Identifying codes of the direct product of two cliques

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Abstract

An identifying code in a graph is a dominating set that also has the property that the closed neighborhood of each vertex in the graph has a distinct intersection with the set. The minimum cardinality of an identifying code in a graph G is denoted $\gamma^{\mathrm{ID}}(G)$. It was recently shown by Gravier, Moncel and Semri that $\gamma^{\mathrm{ID}}(K_n \Box K_n) = \lfloor \frac{3n}{2} \rfloor$. Letting $n, m \geq 2$ be any integers, we consider identifying codes of the direct product $K_n \times K_m$. In particular, we answer a question of Klavžar and show the exact value of $\gamma^{\mathrm{ID}}(K_n \times K_m)$.

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1 Introduction

An identifying code in a graph is a dominating set that also has the property that the closed neighborhood of each vertex has a distinct intersection with the set. Because of this characteristic of the dominating set every vertex can be uniquely located by using this intersection with the identifying code. The first to study identifying codes were Karpovsky, Chakrabarty and Levitin [16] who used them to analyze fault-detection problems in multiprocessor systems. An excellent, detailed list of references on identifying codes can be found on Antoine Lobstein's webpage [19]. The usual invariant of interest is the minimum cardinality of an identifying code in a given graph.

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In this regard various families of graphs have been studied, including trees [3], paths [2, 5, 15], cycles [2, 10, 21, 5, 15], and infinite grids [1, 6, 12].

In terms of graph products, a few of the more recent results have been in the study of hypercubes [4, 13, 14, 17, 20], the Cartesian product of cliques [9, 8], and the lexicographic product of two graphs [7]. A natural problem (posed by Klavžar [18] at the Bordeaux Workshop on Identifying Codes in 2011) is to determine the order of a minimum identifying code in the direct product of two complete graphs. In this paper we completely solve this problem.

The remainder of the paper is organized as follows. We first give some useful definitions and terminology. In Section 2 we state the main results which give the cardinality of a minimum identifying code for the direct product of any two nontrivial cliques. Section 3 is devoted to deriving some important properties that will be useful in showing that a set of vertices is an ID code in a direct product of two cliques. The proofs of the main results are given in Section 4.

1.1 Definitions and Notation

Given a simple undirected graph G and a vertex x of G, we let $N(x)$ denote the *open neighborhood* of x, that is, the set of vertices adjacent to x. The *closed neighborhood* of x is $N[x] = N(x) \cup \{x\}$. By a *code* in G we mean any nonempty subset of vertices in G. The vertices in a code are called *codewords*. A code D in G is a *dominating set* of G if D has a nonempty intersection with the closed neighborhood of every vertex of G. A code D *separates* two distinct vertices x and y if $N[x] \cap D \neq N[y] \cap D$. When $D = \{u\}$ we say that u separates x and y. An *identifying code* (*ID code* for short) of G is a code C that is a dominating set of G with the additional property that C separates every pair of distinct vertices of G. The minimum cardinality of an ID code of G is denoted $\gamma^{\mathrm{ID}}(G)$. Note that any graph having two vertices with the same closed neighborhood (so-called *twins*) does not have an ID code.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *direct product* of G_1 and G_2, denoted $G_1 \times G_2$, is the graph whose vertex set is the Cartesian product, $V_1 \times V_2$, and whose edge set is $E(G_1 \times G_2) = \{(u_1, u_2)(v_1, v_2) \,|\, u_1v_1 \in E_1 \text{ and } u_2v_2 \in E_2\}$. Direct products have been studied for some time, and extensive information on their structural properties can be found in [11].

For a positive integer n we write $[n]$ to denote the set $\{1, 2, \ldots, n\}$, and $[n]$ will be the vertex set of the complete graph K_n. In the direct product $K_n \times K_m$ we refer to a *column* as the set of all vertices having the same first coordinate. A *row* is the set of all vertices with the same second coordinate. In particular, for $i \in [n]$, the i^{th} column is $C_i = \{(i, j) \,|\, j \in [m]\}$. Similarly, for $j \in [m]$ the j^{th} row is the set $R_j = \{(i, j) \,|\, i \in [n]\}$. Using this terminology we see that two vertices of $K_n \times K_m$ are adjacent precisely when they belong to different rows and to different columns. In any figures rows will be horizontal and columns vertical. For ease of reference in this paper we refer to K_n as the *first factor* of $K_n \times K_m$ and K_m as the *second factor*. The two product graphs $K_n \times K_m$ and $K_m \times K_n$ are clearly isomorphic under a natural map. Throughout the remainder of this work we always have the smaller factor first.

Let $G = K_n \times K_m$, and suppose that C is a code in G. The *column span of C* is the set of all columns of G that have a nonempty intersection with C. The number of columns in the column span of C is denoted by $cs(C)$. Similarly, the set of all rows of G that contain at least one member of C is the *row span of* C; its size is denoted $rs(C)$. For a vertex $v = (i, j)$ of G we say that v is *column-isolated in* C if $C \cap C_i = \{v\}$. Similarly, if $C \cap R_j = \{v\}$ then we say that v is *row-isolated in* C. If v is both column-isolated and row-isolated in C, we simply say v is *isolated in* C. When there is no chance of confusion and the set C is clear from the context we shorten these to column-isolated, row-isolated and isolated, respectively.

2 Main Results

Recently, Goddard and the second author determined the minimum cardinality of an identifying code for the Cartesian product of two nontrivial complete graphs [8].

Theorem 1. [8] *For* $2 \le n \le m$, *we have*

$$\gamma^{\mathrm{ID}}(K_n \Box K_m) = \begin{cases} m + \lfloor n/2 \rfloor & \text{if } m \le 3n/2, \\ 2m - n & \text{if } m \ge 3n/2. \end{cases}$$

In this paper we determine the minimum cardinality of an identifying code for the direct product of any two nontrivial complete graphs. Note that the direct product of two complete graphs is the complement of the Cartesian product of those same complete graphs. However, the orders of the identifying codes for these pairs of graphs are quite different. The remainder of this section contains the summary of the exact results.

Note that $K_2 \times K_2$ has vertices with identical closed neighborhoods and so has no ID code.

Theorem 2. *For any positive integer* $m \ge 5$, $\gamma^{\mathrm{ID}}(K_2 \times K_m) = m - 1$. *In addition, if* $3 \le m \le 4$, $\gamma^{\mathrm{ID}}(K_2 \times K_m) = m$.

For $3 \le n \le 5$ and $n \le m \le 2n - 1$ the values of $\gamma^{\mathrm{ID}}(K_n \times K_m)$ were computed by computer program and are given in the following table.

$n\backslash m$	3	4	5	6	7	8	9
3	4	4	5				
4		5	6	7	7		
5			6	7	8	9	9

Table 1: $\gamma^{\mathrm{ID}}(K_n \times K_m)$ for small n and m

The remaining cases are handled based on the order of the second factor relative to the first factor. Theorem 3 presents this number if both cliques have order at least 3 and one clique is sufficiently large compared to the other; its proof is given in Section 4.

Theorem 3. *For positive integers n and m where $n \geq 3$ and $m \geq 2n$,*

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = m - 1\,.$$

In all other cases (that is, for $6 \leq n \leq m \leq 2n-1$), the minimum cardinality of an ID code for $K_n \times K_m$ is one of the values $\lfloor 2(n+m)/3 \rfloor$ or $\lceil 2(n+m)/3 \rceil$. The number $\gamma^{\mathrm{ID}}(K_n \times K_m)$ depends on the congruence of $n+m$ modulo 3. It turns out there are only two general cases instead of three, but one of them has an exception to the easily stated formula. The exact values are given in the following results whose proofs are given in Section 4.

Theorem 4. *Let n and m be positive integers such that $6 \leq n \leq m \leq 2n-1$. If $n+m \equiv 0 \pmod 3$ or $n+m \equiv 2 \pmod 3$, then*

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = \left\lfloor \frac{2m+2n}{3} \right\rfloor\,.$$

Theorem 5. *For a positive integer $n \geq 6$,*

$$\gamma^{\mathrm{ID}}(K_n \times K_{2n-5}) = 2n - 4\,.$$

Theorem 6. *Let n and m be positive integers such that $6 \leq n \leq m \leq 2n-2$ and $m \neq 2n-5$. If $n+m \equiv 1 \pmod 3$, then*

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = \left\lceil \frac{2m+2n}{3} \right\rceil\,.$$

3 Preliminary Properties

In this section we prove a number of results that will be useful in verifying the minimum size of ID codes in the direct product of two complete graphs. It will be helpful in what follows to remember that a vertex is adjacent to (i,j) in $K_n \times K_m$ precisely when its first coordinate is different from i and its second coordinate is different from j. Also, recall that we are assuming throughout that $n \leq m$.

Lemma 7. *If C is an identifying code of $K_n \times K_m$, then $cs(C) \geq n-1$ and $rs(C) \geq m-1$. In particular, $|C| \geq m-1$.*

Proof. Suppose that for some $r \neq s$, $C \cap R_r = \emptyset = C \cap R_s$. For any fixed $i \in [n]$, $C \cap N[(i,r)] = C \setminus C_i = C \cap N[(i,s)]$. Since this violates C being an ID code, $K_n \times K_m$ has at most one row disjoint from C. A similar argument shows that $K_n \times K_m$ has no more than one column disjoint from C. Consequently, $|C| \geq m-1$. □

By considering $N[x]$, the following result is obvious but useful. We omit its proof.

Lemma 8. *If $C \subseteq V(K_n \times K_m)$ and $x = (i,r) \in C$, then C separates x from any $y \in (R_r \cup C_i) \setminus \{x\}$.*

Lemma 8 addresses separating two vertices that belong to the same row or to the same column. The next result concerns vertices that are not in a common row or common column, that is, two vertices at opposite "corners" of a two-row and two-column configuration in $K_n \times K_m$.

Lemma 9. *(4-Corners Property) Suppose C is a dominating set of $K_n \times K_m$. For each $(i,r),(j,s) \in K_n \times K_m$ with $i \neq j, r \neq s$, C separates (i,r) and (j,s) if and only if*

$$C \cap (C_i \cup C_j \cup R_r \cup R_s) \nsubseteq \{i,j\} \times \{r,s\}.$$

Proof. Suppose that $i \neq j$ and $r \neq s$, and let C_i, C_j and R_r, R_s be the corresponding columns and rows of $K_n \times K_m$. Write $x = (i,r), y = (j,s), w = (i,s)$ and $z = (j,r)$, and define

$$\begin{aligned} A &= C \setminus (C \cap (C_i \cup C_j \cup R_r \cup R_s)) \\ B &= [C \cap (C_i \cup C_j \cup R_r \cup R_s)] \setminus \{x,y,w,z\}. \end{aligned}$$

Observe that

$$\begin{aligned} C \cap N[x] &= A \cup (C \cap \{x,y\}) \cup (C \cap ((R_s \cup C_j) \setminus \{x,y,w,z\})) \\ C \cap N[y] &= A \cup (C \cap \{x,y\}) \cup (C \cap ((R_r \cup C_i) \setminus \{x,y,w,z\})). \end{aligned}$$

Therefore, C separates x and y if and only if at least one of the two disjoint sets $C \cap ((R_s \cup C_j) \setminus \{x,y,w,z\})$ or $C \cap ((R_r \cup C_i) \setminus \{x,y,w,z\})$ is non-empty. Since B is the union of these 2 sets, it follows that C separates x and y if and only if $B \neq \emptyset$, or equivalently if and only if

$$C \cap (C_i \cup C_j \cup R_r \cup R_s) \nsubseteq \{i,j\} \times \{r,s\}.$$

□

We will say that a dominating set D of $K_n \times K_m$ has the *4-corners property with respect to columns C_i, C_j and rows R_r, R_s* if

$$D \cap (C_i \cup C_j \cup R_r \cup R_s) \nsubseteq \{i,j\} \times \{r,s\}.$$

Hence, if a dominating set D of $K_n \times K_m$ is an ID code, then D has the 4-corners property with respect to every pair of columns and every pair of rows. Each of the next three results follows immediately from this fact.

Corollary 10. *If C is an identifying code of $K_n \times K_m$, then C has no more than one isolated codeword.*

Corollary 11. *Let C be an identifying code of $K_n \times K_m$. If $cs(C) = n-1$, then there does not exist a column C_j such that $C \cap C_j = \{u,v\}$ where both u and v are row-isolated. Similarly, there is no row R_r containing exactly two codewords each of which is column-isolated if $rs(C) = m-1$.*

Corollary 12. *If C is an identifying code of $K_n \times K_m$ such that $cs(C) = n-1$ and $rs(C) = m-1$, then C has no isolated codeword.*

The next two results will be used to construct ID codes, thereby providing an upper bound for $\gamma^{\text{ID}}(K_n \times K_m)$. Which one is used will depend on the congruence of $n+m$ modulo 3.

Proposition 13. *If $C \subseteq V(K_n \times K_m)$ satisfies the following conditions, then C is an identifying code of $K_n \times K_m$.*

(1) There exist $1 \le n_1 < n_2 < n_3 \le n$ and $1 \le m_1 < m_2 < m_3 \le m$ such that $(n_1, m_1), (n_2, m_2), (n_3, m_3) \in C$;

(2) C contains at most one isolated vertex, and every other vertex in C is row-isolated or column-isolated; and

(3) $rs(C) = m$ and $cs(C) = n$.

Proof. Assume C is as specified. For ease of reference we denote the graph $K_n \times K_m$ by G throughout this proof. By the first assumption above it follows immediately that C dominates G since $\{(n_1, m_1), (n_2, m_2), (n_3, m_3)\}$ does.

We need only to show that C separates every pair x, y of distinct vertices. First assume that x and y are in the same column. If x or y belongs to C, then Lemma 8 shows that C separates them. If neither is in C, then by our assumption that $rs(C) = m$ and $cs(C) = n$ we can choose a vertex $z \in C$ from the same row as x. This vertex z separates x and y. Similarly, C separates any two vertices belonging to a common row.

Now, assume $x = (i, r)$ and $y = (j, s)$ where $1 \le i < j \le n$ and $1 \le r < s \le m$. Any $v = (k, t) \in C$ that is not isolated in C is row-isolated or column-isolated but not both, and it follows that either $|C \cap C_k| \ge 2$ or $|C \cap R_t| \ge 2$.

(a) Suppose $x \in C$ but is not isolated in C. As above, either $|C \cap C_i| \ge 2$ or $|C \cap R_r| \ge 2$. Assume without loss of generality that $|C \cap C_i| \ge 2$. It follows that either $(i, s) \in C$ or there exists $1 \le t \le m$ where $t \notin \{r, s\}$ and $(i, t) \in C$. In the first case where we have $(i, s) \in C$, it follows that (i, s) is row-isolated, and thus $y \notin C$. However, each column of G is in the column span of C so there exists $1 \le p \le m$ where $p \notin \{r, s\}$ and $(j, p) \in C$ since (i, r) and (i, s) are row-isolated. Thus, $(j, p) \in C \cap N[x]$ but $(j, p) \notin C \cap N[y]$. Hence, C separates x and y. On the other hand, if there exists $1 \le t \le m$ where $t \notin \{r, s\}$ and $(i, t) \in C$, then $(i, t) \in C \cap N[y]$ but $(i, t) \notin C \cap N[x]$. Again, this implies that C separates x and y. If we had instead assumed that $|C \cap R_r| \ge 2$, that is we had assumed x is column-isolated and not row-isolated, then a similar argument shows that C separates x and y.

(b) Suppose $x \in C$ and is isolated in C. Since x is both row-isolated and column-isolated, $C = C \cap N[x]$. First assume that $y \notin C$. Since C_j is in the column span of C, there exists $1 \le t \le m$ with $t \notin \{r, s\}$ such that $(j, t) \in C$, and (j, t) separates x and y. On the other hand, if $y \in C$, then either $|C \cap C_j| \ge 2$ or $|C \cap R_s| \ge 2$ since y is not isolated. In either case, $C \cap N[y] \ne C$, and therefore C separates x and y.

(c) Suppose $x, y \in V(G) \setminus C$. If we assume that C does not separate x and y, then because each row of G is in the row span of C and each column of G is in the column span of C, it follows that

$$C \cap (C_i \cup C_j \cup R_r \cup R_s) = \{(i,s),(j,r)\}.$$

Thus, by definition, both (i,s) and (j,r) are isolated in C, contradicting the second assumption. Hence, C separates x and y.

Therefore, C separates every pair of distinct vertices, and thus C is an ID code of $K_n \times K_m$. □

Proposition 14. *If $C \subseteq V(K_n \times K_m)$ satisfies the following conditions, then C is an identifying code of $K_n \times K_m$.*

(1) There exist $1 \leq n_1 < n_2 < n_3 \leq n$ and $1 \leq m_1 < m_2 < m_3 \leq m$ such that $(n_1, m_1), (n_2, m_2), (n_3, m_3) \in C$;

(2) C contains at most one isolated vertex, and every other vertex in C is row-isolated or column-isolated;

(3) $rs(C) = m - 1$ and $cs(C) = n$; and

(4) If R_r has the property that every $v \in C \cap R_r$ is column-isolated but not row-isolated, then $|C \cap R_r| \geq 3$.

Proof. As in the proof of Proposition 13 we see that C dominates $G = K_n \times K_m$.

We show that C separates every pair x, y of distinct vertices in G. Let R_r be the row not in the row span of C. Notice that $G \setminus R_r \cong K_n \times K_{m-1}$ and that C satisfies the hypotheses of Proposition 13 when considered as a subset of $V(G) \setminus R_r$. Thus, C separates x, y if neither is in R_r, and so we may assume that $x \in R_r$, say $x = (i, r)$.

(a) First assume that $y = (j, r)$ with $i \neq j$. Since $cs(C) = n$, there exists $1 \leq s \leq m$ such that $r \neq s$ and $(i, s) \in C$. This vertex (i, s) separates x and y. Next, assume that $y = (i, t)$ for some $1 \leq t \leq m$ with $t \neq r$. If $y \in C$, then y separates x and y. However, if $y \notin C$, then since each row of G other than R_r is in the row span of C, there exists $1 \leq j \leq n$ with $i \neq j$ such that $(j, t) \in C$. It follows that (j, t) separates x and y.

(b) Next, assume that $y = (j, s)$ where $i \neq j$ and $r \neq s$. If we assume that C does not separate x and y, then C does not satisfy the 4-Corners Property with respect to columns C_i, C_j and rows R_r, R_s. In addition, since R_r is not in the row span of C,

$$C \cap (C_i \cup C_j \cup R_r \cup R_s) \subseteq \{(i,s),(j,s)\}.$$

Since both C_i and C_j are in the column span of C, it follows that $C \cap (C_i \cup C_j \cup R_r \cup R_s) = \{(i,s),(j,s)\}$. This means that R_s contains exactly two members of C and they are both column-isolated, contradicting one of the assumptions. Hence, this case cannot occur either, and it follows that C separates x and y.

Therefore, C is an ID code of $K_n \times K_m$. □

4 Proofs of Main Results

In this section we prove all of our main results. The general strategy will be to construct an ID code of the claimed optimal size (by employing Propositions 13 and 14) and prove the given direct product has no smaller ID code.

We treat the smallest case first.

Theorem 1. *For any positive integer $m \geq 5$, $\gamma^{\mathrm{ID}}(K_2 \times K_m) = m - 1$. In addition, if $3 \leq m \leq 4$, $\gamma^{\mathrm{ID}}(K_2 \times K_m) = m$.*

Proof. If C is any ID code of $K_2 \times K_3$, then by Lemma 7 it follows that $rs(C) \geq 2$. No subset of two elements in different rows dominates $K_2 \times K_3$, and so $\gamma^{\mathrm{ID}}(K_2 \times K_3) \geq 3$. It is easy to check that $\{(1,1),(1,2),(1,3)\}$ is an ID code. A similar argument shows that $\gamma^{\mathrm{ID}}(K_2 \times K_4) = 4$.

If $m \geq 5$, it follows from Lemma 7 that $\gamma^{\mathrm{ID}}(K_2 \times K_m) \geq m - 1$, and it is easily checked that $\{(1,1),(1,2)\} \cup \{(2,r) \,|\, 3 \leq r \leq m-1\}$ is an ID code. □

Now we turn our attention to the case when the first factor has order at least 3 and the second factor is sufficiently larger than the first.

Theorem 2. *For positive integers n and m where $n \geq 3$ and $m \geq 2n$,*

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = m - 1\,.$$

Proof. Consider the set

$$D = \{(i, 2i-1), (i, 2i) \,|\, i \in [n-1]\} \;\cup\; \{(n, j) \,|\, 2n-1 \leq j \leq m-1\}.$$

Notice that each v in D is row-isolated but not column-isolated, $rs(D) = m - 1$ and $cs(D) = n$. Furthermore, $(1,1), (2,3)$ and $(3,5)$ are in D. Thus, Proposition 14 guarantees that D is an ID code, and Lemma 7 gives the desired result. □

We now focus on direct products of the form $K_n \times K_m$ where $6 \leq n \leq m \leq 2n-1$ and prove that in all cases

$$\left\lfloor \frac{2m+2n}{3} \right\rfloor \leq \gamma^{\mathrm{ID}}(K_n \times K_m) \leq \left\lceil \frac{2m+2n}{3} \right\rceil\,. \tag{1}$$

For the remainder of this paper, when considering any ID code C of $G = K_n \times K_m$ we define

$$A_C = \{v \in C \,|\, v \text{ is row-isolated in } C\}$$

and

$$B_C = \{v \in C \,|\, v \text{ is column-isolated in } C\}\,.$$

Let $|A_C| = x$, and let p denote the number of columns C_i of G such that $|C \cap C_i| \geq 2$ and $C \cap C_i \subseteq A_C$. Similarly, let $|B_C| = y$, and let q represent the number of rows R_r of G such that $|C \cap R_r| \geq 2$ and $C \cap R_r \subseteq B_C$. Notice that C contains at most one isolated codeword, in which case $|A_C \cap B_C| = 1$. Otherwise, $A_C \cap B_C = \emptyset$. Moreover, we always have $|C| \geq |A_C \cup B_C| \geq x + y - 1$.

The approach we take in the proof of Theorem 3, Theorem 5 and Theorem 6 will be to show that any code of cardinality smaller than the claimed value will violate some consequence of the 4-Corners Property. Which consequence will depend on the particular cardinalities of the row span and column span.

Theorem 3. *If n and m are positive integers such that $6 \leq n \leq m \leq 2n - 1$ and $n + m \equiv 0 \pmod 3$ or $n + m \equiv 2 \pmod 3$, then*

$$\gamma^{\text{ID}}(K_n \times K_m) = \left\lfloor \frac{2m + 2n}{3} \right\rfloor .$$

Proof. Suppose C is an ID code of $G = K_n \times K_m$ such that $|C| \leq \left\lfloor \frac{2n+2m}{3} \right\rfloor - 1$. We consider four cases based on the possible values of $cs(C)$ and $rs(C)$.

Case 1 Suppose $cs(C) = n$ and $rs(C) = m$.
Since $cs(C) = n$ and $|B_C| = y$, there are $n - y$ columns that each contain at least two codewords. Thus, $|C \setminus B_C| \geq 2(n - y)$, which implies $\frac{2m+2n}{3} - 1 \geq |C| \geq 2n - y$. It follows that $y \geq \frac{4n-2m}{3} + 1$. Similarly, we get $x \geq \frac{4m-2n}{3} + 1$. Together these imply that

$$\frac{2m + 2n}{3} - 1 \geq |C| \geq x + y - 1 \geq \frac{2m + 2n}{3} + 1 .$$

This is clearly a contradiction, and hence no such C exists with $cs(C) = n$ and $rs(C) = m$.

Case 2 Suppose $cs(C) = n - 1$ and $rs(C) = m$.
Note that since each codeword in B_C is column-isolated and $cs(C) = n - 1$, there exist at least two codewords in each of the remaining $n - 1 - y$ columns disjoint from the column span of B_C. However, Corollary 11 guarantees that $|C \cap C_j| \geq 3$ for any column C_j for which $|C \cap C_j| \geq 2$ and $C \cap C_j \subseteq A_C$. Since p represents the number of such columns, $|C \setminus B_C| \geq 2(n - 1 - y - p) + 3p = 2n - 2 - 2y + p$. Consequently, $|C| \geq 2n - 2 - y + p$, and it follows that $y \geq \frac{4n-2m}{3} - 1 + p$.

Similarly, since $|A_C| = x$ and $rs(C) = m$, $|C \setminus A_C| \geq 2(m - x)$, which implies $|C| \geq 2m - x$. From Case 1 we see that this gives $x \geq \frac{4m-2n}{3} + 1$. Moreover, $|C| \geq x + y - 1$ so that

$$\frac{2m + 2n}{3} - 1 \geq |C| \geq x + y - 1 \geq \frac{2m + 2n}{3} + p - 1 .$$

Hence, $p = 0$, and we have equality in the above so that

$$\left\lfloor \frac{2m + 2n}{3} \right\rfloor - 1 = |C| = x + y - 1.$$

It follows that $C = A_C \cup B_C$. If there exists $v \in C \setminus B_C$, say $v \in C_i$, then C_i contains an additional codeword that is also row-isolated. Hence, p is at least 1. However, this contradicts $p = 0$ since each codeword is either row-isolated or column-isolated. Consequently, $m = rs(C) \le |C| = |B_C| \le n - 1 \le m - 1$. This contradiction shows that this case cannot occur.

Case 3 Suppose $cs(C) = n$ and $rs(C) = m - 1$.

If we interchange the roles of rows and columns in Case 2, then we are led to $q = 0$ and

$$\left\lfloor \frac{2m + 2n}{3} \right\rfloor - 1 = |C| = x + y - 1 \,.$$

Thus, $C = A_C \cup B_C$. On the other hand, since $cs(C) = n$ it follows as in Case 1 that

$$y \ge \frac{4n - 2m}{3} + 1 \ge \frac{4n - 2(2n - 1)}{3} + 1 = \frac{5}{3} \,.$$

Since y is an integer, we see that C has at least two column-isolated codewords. One of these, say v, is isolated since $|C| = x + y - 1$. Let w be a column-isolated codeword with $w \ne v$, and assume that $w \in R_j$. Since w is not isolated but is column-isolated, R_j contains another codeword besides w. All codewords in R_j are therefore in B_C, and thus $q \ge 1$. This contradiction shows that this case cannot occur.

Case 4 Suppose that $cs(C) = n - 1$ and $rs(C) = m - 1$.
From Case 2 and Case 3, we see that

$$y \ge \frac{4n - 2m}{3} - 1 + p \quad \text{and} \quad x \ge \frac{4m - 2n}{3} - 1 + q \,.$$

Since $cs(C) = n - 1$ and $rs(C) = m - 1$, it follows from Corollary 12 that C does not contain an isolated vertex. It follows that

$$\frac{2m + 2n}{3} - 1 \ge |C| \ge x + y \ge \frac{2m + 2n}{3} - 2 + p + q \,.$$

Hence, $p + q \le 1$.

Suppose $p = 1$. Consequently, we have equality throughout the above inequality, and thus $C = A_C \cup B_C$. Suppose there exists $v \in B_C$, say $v \in R_r$. Since $q = 0$ and there are no isolated codewords, it follows that C contains another codeword u in R_r that is not column-isolated. But $u \notin A_C \cup B_C$, which is a contradiction. Therefore, $C = A_C$. Since $p = 1$ we are led to conclude that $cs(C) = 1$, which is another contradiction.

To show that $q = 1$ is not possible we simply interchange the roles of A_C and B_C in the above.

Finally, suppose $p = 0 = q$. Since $p = 0$, any column that contains a row-isolated codeword would also have to contain a codeword that is not row-isolated. Since there can exist at most one of these to guarantee $|C| \leq \lfloor \frac{2m+2n}{3} \rfloor - 1$, there is a column C_i such that $A_C \subseteq C_i$, and for some r, $(i,r) \in C \setminus (A_C \cup B_C)$. Similarly, since $q = 0$, if there exists a row containing a column-isolated codeword, then that row contains a codeword that is not column-isolated. Since $|C \setminus (A_C \cup B_C)| \leq 1$, such a codeword must be (i,r). This implies that $\frac{2m+2n}{3} - 1 \geq |C| \geq m - 1 + n - 2$, and this implies that $n + m \leq 6$, contradicting our assumption.

Therefore, every ID code of $K_n \times K_m$ has cardinality at least $\lfloor \frac{2m+2n}{3} \rfloor$.

An application of Proposition 13 shows that the following sets are ID codes of cardinality $\lfloor \frac{2m+2n}{3} \rfloor$ and finishes the proof. See Figure 1 for several specific instances of these constructions.

If $n + m \equiv 0 \pmod 3$, let

$$D_1 = \{(i, 2i-1), (i, 2i) | 1 \leq i \leq a\} \cup \{(a+2j-1, 2a+j), (a+2j, 2a+j) | 1 \leq j \leq b\},$$

where $a = \frac{2m-n}{3}$ and $b = \frac{2n-m}{3}$. For $n+m \equiv 2 \pmod 3$ but $m \neq 2n-1$, let $a = \frac{2m-n-1}{3}$, $b = \frac{2n-m-1}{3}$, and

$$D_2 = \{(i, 2i-1), (i, 2i) \big| 1 \leq i \leq a\} \cup \{(a+2j-1, 2a+j), (a+2j, 2a+j) \big| 1 \leq j \leq b\} \cup \{(n, m)\}.$$

Finally, if $m = 2n - 1$, let

$$D_3 = \{(i, 2i-1), (i, 2i) | i \in [n-1]\} \ \cup \ \{(n, 2n-1)\}.$$

□

The following figure illustrates ID codes of optimal order for several of the cases of Theorem 3. The vertices of the direct products in the figure are represented, but the edges are omitted for clarity. Recall that columns are vertical and rows are horizontal. Solid vertices indicate the members of an optimal ID code in each case.

(a) $K_6 \times K_6$

(b) $K_6 \times K_8$

Figure 1: Examples of ID codes when $n + m \equiv 0, 2 \pmod 3$

For a fixed $n \geq 6$, the lone exception to the formula $\lceil \frac{2m+2n}{3} \rceil$ for $\gamma^{\text{ID}}(K_n \times K_m)$ where $n \leq m \leq 2n-2$ and $n+m$ congruent to 1 modulo 3 is the instance $m = 2n-5$. We now prove Theorem 5, which shows the correct value is $\lfloor \frac{2(2n-5)+2n}{3} \rfloor$. We restate it here for convenience.

Theorem 4. *For a positive integer $n \geq 6$,*

$$\gamma^{\mathrm{ID}}(K_n \times K_{2n-5}) = 2n - 4\,.$$

Proof. Assume there exists an ID code C for $K_n \times K_{2n-5}$ such that $|C| \leq 2n-5$. Since $rs(C) \geq 2n-6$, we consider the following two cases.

Case 1 Suppose that $rs(C) = 2n-6$.
Since each codeword in A_C is row-isolated and $rs(C) = 2n-6$, there exist at least two codewords in each of the remaining $2n-6-x$ rows disjoint from the row span of A_C. However, Corollary 11 guarantees that $|C \cap R_r| \geq 3$ for any row R_r where $C \cap R_r \subseteq B_C$. Since q represents the number of these rows, $|C \setminus A_C| \geq 2(2n-6-x-q)+3q$, which implies $|C| \geq 4n-12-x+q$. Consequently, $2n-5 \geq 4n-12-x+q$, which implies $x \geq 2n-7+q$.

Similarly, since $cs(C) \geq n-1$ and each codeword in B_C is column-isolated, there are at least $n-1-y$ columns disjoint from the column span of B_C that each contain at least two codewords. Thus, $|C \setminus B_C| \geq 2(n-1-y)$, which implies that $|C| \geq 2n-2-y$. Therefore, $y \geq 3$. It follows that

$$2n-5 \geq |C| \geq x+y-1 \geq 2n-5+q.$$

Thus, $q = 0$. Moreover, we have equality in the above, and therefore $C = A_C \cup B_C$. On the other hand, $y \geq 3$ and only one of these column-isolated codewords can be isolated. Consequently, $q \geq 1$ since each codeword of C is either row-isolated or column-isolated, which is a contradiction.

Case 2 Suppose $rs(C) = 2n-5$.
Using a similar argument as in Case 1, we have $|C \setminus A_C| \geq 2(2n-5-x)$, which implies $|C| \geq 4n-10-x$. This implies $2n-5 \geq |C| \geq x \geq 2n-5$. Therefore, it follows that $C = A_C$, and thus $cs(C) = cs(A_C) \leq \frac{2n-6}{2}+1 = n-2$, contradicting Lemma 7.

Therefore, no such identifying code C exists with $|C| \leq 2n-5$. It follows that $\gamma^{\mathrm{ID}}(G) \geq 2n-4$.

An application of Proposition 14 shows that the set

$$D = \{(i,2i-1),(i,2i) \big| 1 \leq i \leq n-4\} \cup \{(n-3,2n-7),(n-2,2n-7),(n-1,2n-7),(n,2n-6)\}$$

is an ID code of $K_n \times K_{2n-5}$ of cardinality $2n-4$. □

Theorem 5. *Let n and m be positive integers such that $6 \leq n \leq m \leq 2n-2$ and $m \neq 2n-5$. If $n+m \equiv 1 \pmod 3$, then*

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = \left\lceil \frac{2m+2n}{3} \right\rceil\,.$$

Proof. Notice that $\lceil \frac{2m+2n}{3} \rceil = \frac{2m+2n+1}{3}$. Assume that there exists an ID code C for $K_n \times K_m$ such that $|C| \leq \frac{2n+2m+1}{3} - 1$. We again consider four cases based on the possible values of $cs(C)$ and $rs(C)$.

Case 1 Suppose $cs(C) = n$ and $rs(C) = m$.

Using reasoning similar to that in Case 1 of the proof of Theorem 3, we have $|C \setminus B_C| \geq 2(n-y)$. This implies that $|C| \geq 2n - y$, and hence

$$\frac{2m + 2n + 1}{3} - 1 \geq |C| \geq 2n - y\,.$$

It follows that $y \geq \frac{4n-2m+2}{3}$. Similarly, we have that $x \geq \frac{4m-2n+2}{3}$. On the other hand, we know $|C| \geq x + y - 1$. Consequently, $\frac{2m+2n+1}{3} - 1 \geq x + y - 1 \geq \frac{2m+2n+1}{3}$, which is clearly a contradiction.

Case 2 Suppose $cs(C) = n - 1$ and $rs(C) = m$.

Since $|B_C| = y$ and $cs(C) = n-1$, there exist at least two codewords in each of the remaining $n - 1 - y$ columns that are disjoint from the column span of B_C. However, Corollary 11 guarantees $|C \cap C_j| \geq 3$ for any such column C_j where $C \cap C_j \subseteq A_C$. Since p represents the number of these columns, $|C \setminus B_C| \geq 2(n - 1 - y - p) + 3p = 2n - 2 - 2y + p$. As a result it follows that $y \geq \frac{4n-2m-4}{3} + p$.

Similarly, since $rs(C) = m$ and $x = |A_C|$ we get $|C \setminus A_C| \geq 2(m - x)$, which implies $|C| \geq 2m - x$. As in Case 1 it follows that $x \geq \frac{4m-2n+2}{3}$. This yields

$$\frac{2m + 2n + 1}{3} - 1 \geq |C| \geq x + y - 1 \geq \frac{2m + 2n + 1}{3} + p - 2\,.$$

Thus, $p \leq 1$. Assume first that $p = 1$. This yields equality in the above, and thus $C = A_C \cup B_C$, $y = \frac{4n-2m-1}{3}$ and $x = \frac{4m-2n+2}{3}$. Furthermore, C contains an isolated codeword, call it v. Since $p = 1$, there exists a column C_i such that $A_C \setminus \{v\} = C \cap C_i$. It follows that $cs(A_C) = 2$. On the other hand, $cs(C) = n - 1$ so $B_C \setminus \{v\}$ spans the remaining $n - 3$ columns. Therefore, $n - 3 = \frac{4n-2m-1}{3} - 1$, which contradicts the assumption that $n \leq m$.

Therefore, we conclude that $p = 0$. First assume that C contains no isolated codeword. This implies

$$\frac{2m + 2n + 1}{3} - 1 \geq |C| \geq x + y \geq \frac{2m + 2n + 1}{3} + p - 1\,.$$

Since $p = 0$ we get equality throughout the above, and hence $C = A_C \cup B_C$. As in the proof of Case 2 of Theorem 3 we arrive at a contradiction. Therefore, C contains an isolated codeword, say v. Because $p = 0$, any column that contains a row-isolated codeword other than v would also have to contain a codeword that is not row-isolated. Furthermore, the fact that $p = 0$, together with

$$\frac{2m + 2n + 1}{3} - 1 \geq |C| \geq x + y - 1 \geq \frac{2m + 2n + 1}{3} + p - 2\,,$$

implies that there exists at most one such codeword that is neither row-isolated nor column-isolated. Note that $x \geq \frac{4m-2n+2}{3} \geq 5$. Therefore, the row-isolated vertices other than v are contained in precisely one column, say C_i. Hence, $A_C \setminus \{v\} \subseteq C \cap C_i$. We let (i, r) denote the codeword that is neither row-isolated nor column-isolated. This means $C = A_C \cup B_C \cup \{(i, r)\}$ and so $y = \frac{4n-2m-4}{3}$. It follows that $cs(A_C) = 2$. On the other hand, $cs(C) = n-1$ so $B_C \setminus \{v\}$ spans the remaining $n-3$ columns. Therefore, $n-3 = \frac{4n-2m-4}{3} - 1$, which implies $2m = n+2$, again contradicting the assumption that $n \leq m$.

Case 3 Suppose $cs(C) = n$ and $rs(C) = m - 1$.

Since $|A_C| = x$ and $rs(C) = m - 1$, there exist at least 2 codewords in each of the remaining $m-1-x$ rows disjoint from the row span of A_C. However, Corollary 11 guarantees $|C \cap R_r| \geq 3$ for any such row R_r where $C \cap R_r \subseteq B_C$. Since q represents the number of these rows, $|C \setminus A_C| \geq 2(m-1-x-q) + 3q = 2m - 2 - 2x + q$. This implies that $x \geq \frac{4m-2n-4}{3} + q$. Similarly, since $cs(C) = n$ and $|B_C| = y$ we get $|C \setminus B_C| \geq 2(n-y)$, which implies $|C| \geq 2n - y$. As in Case 1 it follows that $y \geq \frac{4n-2m+2}{3}$. Consequently,

$$\frac{2m+2n+1}{3} - 1 \geq |C| \geq x + y - 1 \geq \frac{2m+2n+1}{3} + q - 2.$$

Thus, $q \leq 1$. Assume first that $q = 1$. This gives equality in the above, and thus $C = A_C \cup B_C$, $y = \frac{4n-2m+2}{3}$ and $x = \frac{4m-2n-1}{3}$. Furthermore, C contains an isolated codeword, call it v. Since $q = 1$, there exists a row R_r such that $B_C \setminus \{v\} = C \cap R_r$. Thus, $rs(B_C) = 2$. On the other hand, $rs(C) = m - 1$ so $A_C \setminus \{v\}$ spans the remaining $m - 3$ rows. Therefore, $m - 3 = \frac{4m-2n-1}{3} - 1$, which contradicts the assumption that $m \neq 2n - 5$.

Therefore, $q = 0$. First assume C contains no isolated codeword. Consequently, $C = A_C \cup B_C$ and since $q = 0$, it follows that $C = A_C$. Since $cs(C) = n$ and no isolated codeword exists, it follows that $|C| \geq 2n$. Therefore, $\frac{2m+2n+1}{3} - 1 \geq 2n$, which implies $m \geq 2n + 1$. Because of this contradiction we conclude that C contains an isolated codeword, say v.

Because $q = 0$, any row that contains a column-isolated codeword other than v would also have to contain a codeword that is not column-isolated.

Furthermore, the fact that $q = 0$, together with

$$\frac{2m+2n+1}{3} - 1 \geq |C| \geq x + y - 1 \geq \frac{2m+2n+1}{3} + q - 2\,,$$

implies that there exists at most one such codeword that is neither row-isolated nor column-isolated. Note that $y \geq \frac{4n-2m+2}{3} \geq 2$. Therefore, the column-isolated vertices other than v are contained in precisely one row, say R_r, and hence $B_C \setminus \{v\} \subseteq C \cap R_r$. We let (i, r) denote the codeword that is neither row-isolated nor column-isolated. This means $C = A_C \cup B_C \cup \{(i, r)\}$ and so $x = \frac{4m-2n-4}{3}$. It follows that $rs(B_C) = 2$. On the other hand, $rs(C) = m - 1$ so $A_C \setminus \{v\}$ spans the remaining $m - 3$ rows. Therefore, $m - 3 = \frac{4m-2n-4}{3} - 1$, which implies $m = 2n - 2$. However, in this specific case $x = 2n - 4$ and $y = 2$. Since $A_C \cap B_C = \{v\}$, it follows that

$$n = cs(C) \leq \frac{|A_C \setminus \{v\}|}{2} + |B_C| = \frac{2n-4-1}{2} + 2 = n - \frac{1}{2}\,,$$

which is a contradiction.

Case 4 Suppose that $cs(C) = n - 1$ and $rs(C) = m - 1$.
From Case 2 and Case 3, we see that

$$y \geq \frac{4n-2m-4}{3} + p \quad \text{and} \quad x \geq \frac{4m-2n-4}{3} + q.$$

Since $cs(C) = n-1$ and $rs(C) = m-1$, it follows from Corollary 12 that C does not contain an isolated codeword. Thus,

$$\frac{2m+2n+1}{3} - 1 \geq |C| \geq x + y \geq \frac{2m+2n+1}{3} - 3 + p + q.$$

Hence, $p + q \leq 2$.

(i) Suppose that $p = 0$. For each column C_i where $A_C \cap C_i \neq \emptyset$, there will exist another codeword in C_i that is not row-isolated. To guarantee that $\frac{2m+2n+1}{3} - 1 \geq |C|$, C contains at most two such codewords. Therefore, $cs(A_C) \leq 2$. If $cs(A_C) = 2$, then $y = \frac{4n-2m-4}{3}$, and it follows that

$$n - 1 = cs(C) = cs(A_C) + cs(B_C) = 2 + \frac{4n-2m-4}{3}.$$

This contradicts the assumption that $m \geq n$, and thus $cs(A_C) < 2$. On the other hand, $x \geq \frac{4m-2n-4}{3} + q \geq \frac{8}{3}$. Hence, C contains precisely one codeword, say v, that is neither row-isolated nor column-isolated. This implies that $cs(A_C) = 1$, and if we let C_i represent the column containing these row-isolated vertices, then $v \in C_i$ and $cs(A_C \cup \{v\}) = 1$. Since

$$n - 1 = cs(C) = cs(A_C \cup \{v\}) + cs(B_C) = 1 + cs(B_C)\,,$$

we know $cs(B_C) = n-2$. Therefore, $y = n-2$ since each vertex of B_C is column-isolated. On the other hand, to guarantee $\frac{2m+2n+1}{3} - 1 \geq |C|$, it is the case that $y \leq \frac{4n-2m-4}{3} + 1$. Consequently, $n - 2 \leq \frac{4n-2m-4}{3} + 1$, which again implies that $m < n$. This contradiction shows that $p \neq 0$.

(ii) Suppose that $q = 0$. For each row R_r where $B_C \cap R_r \neq \emptyset$, there will exist another codeword in R_r that is not column-isolated. Since $p \neq 0$, C contains at most one such codeword and it follows that $rs(B_C) \leq 1$. On the other hand, $y \geq \frac{4n-2m-4}{3} + p \geq p \geq 1$. This implies $rs(B_C) = 1$, and C contains precisely one codeword, say v, that is neither row-isolated nor column-isolated. Since v is in the same row as the vertices of B_C, $rs(B_C \cup \{v\}) = 1$. This implies

$$m - 1 = rs(C) = rs(A_C) + rs(B_C \cup \{v\}) = rs(A_C) + 1\,,$$

and consequently $m - 2 = rs(A_C)$. Therefore, $x = m - 2$ since each vertex of A_C is row-isolated. On the other hand, since v is not column-isolated and $p = 1$, it follows that $cs(A_C \cup \{v\}) = 2$. Therefore,

$$n - 1 = cs(C) = cs(A_C \cup \{v\}) + cs(B_C) = 2 + cs(B_C)\,,$$

which implies $y = cs(B_C) = n - 3$. Combining these facts we get

$$|C| = |A_C \cup B_C \cup \{v\}| = x + y + 1 = m + n - 4\,.$$

However, $\frac{2m+2n+1}{3} - 1 \geq |C| = m + n - 4$, which implies $m + n \leq 10$. This contradicts our assumption that $n \geq 6$.

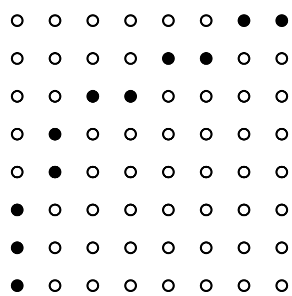
(iii) Since $p = 1$ and $q = 1$, then $x \geq \frac{4m-2n-4}{3} + 1$ and $y \geq \frac{4n-2m-4}{3} + 1$. It follows that

$$\frac{2m+2n+1}{3} - 1 \geq |C| \geq x + y \geq \frac{2m+2n+1}{3} - 1.$$

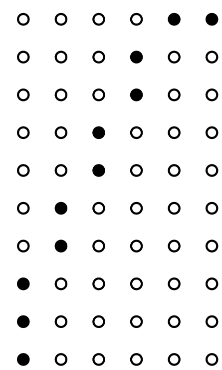
Thus, $C = A_C \cup B_C$. On the other hand, $cs(A_C) = 1$ since $p = 1$. Therefore, B_C spans the remaining $n-2$ columns since $cs(C) = n-1$. Hence, $n - 2 = \frac{4n-2m-4}{3} + 1$, which contradicts $m \geq n$.

Therefore, every ID code of $K_n \times K_m$ has cardinality at least $\lceil \frac{2m+2n}{3} \rceil$.

We now present ID codes to show that this lower bound is realized. Figure 2 contains examples of minimum cardinality ID codes for some cases covered in Theorem 6. As in Figure 1 the code consists of the solid vertices.



(a) $K_8 \times K_8$



(b) $K_6 \times K_{10}$

Figure 2: Several ID codes when $n + m \equiv 1 \pmod 3, m \neq 2n - 5$

If $m \neq 2n - 2$, let

$$D_1 = \{(1,1)\} \cup \{(i,2i),(i,2i+1) \,|\, 1 \leq i \leq a\} \cup \{(a+2j-1, 2a+j+1),(a+2j, 2a+j+1) \,|\, 1 \leq j \leq b\},$$

where $a = \frac{2m-n-2}{3}$ and $b = \frac{2n-m+1}{3}$. It is straightforward to check that D_1 satisfies the properties of Proposition 13 and is therefore an ID code of $K_n \times K_m$.
If $m = 2n - 2$, let

$$D_2 = \{(1,1)\} \cup \{(i,2i),(i,2i+1) \,|\, 1 \leq i \leq n-2\} \cup \{(n-1, 2n-2),(n, 2n-2)\}.$$

Again, one can verify that D_2 satisfies all properties of Proposition 13 and is therefore an ID code of $K_n \times K_{2n-2}$.

Therefore, if $m \neq 2n - 5$ but $n + m \equiv 1 \pmod 3$ and $6 \leq n \leq m \leq 2n - 2$, then

$$\gamma^{\mathrm{ID}}(K_n \times K_m) = \left\lceil \frac{2m+2n}{3} \right\rceil.$$

□

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