Domination game played on trees and spanning subgraphs

Boštjan Brešar*
Faculty of Natural Sciences and Mathematics
University of Maribor, Slovenia
bostjan.bresar@uni-mb.si

Sandi Klavžar*
Faculty of Mathematics and Physics
University of Ljubljana, Slovenia
and
Faculty of Natural Sciences and Mathematics
University of Maribor, Slovenia
sandi.klavzar@fmf.uni-lj.si

Douglas F. Rall†
Herman N. Hipp Professor of Mathematics
Department of Mathematics, Furman University
Greenville, SC, USA
doug.rall@furman.edu

January 7, 2013

Abstract

The domination game, played on a graph G, was introduced in [2]. Vertices are chosen, one at a time, by two players Dominator and Staller. Each chosen vertex must enlarge the set of vertices of G dominated to that point in the game. Both players use an optimal strategy—Dominator plays so as to end the game as quickly as possible, and Staller plays in such a way that the game lasts as many steps as possible. The game domination number $\gamma_g(G)$ is the number of vertices chosen when Dominator starts the game and the Staller-start game domination number $\gamma_g'(G)$ is the result when Staller starts the game.

*Supported by the Ministry of Science of Slovenia under the grants P1-0297 and J1-2043. The author is also with the Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana.

†Supported by the Wylie Enrichment Fund of Furman University and by a grant from the Simons Foundation (Grant Number 209654 to Douglas F. Rall). Part of the research done during a sabbatical visit at the University of Ljubljana and University of Maribor.

In this paper these two games are studied when played on trees and spanning subgraphs. A lower bound for the game domination number of a tree in terms of the order and maximum degree is proved and shown to be asymptotically tight. It is shown that for every k, there is a tree T with $(\gamma_g(T), \gamma_g'(T)) = (k, k+1)$ and conjectured that there is none with $(\gamma_g(T), \gamma_g'(T)) = (k, k-1)$. A relation between the game domination number of a graph and its spanning subgraphs is considered. It is proved that there exist 3-connected graphs G having a 2-connected spanning subgraph H such that the game domination number of H is arbitrarily smaller than that of G. Similarly, for any integer $\ell \geq 1$, there exists a graph G and a spanning tree T such that $\gamma_g(G) - \gamma_g(T) \geq \ell$. On the other hand, there exist graphs G such that the game domination number of any spanning tree of G is arbitrarily larger than that of G.

Keywords: domination game, game domination number, tree, spanning subgraph

AMS subject classification (2010): 05C57, 91A43, 05C69

1 Introduction

The domination game played on a graph G consists of two players, Dominator and Staller, who alternate taking turns choosing a vertex from G such that whenever a vertex is chosen by either player, at least one additional vertex is dominated. Dominator wishes to dominate the graph as fast as possible, and Staller wishes to delay the process as much as possible. The *game domination number*, denoted $\gamma_g(G)$, is the number of vertices chosen when Dominator starts the game provided that both players play optimally. Similarly, the *Staller-start game domination number*, written as $\gamma_g'(G)$, is the result of the game when Staller starts the game. The Dominator-start game and the Staller-start game will be briefly called *Game 1* and *Game 2*, respectively.

This game was first studied in 2010 ([2]) but was brought to the authors' attention back in 2003 by Henning [3]. Among other results, the authors of [2] proved a lower bound for the game domination number of the Cartesian product of graphs and established a connection with Vizing's conjecture; for the latter see [1]. The Cartesian product was further investigated in [6], where the behavior of $\lim_{\ell\to\infty} \gamma_g(K_m \,\square\, P_\ell)/\ell$ was studied in detail.

In the rest of this section we give some notation and definitions, and we recall results needed later. In Section 2 we prove a general lower bound for the game domination number of a tree. In Section 3 we consider which pairs of integers (r, s) can be realized as $(\gamma_g(T), \gamma_g'(T))$, where T is a tree. It is shown that this is the case for all pairs but those of the form $(k, k-1)$. This enlarges the set of pairs known to be realizable by *connected* graphs. We conjecture that the pairs $(k, k-1)$ cannot be realized by trees. In the final section we study relations between the game domination number of a graph and its spanning subgraphs. We construct graphs G having a spanning tree T with $\gamma_g(G) - \gamma_g(T)$ arbitrarily large and build

3-connected graphs having a 2-connected spanning subgraph exhibiting the same phenomenon. This is rather surprising because the domination number of a spanning tree (or a spanning subgraph) can never be smaller than the domination number of its supergraph. We also present graphs G_{2r} ($r \geq 1$) such that $\gamma_g(T) - \gamma_g(G_{2r}) \geq r - 1$ holds for any spanning tree T of G_{2r}. This is again different from the usual domination because it is known (see [5, Exercise 10.14]) that every graph contains a spanning tree with the same domination number.

Throughout the paper we will use the convention that $d_1, d_2, \ldots$ denotes the list of vertices chosen by Dominator and $s_1, s_2, \ldots$ the list chosen by Staller. We say that a pair (r, s) of integers is *realizable* if there exists a graph G such that $\gamma_g(G) = r$ and $\gamma_g'(G) = s$. Following [6], we make the following definitions. A *partially dominated graph* is a graph in which some vertices have already been dominated in some turns of the game already played. A vertex u of a partially dominated graph G is *saturated* if each vertex in $N[u]$ is dominated. The *residual graph* of G is the graph obtained from G by removing all saturated vertices and all edges joining dominated vertices. If G is a partially dominated graph, then $\gamma_g(G)$ and $\gamma_g'(G)$ denote the optimal number of moves remaining in Game 1 and Game 2, respectively (it is assumed here that Game 1, respectively Game 2, refers to Dominator, respectively Staller, being the first to play in the partially dominated graph G).

The game domination number of a graph G can be bounded in terms of the domination number $\gamma(G)$ of G:

Theorem 1.1 ([2]) *For any graph G, $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$.*

It was demonstrated in [2] that, in general, Theorem 1.1 cannot be improved. More precisely, for any positive integer k and any integer r such that $0 \leq r \leq k-1$, there exists a graph G with $\gamma(G) = k$ and $\gamma_g(G) = k + r$.

The game domination number and the Staller-start game domination number never differ by more than 1 as the next result asserts.

Theorem 1.2 ([2, 6]) *If G is any graph, then $|\gamma_g(G) - \gamma_g'(G)| \leq 1$.*

By Theorem 1.2 only pairs of the form (r, r), $(r, r+1)$, and $(r, r-1)$ are realizable. See [4] for a study of realizable pairs.

The following lemma, due to Kinnersley, West, and Zamani [6] in particular implies $\gamma_g'(G) \leq \gamma_g(G) + 1$, which is one half of Theorem 1.2. The other half was earlier proved in [2].

Lemma 1.3 (Continuation Principle) *Let G be a graph and let A and B be subsets of $V(G)$. Let G_A and G_B be partially dominated graphs in which the sets A and B have already been dominated, respectively. If $B \subseteq A$, then $\gamma_g(G_A) \leq \gamma_g(G_B)$ and $\gamma_g'(G_A) \leq \gamma_g'(G_B)$.*

We wish to point out that the Continuation Principle is a very useful tool for proving results about game domination number. In particular, suppose that at some stage of the game a vertex x is an optimal move for Dominator. If for some vertex y such that the undominated part of $N[x]$ is contained in $N[y]$, then y is also an optimal selection for Dominator. We can thus assume (if desired) that he plays y.

2 A lower bound for trees

In this section we give a lower bound on the game domination number of trees and show that it is asymptotically sharp. Before we can state the main result, we need the following:

Lemma 2.1 *In a partially dominated tree F, Staller can make a move that dominates at most two new vertices.*

Proof. Let A be the set of saturated vertices of F and let B be the set of vertices of F that are dominated but not saturated. Let $C = V(F) - (A \cup B)$. Let F' be the subforest of F induced by $B \cup C$ but with edges induced by B removed (that is, F' is the residual graph). We may assume that $C \neq \emptyset$. Now F' has a leaf x. If Staller plays x, then she dominates at most two vertices in C. If $x \in B$, then Staller dominates exactly one vertex in C. $\square$

The move guaranteed by Lemma 2.1 may not be an optimal move for Staller. For instance, the optimal first move of Staller when playing on P_5 is the middle vertex of P_5, thus dominating three new vertices. Also, we will see later that an optimal first move for Staller when playing Game 2 on the tree T_r from Figure 2 is w, thus dominating $r+1$ new vertices.

Theorem 2.2 *If T is a tree on n vertices, then*

$$\gamma_g(T) \geq \left\lceil \frac{2n}{\Delta(T)+3} \right\rceil - 1\,.$$

Proof. By Lemma 2.1, Staller can move in such a way that at most two new vertices are dominated on each of her moves. Let us suppose that Dominator plays optimally when Staller plays to dominate at most two new vertices on each move. Let $d_1, s_1, d_2, s_2, \ldots, d_t, s_t$ be the resulting game, where we assume that s_t is the empty move if T is dominated after the move d_t. Let $f(d_i)$ (resp. $f(s_i)$) denote the number of newly dominated vertices when Dominator plays d_i (resp. when Staller plays s_i). If the game ends on a move by Staller, then

$$n = \sum_{i=1}^{t} (f(d_i) + f(s_i)) \leq \sum_{i=1}^{t} ((\deg(d_i)+1)+2) = \sum_{i=1}^{t} \deg(d_i) + 3t\,.$$

Since this strategy may not be an optimal one for Staller, it follows that $\gamma_g(T) \geq 2t$. Similar arguments give $\gamma_g(T) \geq 2t - 1$ if the game ends on Dominator's move.

If Staller ends the game, then $n \leq t(\Delta(T)+3) \leq \frac{1}{2}\gamma_g(T)(\Delta(T)+3)$, and hence $\gamma_g(T) \geq \left\lceil \frac{2n}{\Delta(T)+3} \right\rceil$ since $\gamma_g(T)$ is integral. If the game ends on Dominator's move, then $\gamma_g(T) \geq 2t - 1$, and hence

$$n \leq t(\Delta(T)+3) - 2 \leq t(\Delta(T)+3) \leq \frac{\gamma_g(T)+1}{2}(\Delta(T)+3)\,.$$

This is equivalent to $2n \leq (\gamma_g(T)+1)(\Delta(T)+3)$, which in turn implies that

$$\gamma_g(T) \geq \left\lceil \frac{2n}{\Delta(T)+3} - 1 \right\rceil = \left\lceil \frac{2n}{\Delta(T)+3} \right\rceil - 1\,,$$

as claimed. □

To see that Theorem 2.2 is asymptotically optimal, consider the caterpillars $T(s,t)$ shown in Figure 1.

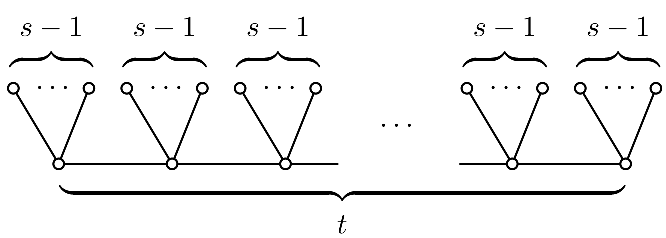

Figure 1: Caterpillar $T(s,t)$

Clearly, $T(s,t)$ has st vertices. Let $s \geq t+1$. It is easy to see that $\gamma_g(T(s,t)) = 2t-1$. Indeed, since $s-1 \geq t$, Staller can select a leaf after each of the first $t-1$ moves of Dominator. Hence after Dominator selects the t vertices of high degree, the game is over. By Theorem 2.2, $\gamma_g(T(s,t)) \geq \frac{2st}{s+4} - 1$, which for a fixed t and $n = st$ tends to $\frac{2n}{\Delta(T(s,t))+3} - 1 = \frac{2st}{s+4} - 1 \sim 2t-1$ as $s \to \infty$.

3 Pairs realizable by trees

In this section we are interested in which of the possible realizable pairs (r,r), $(r,r+1)$, and $(r,r-1)$ can be realized by trees. It was observed in [2] that (k,k) is realizable by a tree, for $k \geq 1$. We now show that pairs $(k,k+1)$ are also realizable by trees. On the other hand, we prove that the pairs $(3,2)$ and $(4,3)$ cannot be realized by trees and conjecture that no pair of the form $(k+1,k)$ is realizable by a tree. (Clearly, no graph realizes the pair $(2,1)$.)

Theorem 3.1 *For any positive integer k, there exists a tree T such that $\gamma_g(T) = k$ and $\gamma_g'(T) = k + 1$.*

Proof. Stars confirm the result for $k = 1$. For $k = 2$ consider the tree on five vertices obtained from $K_{1,3}$ by subdividing one edge. In the rest of the proof assume that $k \geq 3$. We distinguish three cases based on the congruence class of $k \pmod 3$.

Case 1: $(3r, 3r + 1)$.
Let $r \geq 1$ and consider the tree T_r of order $5r + 1$ from Figure 2.

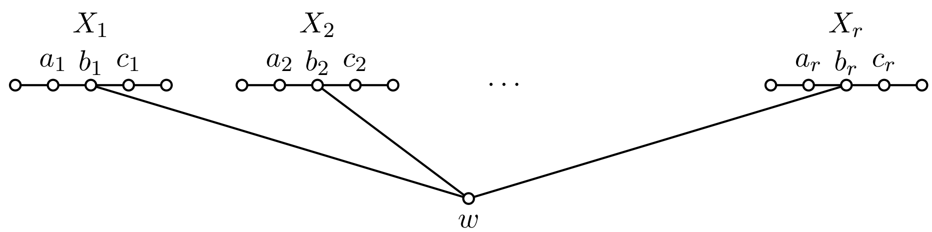

Figure 2: Tree T_r

We will prove that $\gamma_g'(T_r) \geq 3r+1$ and $\gamma_g(T_r) \leq 3r$. These two inequalities together with Theorem 1.2 show that T_r realizes $(3r, 3r + 1)$. We give a strategy for Staller that will force at least $3r + 1$ vertices to be played in T_r. Staller begins by playing w. Her strategy is now to play in such a way that b_t is played for every t such that $1 \leq t \leq r$.

She can accomplish this as follows. Dominator's first move will be to play a vertex from some X_i. By the Continuation Principle we see that $d_1 \in \{a_i, b_i, c_i\}$. If Dominator plays a_i or c_i, then Staller plays b_i. On the other hand, if $d_1 = b_i$, then Staller plays a_i. If Dominator now plays his second move in X_i, then Staller plays b_j for some j different from i. Otherwise, if Dominator plays his second move in X_t for $t \neq i$, then Staller plays in X_t using the same approach as she did in responding to Dominator's first move in X_i. By continuing this strategy Staller can ensure that all of the vertices in the set $\{b_1, \ldots, b_r\}$ are played in the course of the game. This guarantees that at least $3r + 1$ moves will be made, and thus $\gamma_g'(T_r) \geq 3r + 1$.

Now consider Game 1 on T_r. Dominator begins by playing a_1. By symmetry and the Continuation Principle, Staller can choose essentially five different vertices for s_1. These are b_1, b_2, w, the leaf u adjacent to c_1, and the leaf v adjacent to a_2. For ease of explanation, let P_3' denote a partially dominated path of order 3 where one of the leaves is dominated, and let P_5' denote a partially dominated path of order 5 with the center vertex dominated. It is easy to see that $\gamma_g(P_3') = 1$, $\gamma_g'(P_3') = 2$, and $\gamma_g(P_5') = 3 = \gamma_g'(P_5')$.

If $s_1 = b_1$, then the residual graph after these two moves is the disjoint union of two partially dominated trees, a path of order 2 with one of its vertices dominated and T_{r-1} with w dominated. In this case it follows by the Continuation Principle and induction that at most $3r$ moves will be made altogether.

If $s_1 = w$, then the residual graph is the disjoint union of P_3' and $r-1$ copies of P_5'. Dominator responds with c_1 in P_3', and after that at most $3(r-1)$ more vertices will be played. If Staller plays u on her first move, then Dominator responds with w. The residual graph is now the disjoint union of $r-1$ copies of P_5', and once again we see that at most $3r$ vertices will be played. If $s_1 = b_2$, then Dominator plays c_1. By this move Dominator has limited the number of vertices played in X_1 to 2 and can then play in such a way to ensure that no more than three vertices are played from any X_j with $j > 1$. The vertex w might be played in the remainder of the game, but we see again that a total of at most $3r$ moves will be made in the game.

Finally, assume that $s_1 = v$. Dominator responds with w. In this case the residual graph F after these three moves is the disjoint union of two copies of P_3' and $r-2$ copies of P_5'. Regardless of where Staller plays her second move, Dominator plays in one of the copies of P_3'. Since no additional move on it will be played, it follows that on the corresponding P_5 only two moves are played in the course of the game. This ensures that at most $3(r-1)$ vertices will be played in F, and again the total number of moves in the game is no more than $3r$. In all cases Dominator can limit the total number of vertices played to $3r$, and hence $\gamma_g(T_r) \le 3r$.

As we noted at the beginning, it now follows that T_r realizes $(3r, 3r+1)$.

Case 2: $(3r+1, 3r+2)$.
For $r \ge 1$ let T_r' be the graph of order $5r+3$ obtained from T_r (the tree from Figure 2) by attaching a path of length 2 to w with new vertices y and z, where z is a pendant vertex and y is adjacent to w and z. Proceeding as we did above in Case 1, we show that $\gamma_g'(T_r') \ge 3r+2$ and $\gamma_g(T_r') \le 3r+1$. Theorem 1.2 along with these two inequalities then imply that T_r' realizes $(3r+1, 3r+2)$. In Game 2 Staller first plays w, which leaves a residual graph that is the disjoint union of r copies of P_5' and a path of order two with one of its vertices dominated. Since Staller can force at least three vertices to be played from each P_5', it follows that at least $3r+1$ more moves will be made on this residual graph, and hence $\gamma_g'(T_r') \ge 1 + (3r+1) = 3r+2$.

To begin Game 1 on T_r', Dominator plays a_1. Using symmetry and the Continuation Principle, we conclude that Staller has six different vertices to play as her first move. That is, we may assume $s_1 \in \{b_1, u, w, z, b_2, v\}$, where u and v are the vertices of degree 1 as described in Case 1. If $s_1 = b_1$, Dominator plays a_2; if $s_1 \in \{u, v\}$, then Dominator responds with y; if $s_1 \in \{w, z, b_2\}$, then Dominator plays c_1. With this second move by Dominator, he can limit the total number of moves in Game 1 to at most $3r+1$. The proof of this in the six different cases is too detailed to include here, but it is similar to our analysis of Game 1 in Case 1. It now follows that $\gamma_g(T_r') \le 3r+1$, and thus T_r' realizes $(3r+1, 3r+2)$.

Case 3: $(3r+2, 3r+3)$.
In this case let T_r'' be the tree obtained from the tree T_r (of Figure 2) by attaching two paths of length 2 to b_1, say $P = b_1, p, q$ and $Q = b_1, m, n$. We denote by F the

(partially dominated) subtree of T_r'' of order nine that is the component of $T_r'' - wb_1$ that contains b_1 and in which b_1 is dominated. It can be shown that F realizes $(5,5)$ and that T_1'' realizes $(5,6)$. Because of the latter fact we assume that $r \geq 2$. Let S denote the component of $T_r'' - wb_1$ that contains vertex w.

In Game 2 Staller first plays w, leaving the residual graph $F \cup (r-1)P_5'$. It follows that $\gamma_g'(T_r'') \geq 1 + 5 + 3(r-1) = 3r+3$.

Dominator begins Game 1 on T_r'' by playing a_2. After this opening move, Dominator's goal is to play in such a way as either to prevent the vertex w from being played in the course of the game or to limit the number of moves made on some P_5 to 2. (By "some P_5" here we mean an induced path of order 5 that contains both of a_j and c_j for some j with $2 \leq j \leq r$.) Suppose that $s_1 = w$. Dominator then plays c_2, and the resulting residual graph is $F \cup (r-2)P_5'$. In this case a total of at most $1 + 1 + 1 + 5 + 3(r-2) = 3r+2$ moves will be made in the game.

Suppose instead that $s_1 = b_2$. In this case Dominator responds with a_3. If Staller follows Dominator's moves by playing on the same P_5, then Dominator continues to play a_j from some P_5 that has not had one of its vertices played. If, at some point in the game, Staller plays w before all of $c_2, c_3, \ldots, c_r$ are dominated, then Dominator can achieve his goal by playing a second vertex on the (same) P_5 where he made his previous move to dominate it in two moves. Otherwise, Staller will be the last player to play on S. In this case, Dominator plays the vertex a_1 thereby preventing the vertex w from ever being played. (It is easy to see that on any move by Staller in F Dominator can follow in F in such a way that the total of at most 5 moves will be played in F during the game.) Therefore, in all cases Dominator can ensure that at most $3r+2$ total moves are made in Game 1.

Again, we employ Theorem 1.2 to conclude that T_r'' realizes $(3r+2, 3r+3)$. $\square$

For the $(k, k-1)$ case we pose:

Conjecture 3.2 *No pair of the form $(k, k-1)$ can be realized by a tree.*

In the rest of this section we prove the first two cases of the conjecture:

Theorem 3.3 *No tree realizes the pair $(3,2)$ or the pair $(4,3)$.*

Proof. Suppose that a tree T realizes $(3,2)$. It is easy to see that $\gamma_g'(T) = 2$ implies that T is either a star $K_{1,n}$ for $n \geq 2$ or a P_4. In both cases $\gamma_g(T) \leq 2$, so $(3,2)$ is not realizable on trees.

Suppose T is a tree that realizes $(4,3)$, and let x be an optimal first move for Dominator. The residual graph T' has at most 3 components, each of which is a partially dominated subtree of T. Note that if one of these partially dominated components F has $\gamma_g'(F) = 1$, then F has exactly one undominated vertex.

Suppose first that T' has three partially dominated components T_1, T_2, T_3 with T_i rooted at the dominated vertex v_i. If at least one of these subtrees, say T_1, has more than one undominated vertex, then Staller can force at least two moves in

T_1. Because the other two subtrees each require at least one move, it follows that $\gamma_g(T) \geq 5$, a contradiction. Hence, each of T_1, T_2, T_3 has exactly one undominated vertex, and T is a tree formed by identifying a leaf from three copies of P_3 and attaching some pendant vertices at the vertex of high degree. However, this tree has Staller-start game domination number at least 4, again contradicting our initial assumption.

Now suppose that T' is the disjoint union of T_1 and T_2. Note that in this case x cannot be a neighbor of a leaf in the original tree T. Indeed, if x is adjacent to a leaf y, then when Game 2 is played on T, Staller can play first at y, which is easily shown to force at least four moves. Thus, $\deg(x) = 2$. If $\gamma_g'(T_1) = 1 = \gamma_g'(T_2)$, then $T = P_5$ and $\gamma_g(T) = 3$, a contradiction.

Note that the Staller-start game domination number of either of these two partially dominated trees cannot exceed 2. We may thus assume without loss of generality that $2 = \gamma_g'(T_1) \geq \gamma_g'(T_2)$. Suppose that $\gamma_g'(T_2) = 2$. Staller can then play at vertex x when Game 2 is played on T. After Dominator's first move at least one of T_1 or T_2 is part of the residual graph, and Staller can then force at least two more moves, again contradicting the assumption that $\gamma_g'(T) = 3$. Therefore, T_2 is the path of order 2 with one of its vertices dominated.

If T_1 is a star with v_1 as its center or as one of its leaves, then $\gamma(T) = 2$ and hence $4 = \gamma_g(T) \leq 2 \cdot 2 - 1$, an obvious contradiction. Therefore, $\gamma_g'(T_1) = 2$, but T_1 is not a star. A short analysis shows that T_1 must be one of the partially dominated trees in Figure 3. Each of these candidates for T_1 together with $T_2 = P_2$ yields a tree T with either $\gamma_g(T) \neq 4$ or $\gamma_g'(T) \neq 3$, again contradicting our assumption about T. This implies that the residual graph T' has exactly one component.

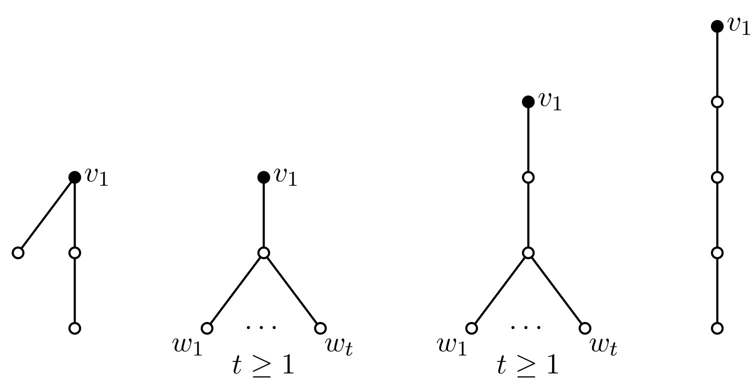

Figure 3: Possible partially dominated trees

Hence we are left with a tree T, a vertex x that is an optimal first move for Dominator, and the residual tree T' which has one component. Besides the neighbor v_1 in T', the vertex x is adjacent to some leaves $y_1, \ldots, y_k$. We may assume that $k \geq 1$, because otherwise (by the Continuation Principle) Dominator would rather select v_1 than x in his first move. Since x is an optimal first move by Dominator in Game 1, it follows that $\gamma_g'(T_1) = 3$ in addition to $\gamma_g'(T) = 3$.

Consider Game 2 played on the partially dominated tree T_1. Let w be an optimal first move by Staller in this game, and let v in T_1 be an optimal response by Dominator. At least one vertex, say u, in T_1 is not dominated by $\{w, v\}$. Note that $u \neq v_1$, since T_1 is a partially dominated tree with v_1 dominated. We can now show that $\gamma_g'(T) \geq 4$. Staller starts Game 2 on the original tree T by making the move $s_1 = w$. Dominator either plays d_1 in T_1 or $d_1 = x$. If Dominator responds in T_1, then y_1 and at least one vertex in T_1 (other than v_1) are not yet dominated. Thus in this case at least four total moves are required in Game 2. On the other hand, if $d_1 = x$, then Staller plays $s_2 = v$, and u is not yet dominated. Again Game 2 lasts at least a total of four moves. This now implies that $\gamma_g'(T) \geq 4$, a contradiction. $\square$

4 Game on spanning subgraphs

We now turn our attention to relations between the game domination number of a graph and its spanning subgraphs, in particular spanning trees.

Note that since any graph is a spanning subgraph of the complete graph of the same order, the ratio $\gamma_g(H)/\gamma_g(G)$ where H is a spanning subgraph of G can be arbitrarily large. On the other hand the following result shows that this ratio is bounded below by one half.

Proposition 4.1 *If G is a graph and H is a spanning subgraph of G, then*

$$\gamma_g(H) \geq \frac{\gamma_g(G) + 1}{2}\,.$$

In particular, if T is a spanning tree of connected G, then $\gamma_g(T) \geq (\gamma_g(G) + 1)/2$.

Proof. Clearly, $\gamma(H) \geq \gamma(G)$. By Theorem 1.1, $\gamma_g(H) \geq \gamma(H)$ and $\gamma_g(G) \leq 2\gamma(G) - 1$. Then $\gamma_g(H) \geq \gamma(H) \geq \gamma(G) \geq (\gamma_g(G) + 1)/2$. $\square$

To see that a spanning subgraph can indeed have game domination number much smaller than its supergraph, consider the graph G_t consisting of t blocks isomorphic to $\overline{P_5}$ (the graph obtained from C_5 by adding an edge) and its spanning subgraph H_t, see Figure 4. Let x be the vertex where the houses of G_t are amalgamated. Note that Dominator needs at least two moves to dominate each of the blocks of G_t. Hence his strategy is to play x and then finish dominating one block on each move. On the other hand, if not all blocks are already dominated, Staller can play the vertex of degree 2 adjacent to x of such a block B in order to force one more move on B. So in half of the blocks two vertices will be played (not counting the move on x), which in turn implies that $\gamma_g(G_t)$ is about $3t/2$. On the other hand,

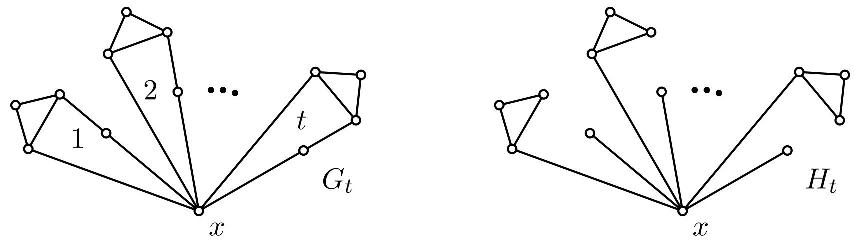

Figure 4: Graph G_t and its spanning subgraph H_t

playing Game 1 on H_t, the optimal first move for Dominator is x. After that Staller and Dominator will in turn dominate each of the triangles, so $\gamma_g(H_t) = t + 1$.

The example of Figure 4 might lead to a suspicion that 2-connected spanning subgraphs cannot have smaller game domination number than their 2-connected supergraphs. However:

Theorem 4.2 *For $m \geq 3$, there exists a 3-connected graph G_m having a 2-connected spanning subgraph H_m such that $\gamma_g(G_m) \geq 2m - 2$ and $\gamma_g(H_m) = m$.*

Proof. We form a graph G_m of order $m(m+2)$ as follows. Let $X_i = \{a_{i,1}, \ldots, a_{i,m}\} \cup \{x_i, y_i\}$ for $1 \leq i \leq m$, and then set

$$V(G_m) = \bigcup_{i=1}^{m} X_i \,.$$

The edges are the following. Let $\{x_1, y_1, \ldots, x_m, y_m\}$ induce a complete graph of order $2m$. For $1 \leq p \leq m$, let X_i induce a complete graph of order $m + 2$. In addition, for $1 \leq i \leq m - 1$ and $i \leq j \leq m - 1$, add the edge $a_{i,j}a_{j+1,i}$. See Figure 5 for G_4.

If $d_1 = x_1$, then Staller plays in X_1, say at $a_{1,1}$. In each of the subsequent rounds, the Continuation Principle implies that Dominator must play in some X_i that has not been played in before and on a vertex of X_i that has an undominated neighbor outside X_i. It will always be possible for Staller to follow Dominator and also play in X_i in each of her first $m - 2$ moves. Hence by this time, $2m - 4$ moves are made. At this stage, there are two undominated vertices in different X_i's with no common neighbor. Hence two more moves are needed to end the game, which thus ends in no less than $2m - 2$ moves.

Assume next that $d_1 = a_{1,1}$. Now Staller plays x_1 and we are in the first case. Note that $d_1 = a_{1,m}$ need not be considered due to the Continuation Principle, so $d_1 \in \{x_1, a_{1,1}\}$ covers all cases by symmetry. Hence $\gamma_g(G_m) \geq 2m - 2$.

The spanning subgraph H_m of G_m is obtained by removing all the edges $a_{i,j}a_{j+1,i}$. By the Continuation Principle, we may without loss of generality assume that $d_1 =$

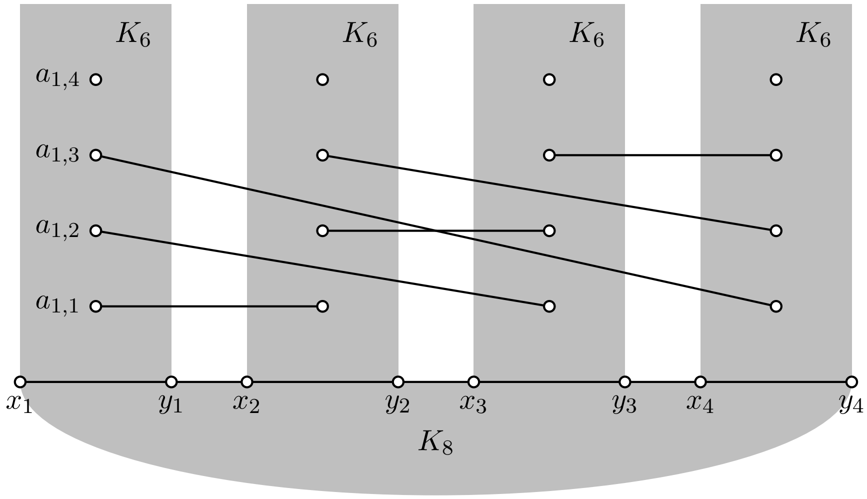

Figure 5: Graph G_4

x_1 when Game 1 is played on H_m. In this case, each successive move of either player completely dominates the X_i in which it is played. Hence $\gamma_g(H_m) = m$. $\square$

If $\gamma_g(G)$ attains one of the two possible extremal values, $\gamma(G)$ or $2\gamma(G) - 1$, then we can say more.

Proposition 4.3 *(i) If G is a graph with $\gamma_g(G) = \gamma(G)$ and H is a spanning subgraph of G, then $\gamma_g(H) \geq \gamma_g(G)$.*

(ii) If G is a graph with $\gamma_g(G) = 2\gamma(G) - 1$ and H is a spanning subgraph of G with $\gamma(H) = \gamma(G)$, then $\gamma_g(H) \leq \gamma_g(G)$.

Proof. Assertion (i) follows because $\gamma_g(H) \geq \gamma(H) \geq \gamma(G) = \gamma_g(G)$; while (ii) follows since $\gamma_g(H) \leq 2\gamma(H) - 1 = 2\gamma(G) - 1 = \gamma_g(G)$. $\square$

Since every graph G has a spanning forest F such that $\gamma(G) = \gamma(F)$, see [5, Exercise 10.14], we also infer that if G is a graph with $\gamma_g(G) = 2\gamma(G) - 1$, then G contains a spanning forest F (spanning tree if G is connected) such that $\gamma_g(F) \leq \gamma_g(G)$.

In the rest of this section we focus on spanning trees. First we show that **all** spanning trees of a graph G may have game domination number much larger than G.

Theorem 4.4 *For $m = 2r$, $r \geq 1$, there exists a graph G_m such that*

$$\gamma_g(T) - \gamma_g(G_m) \geq r - 1$$

holds for any spanning tree T of G_m.

Proof. Let $n \geq 3$ and let S be the star with center x and leaves $v_1, \ldots, v_m$. Let G_m be the graph (of order $nm + 1$) constructed by identifying a vertex of a complete graph of order n with v_i, for $1 \leq i \leq m$; see Figure 6.

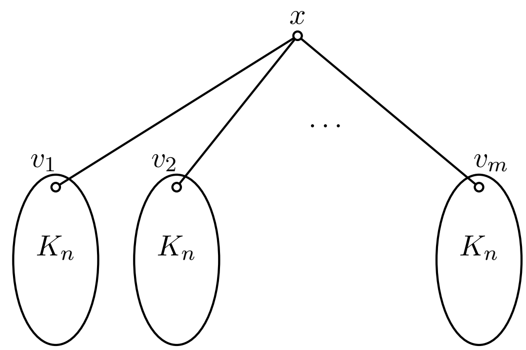

Figure 6: Graph G_m

We first note that $\gamma_g(G_m) = m + 1$. Let T be any spanning tree of G_m. T has at least one leaf ℓ_i in the subtree T_i of T rooted at v_i when the edge xv_i is removed from T (choose $\ell_i \neq v_i$). When Game 1 is played on T, Staller can choose at least half of these leaves $(\ell_1, \ldots, \ell_m)$ or let Dominator choose them. Thus in at least half of $T_1, \ldots, T_m$, two vertices will be chosen. Therefore $\gamma_g(T) \geq m + m/2 = 3m/2$, so

$$\gamma_g(T) - \gamma_g(G_m) \geq \frac{3}{2}m - m - 1 = r - 1\,.$$

□

Recall that the domination number of a spanning subgraph of a graph G cannot be smaller than that of G. In contrast we now give an example of a graph G with a spanning tree T such that $\gamma_g(T) < \gamma_g(G)$. Consider the graph G and the spanning tree T from Figure 7.

Figure 7: Graph G and its spanning tree T

For each of the following pairs (x, y) of vertices from G, if Dominator plays x then Staller can play y and then the game domination number of the resulting residual graph G' will be 2: $(1, 6); (2, 3); (3, 2); (4, 8); (8, 4); (7, 3); (6, 1); (5, 1)$. Therefore,

$\gamma_g(G) \geq 4$. Consider now the spanning tree T, and let Dominator play 2 on T. For each of the following vertices a, the residual graph T' when Staller plays a is listed in Figure 8. For instance, the left case is when Staller plays 5; in this case the residual graph is induced by vertices $6, 7, 8$ and the vertex 6 of the residual graph is already dominated, as indicated by the filled vertex.

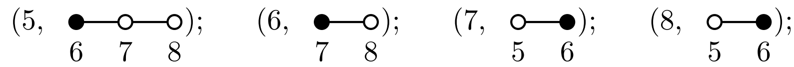

Figure 8: Staller's possible moves

In each case we find that the residual graph has game domination number 1, so

$$\gamma_g(T) \leq 3 < \gamma_g(G)\,.$$

This rather surprising fact demonstrates the intrinsic difficulty and unusual behavior of the game domination number. Even more can be shown:

Theorem 4.5 *For any positive integer ℓ, there exists a graph G, having a spanning tree T such that $\gamma_g(G) - \gamma_g(T) \geq \ell$.*

Proof. We introduce the family of graphs G_k and their spanning trees T_k in the following way. Let k be a positive integer, and for each i between 1 and k, $x_i^1, x_i^2, x_i^3, x_i^4, x_i^5$ are non-adjacent vertices in T_k, and $Q_i : y_i^1 y_i^2 y_i^3 y_i^4 y_i^5$ is a path isomorphic to P_5 in T_k. Finally x and y are two vertices, such that x is adjacent to $x_i^1, x_i^2, x_i^3, x_i^4$ and x_i^5 for all $i \in \{1, \ldots, k\}$, while y is adjacent to y_i^1 for all $i \in \{1, \ldots, k\}$, and x and y are also adjacent. The resulting graph T_k is a tree on $10k + 2$ vertices. We obtain G_k by adding edges between x_i^j and y_i^j for $1 \leq i \leq k$, $1 \leq j \leq 5$. See Figure 9 for G_4, from which T_4 is obtained by removing all vertical edges except xy.

To complete the proof it suffices to show that for any integer $k \geq 1$,

$$\gamma_g(G_k) \geq \frac{5}{2}k - 1 \quad \text{and} \quad \gamma_g(T_k) \leq 2k + 3\,.$$

Let us first verify the second inequality, concerning trees T_k. To prove it we need to show that Dominator has a strategy by which at most $2k + 3$ moves will be played during Game 1. His strategy is as follows. In his first two moves, he ensures that x and y are chosen. He plays x in his first move, and y in his second move, unless already Staller played y (we will consider this case later). Now, $s_1 \neq y$ implies that s_1 is in some Q_i; without loss of generality let this be Q_1. Hence in Dominator's third move, since y_i^1 is dominated for each i, he can dominate all vertices of Q_1. One by one, Staller will have to pick a new Q_i to play in, which Dominator will entirely dominate in his next move. Altogether, in each Q_i (with a possible exception of

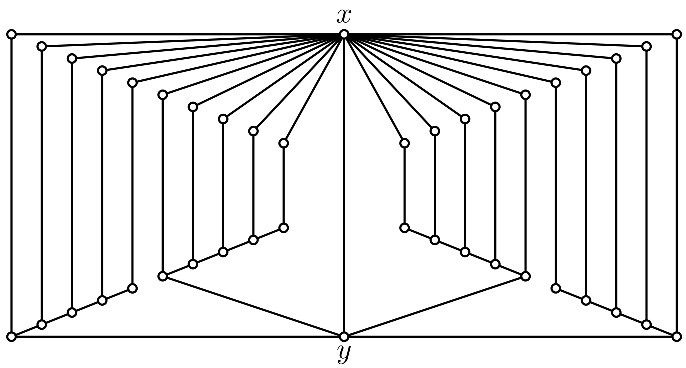

Figure 9: Graph G_4

one Q_i, where Staller can force three vertices to be played), there will be only two vertices played, which yields $2k+3$ as the total number of moves in this game. On the other hand, if $s_1 = y$, then $d_2 = y_1^3$ ensures that in Q_1 only two vertices will be played. In addition, by a similar strategy as above Dominator can force that only two moves will be played in each of Q_is. Hence only $2k+2$ moves will be played.

To prove the first inequality we need to show that Staller has a strategy to enforce at least $\frac{5}{2}k - 1$ moves played during Game 1 in G_k. Her strategy in each of the first k moves of the game is to play an x_i^4 such that no vertex from $Q_i \cup \{x_i^1, x_i^2, x_i^3, x_i^5\}$ has yet been played. Using this strategy she ensures that at least two more moves will be needed to dominate each of these $\lfloor \frac{k}{2} \rfloor$ Q_is (since at least y_i^2, y_i^3 and y_i^5 are left undominated). The remaining paths Q_i require at least two moves each as well. Hence altogether, there will be at least $2k + \lfloor \frac{k}{2} \rfloor$ moves played during Game 1, which implies $\gamma_g(G_k) \geq \frac{5}{2}k - 1$. $\square$

Acknowledgements

The authors extend their appreciation to anonymous referees for their careful reading of our manuscript and for the resulting suggestions for improvement. In particular, several of the proofs in the paper were clarified with their help. We are also grateful to Editor-in-Chief Douglas B. West for investing considerable time and effort in pointing out grammatical (and some other) corrections.

References

[1] B. Brešar, P. Dorbec, W. Goddard, B. L. Hartnell, M. A. Henning, S. Klavžar, D. F. Rall, Vizing's conjecture: a survey and recent results, J. Graph Theory 69 (2012) 46–76.

[2] B. Brešar, S. Klavžar, D. F. Rall, Domination game and an imagination strategy, SIAM J. Discrete Math. 24 (2010) 979–991.

[3] M. Henning, personal communication, 2003.

[4] G. Košmrlj, Realizations of the game domination number, to appear in J. Comb. Optim., DOI: 10.1007/s10878-012-9572-x.

[5] W. Imrich, S. Klavžar, D. F. Rall, *Topics in Graph Theory: Graphs and Their Cartesian Product*, A K Peters, Wellesley, MA, 2008.

[6] B. Kinnersley, D. B. West, R. Zamani, Extremal problems for game domination number, manuscript, 2011.