mid u \in V(G) - A, v \equiv 1 \pmod 3\}$. If $m \not\equiv 1 \pmod 3$, then $X \cup Y$ is an ID-code of $G \square P_m$. If $m \equiv 1 \pmod 3$, then let $Y' = \{(u, v) \mid u \in V(G) - A, v \equiv 1 \pmod 3 \text{ or } v = m - 1\}$. The set $X \cup Y'$ is an ID code of $G \square P_m$. In either case, we have constructed an ID code of cardinality $m\gamma(G) + \left\lceil \frac{m}{3} \right\rceil (n - \gamma(G))$. $\square$

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