# Domination game: extremal families of graphs for the 3/5-conjectures

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**Abstract**

Two players, Dominator and Staller, alternate choosing vertices of a graph $G$, one at a time, such that each chosen vertex enlarges the set of vertices dominated so far. The aim of the Dominator is to finish the game as soon as possible, while the aim of the Staller is just the opposite. The game domination number $\gamma_g(G)$ is the number of vertices chosen when Dominator starts the game and both players play optimally. It has been conjectured in [7] that $\gamma_g(G) \leq \frac{3|V(G)|}{5}$ holds for an arbitrary graph $G$ with no isolated vertices, which is in particular open when $G$ is a forest. In this paper we present constructions that lead to large families of trees that attain the conjectured 3/5-bound. Some of these families can be used to construct graphs with game domination number

3/5 of their order by gluing them to an arbitrary graph. All extremal trees on up to 20 vertices were found by computer. In particular, there are exactly ten trees $T$ on 20 vertices with $\gamma_g(T) = 12$ all of which belong to the constructed families.

**Keywords:** domination game, game domination number, 3/5-conjecture, computer search

**AMS subject classification (2010)**: 05C57, 91A43, 68-04, 05C69

# 1 Introduction

The domination game, introduced in [2], is played by two players on an arbitrary graph $G$. The two players are called Dominator and Staller, which indicates the role they are supposed to play in the game. They are taking turns choosing a vertex from $G$ such that whenever they choose a vertex the set of vertices dominated so far increases. The game ends when all vertices of $G$ are dominated, and the aim of Dominator is that the total number of moves played in the game is as small as possible, while Staller wishes to maximize this number. By *Game 1* we mean a game in which Dominator has the first move, while *Game 2* refers to a game in which Staller begins. Assuming that both players play optimally, the *game domination number* $\gamma_g(G)$, respectively the *Staller-start game domination number* $\gamma_g'(G)$, of a graph $G$, denotes the number of moves played, equivalently the number of vertices chosen, in Game 1, respectively Game 2.

Similarly as in the case of the game chromatic number (see e.g. [1] for a survey on this related graph invariant and [4] for the general framework of combinatorial games), the game domination number is intrinsically different from the ordinary domination number. In particular it is not trivial to determine $\gamma_g(G)$ even in the simple case when $G$ is a path [8]. In addition, in [2] it was wrongly asserted that $\gamma_g(X) = \gamma_g'(X)$ for a comb $X$ (a graph obtained from a path by attaching a leaf to each vertex, also known as the corona of a path). The correct values for combs were determined in [9]. It is thus clear that the game is very non trivial even when played on trees.

The main purpose of this paper is to explore extremal trees that attain the bound in the following conjecture.

**Conjecture 1.1** ([7, Conjecture 5.1]) *If $G$ is an isolate-free forest of order $n$, then*

$$\gamma_g(G) \leq 3n/5\,.$$

Along the way we will encounter an infinite family of trees that can be glued to an arbitrary graph so that the obtained graphs achieve the upper bound in the following more general conjecture.

**Conjecture 1.2** ([7, Conjecture 6.2]) *If $G$ is an isolate-free graph of order $n$, then*

$$\gamma_g(G) \leq 3n/5\,.$$

Note that the truth of Conjecture 1.2 implies the same for Conjecture 1.1, however it is not obvious at all whether they are equivalent. In particular the game domination number of a spanning tree $T$ of a connected graph $G$ can be arbitrarily smaller than $\gamma_g(G)$ [3].

Isolate-free graphs and trees with game domination number equal to 3/5 of their order will be called 3/5-*graphs* and 3/5-*trees*, respectively. An infinite class of such graphs (in particular trees) can be constructed in the following way. Let $G'$ be the graph obtained from an arbitrary graph $G$ of order $n$, where for each vertex $v \in V(G)$, a path of order 5 is added and the center of the new path is identified with $v$. Clearly $G'$ is of order $5n$ and it is not difficult to see that $\gamma_g(G') = 3n$. This construction was independently discovered by several authors [6, 2, 7] and eventually culminated in the above two conjectures.

Attempts to settle Conjecture 1.1 led to a search for graphs that would achieve the conjectured bound. Using computer we found out that there are only one, two, and four 3/5-trees on 5, 10, and 15 vertices, respectively. Three of these seven do not belong to the family mentioned above. In all of these three trees the so-called fork appears as a subgraph. In order to better understand this phenomenon, in particular, to understand the role of the fork, the computation was extended to all trees of order 20. As there are 823065 non-isomorphic such trees, certain optimizations (see Section 4) were needed to finish the computation in reasonable time. It turned out that there are ten 3/5-trees, eight of them being new (that is, not covered by the previously known construction). The variety of these examples was then large enough to grasp the patterns that are the core of this paper.

The paper is organized as follows. In the rest of this section we present concepts, conventions and known results needed. Then, in the next section, we present a construction using a path and two special trees ($P_5$ and the so-called fork) that yields an infinite family of trees that attain the bound in Conjecture 1.1. In Section 3 we give a different approach that uses an arbitrary graph instead of a path yielding additional extremal graphs with respect to the two conjectures. The list of all extremal trees on up to 20 vertices is presented in Section 4. They were obtained by computer and can all be constructed by the methods of this paper. We conclude the paper with some open problems.

Recall that a set $D \subset V(G)$ is *dominating* if every vertex from $V(G) - D$ has a neighbor in $D$. The minimum size of a dominating set of a graph $G$ is called the *domination number* of $G$, denoted $\gamma(G)$; we refer to the monograph [5] on domination theory. Throughout the paper we will use the convention that $d_1, d_2, \ldots$ denotes the sequence of vertices chosen by Dominator and $s_1, s_2, \ldots$ the sequence chosen by Staller. A *partially-dominated graph* is a graph together with a declaration that some vertices are already dominated, that is they need not be dominated in the rest of the game. For a vertex subset $S$ of a graph $G$, let $G|S$ denote the partially dominated graph in which vertices from $S$ are already dominated. In particular, if $S = \{x\}$ we will write $G|x$. The *Staller-pass game* is the domination game in which, on each turn, Staller may pass her move. Let $\hat{\gamma}_g(G)$ be the number of moves in such a game played optimally on $G$ when Dominator starts, and $\hat{\gamma}'_g(G)$ when Staller starts. The turns when Staller passes do not count as moves. With these concepts in hand we now recall several very useful results due to Kinnersley, West, and Zamani [7].

**Lemma 1.3** (Continuation Principle) [7, Lemma 2.1] *Let $G$ be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \le \gamma_g(G|B)$ and $\gamma'_g(G|A) \le \gamma'_g(G|B)$.*

**Theorem 1.4** [7, Theorem 4.6] *Let $F$ be a forest and $S \subseteq V(F)$. Then $\gamma_g(F|S) \le \gamma'_g(F|S)$.*

**Lemma 1.5** [7, Corollary 4.7] *Let $F$ be a forest and $S \subseteq V(F)$. Then $\hat{\gamma}_g(F|S) = \gamma_g(F|S)$ and $\hat{\gamma}'_g(F|S) = \gamma'_g(F|S)$.*

## 2 Construction using paths and forks

In this section we present a large family of trees which attain the conjectured 3/5 bound using two special trees. One is the so-called "fork" that we denote by $F$, and the other is the path of order 5. For convenience we will refer to them as *basic trees*. Both are shown in Fig. 1 together with the labels that will be used in the rest of the paper.

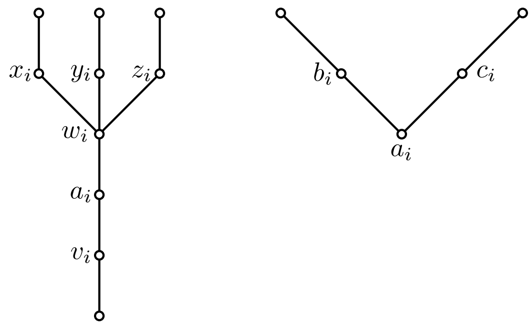


Figure 1: The fork $F$ and $P_5$ each labeled as $T_i$

It can be easily verified by inspection that if $G$ is one of the fork $F$ or $P_5$ then $\gamma_g(G|a_i) = \gamma_g(G) = \gamma_g'(G)$, where $a_i$ is as in Fig. 1.

Let $k \geq 6$ and $3 \leq \ell \leq k-2$. For a path of order $k$ we denote the first two vertices by $s$ and $s'$ and the last two in natural order by $t'$ and $t$. In addition, the vertex at distance $\ell - 1$ from $s$ we denote by $x$. Denote the remaining vertices, starting from the degree 2 neighbor of $s'$, by $u_1, u_2, \ldots, u_{k-5}$. We refer to this labeled path as $P$. Then the tree $T_k^{(\ell)}[T_1, \ldots, T_{k-5}]$ is constructed from this labeled path as follows. For any vertex $u_i$, $i = 1, 2, \ldots, k-5$, let $T_i$ be a basic tree and identify the vertex $a_i \in V(T_i)$ with $u_i \in V(P)$. See Fig. 2.

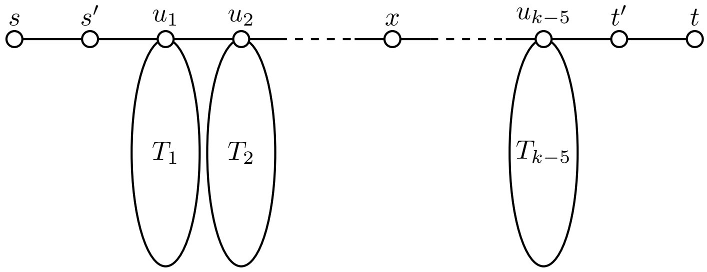


Figure 2: Construction for Theorem 2.1

**Theorem 2.1** *For any $k \geq 6$, any $3 \leq \ell \leq k-2$, and any list of basic trees $T_1, \ldots, T_{k-5}$, $T_k^{(\ell)}[T_1, \ldots, T_{k-5}]$ is a 3/5-tree.*

**Proof.** Let $n = \left|V\left(T_k^{(\ell)}[T_1, \ldots, T_{k-5}]\right)\right|$ and $n_i = |V(T_i)|$, where $1 \leq i \leq k-5$. We first show that Dominator has a strategy to ensure that no more than $3n/5$ vertices are played.

This goal of Dominator will be accomplished if for every $1 \leq i \leq k-5$, at most $3n_i/5$ vertices are played from $T_i$ and at most three of $\{s, s', x, t, t'\}$ are played.

Suppose first that $\ell = 3$, that is $x$ is adjacent to $s'$. Dominator begins by playing $x$. This will guarantee that at most two of $\{s, s', x\}$ are played during the game. As long as Staller plays a vertex from some $T_i$, Dominator follows with an optimal strategy in $T_i$ if possible. In particular, after the first move of Staller in some $T_i$, Dominator responds in $T_i$ with a move that dominates $u_i$. Note that in the fork $F$ and the path of order five this is always possible. If Staller's move in $T_i$ ends the game there, then Dominator plays optimally in some other $T_j$ in which the game is not yet finished. If this is the first move in $T_j$, then Dominator will play $u_j$ which is an optimal move in $T_j$ whether it is a fork or path of order 5. Note that if this was not the first vertex played in $T_j$, then a Staller-pass game is being played in $T_j$, which by Lemma 1.5 is no worse for Dominator.

Suppose at some point in the game Staller plays $t$. If no vertex of $T_{k-5}$ has been played, then Dominator responds by playing $u_{k-5}$. Since this is an optimal move by Dominator when beginning Game 1 on $T_{k-5}$, whether it is a fork or a path of order 5, at most $3n_{k-5}/5$ vertices in $T_{k-5}$ will be played. Moreover, his goal of preventing both of $t'$ and $t$ from being played has been achieved. On the other hand, if some vertices from $T_{k-5}$ have already been played, then $u_{k-5}$ has been dominated and hence $t'$ can never be played.

If $\ell = k-2$, then $x$ is adjacent to $t'$ and the argument is the same as above. Assume finally that $3 < \ell < k-2$. Dominator begins as in the first case by playing $x$. He then follows Staller in any $T_i$ using the same strategy as above which guarantees that on each $T_j$ at most $3n_j/5$ vertices are played. More precisely, when he follows Staller for the first time on $T_j$, Dominator's move will be $u_j$. Moreover, just as above he can guarantee that at most one of $t'$ and $t$ is played and that at most one of $s$ and $s'$ is played.

Now we demonstrate a strategy for Staller that will guarantee that at least $3n/5$ moves are made in Game 1.

Suppose first that $d_1 = x$. In this case at least three vertices from $\{s, s', x, t', t\}$ will be played, and in fact, an optimal first move for Staller is to play $s$. After her first move, the strategy of Staller is to follow Dominator, if possible, in any $T_i$ using an optimal strategy for $T_i$. If such a move is not possible, then either the game is over (or she plays $t$ as the final move in the game) or Staller can select a vertex from some partially dominated $T_j$; either this is the first move in $T_j$ or the last move previously made in $T_j$ was also made by her.

Note that if Staller is the first to play in some $T_i$ and if the usual game is played in that $T_i$ (i.e., Dominator follows Staller in all moves in $T_i$), then exactly $3n_i/5$ vertices from $T_i$ will eventually be played. This follows because $\gamma_g(T_i|u_i) = \gamma_g(T_i) = \gamma'_g(T_i)$. (Recall that $a_i$ is identified with $u_i$.) On the other hand, if Dominator allows Staller to play two consecutive moves in a partially dominated $T_i$, then by Theorem 1.4 the total number of vertices played in such a $T_i$ will be at least $3n_i/5$.

Therefore, if $d_1 = x$, then Staller can force at least $3n/5$ vertices to be played.

By the Continuation Principle $d_1 \neq s$ and $d_1 \neq t$. If $d_1 = s'$, then Staller plays $x$ and the situation is essentially the same as when $d_1 = x$ and $s_1 = s$. By the assumptions imposed on $T_1$, we can argue the same as above to ensure that at least $3n/5$ vertices are played. Similarly, a parallel argument works if $d_1 = t'$.

In the last case the first move of Dominator is in some $T_m$. Then Staller replies with $s_1 = x$. For convenience let us call a basic tree *open* if exactly one of its vertices has been played and this was on a move of Dominator. Hence after the first two moves the tree $T_m$ is the only open tree. In the course of the game there are four essentially different moves by Dominator.

1. Dominator creates a new open tree,
2. Dominator plays $s'$ or $t'$,
3. Dominator plays in a tree in which the last move was made by Staller, and
4. Dominator plays in an open tree.

We now treat each of these cases separately and along the way describe the strategy for Staller.

Suppose a move by Dominator creates an open tree $T_i$. If this is the only open tree, then Staller responds with an optimal move in $T_i$. Otherwise, suppose there is another open tree $T_j$. If both $i$ and $j$ are smaller than $\ell$, say, $i < j < \ell$, then Staller makes an optimal move in $T_j$. On the other hand, if one of $i$ and $j$ is bigger than $\ell$, say $i > \ell$, then Staller plays optimally in $T_{\min\{i,j\}}$. Note that this strategy of Staller ensures that there are never more than two open trees after a move of Dominator and at most one open tree after a move of Staller.

If Dominator plays $s'$ or $t'$ and there are no open trees, then Staller plays the other one (that is, $t'$ or $s'$). (By Continuation Principle, an even better move for Staller would be to play $t$ or $s$, but for her purposes this makes no difference.) On the other hand, if there is an open tree, then she plays in it.

In the third case when Dominator played in some $T_i$ in which Staller was the last player to have made a move, Staller plays in $T_i$ if this is possible. If not, then she plays in an open tree if one exists. If there is no open tree, then Staller either makes the first move in some basic tree or plays in a basic tree, say $T_j$, in which she was the last one to have made a move. This latter type of move leads to a game on $T_j$ in which Staller has made two consecutive moves. Using Theorem 1.4 we see that the total number of vertices played in $T_j$ will be at least $3n_j/5$.

Finally suppose that Dominator plays in the (only) open tree $T_i$. That $T_i$ is indeed the only open tree follows from the strategy of Staller.

**Claim.** *If the first two moves on $T_i$ were made by Dominator and neither is $u_i$, then $u_i$ is a legal move for Staller.*

Let the first two moves on $T_i$ made by Dominator be $d_p$ and $d_q$, where $q > p$ and $d_p, d_q \neq u_i$. We may assume without loss of generality that $i > \ell$. Suppose that the right neighbor $y$ of $u_i$ on $P$ was dominated before the move $d_q$. Note that either $y = u_{i+1}$ or $y = t'$. If $y = t'$ and it was played before by Dominator, then by the strategy of Staller (case 2) moves $d_p$ and $d_q$ of Dominator in $T_i$ were not possible in the first place. Suppose next that $y = u_{i+1}$. We distinguish two subcases. Assume first that the right neighbor of $y$ was played earlier in the game. Then $d_p$ was not played before the right neighbor of $y$, because otherwise Staller would have replied with an optimal move in $T_i$ (and thus Dominator would not have two consecutive moves in $T_i$). Hence, after $d_p$ was played, $T_i$ is the most left open tree so Staller would have replied in $T_i$. Using the same argument we also get a contradiction when $y$ was dominated before with a move within $T_{i+1}$. This proves the claim.

If $T_i$ is $P_5$ and Dominator played $b_i$ and $c_i$, then Staller plays $u_i$. (This is a legal move by the above claim.) If Dominator plays $u_i$ in one of these two moves, then at least one of the leaves of $T_i$ is not dominated and Staller plays it. Suppose next that $T_i$ is $F$. If $\{d_p, d_q\}$ is one of $\{x_i, w_i\}$, $\{x_i, v_i\}$, or $\{v_i, w_i\}$, then Staller playing $u_i$ guarantees six moves in $T_i$. (Recall again that this is a legal move by the above claim.) On the other hand, if $\{d_p, d_q\}$ is either $\{x_i, u_i\}$ or $\{x_i, y_i\}$, then Staller plays $w_i$. In the former case this already guarantees six moves on $T_i$, which also obviously happens in the latter case unless the next move of Dominator in $T_i$ is $v_i$. Using symmetry and the Continuation Principle this covers all possible cases. □

If the above construction is extended to $k = 5$ and the condition $3 \leq \ell \leq k-2$ is interpreted that no $T_i$ is attached to the starting $P_k = P_5$, we get the path on five vertices which is a 3/5-tree.

In Theorem 2.1 it would be good to extend the definition of basic trees to contain more than two graphs. However, in the proof we intrinsically use specific properties of $P_5$ and $F$, see for instance the last paragraph of the proof. Moreover, while designing the strategy of Dominator, we also need the property that if Staller first plays in some basic tree $T_i$, then Dominator can respond with an optimal move that dominates $u_i$. This is another obstruction in an attempt to extend the set of basic trees.

In order to see the sensitive nature of the theorem's construction, consider the tree $S$ which is shown in Fig. 3 in bold. It can be verified directly (cf. also Theorem 3.7) that $\gamma_g(S) = 9 = \frac{3}{5}|V(S)| = \gamma_g'(S)$ and that $\gamma_g(S|a_i) = \gamma_g(S)$. Now consider the graph $T_6^{(4)}[S]$ (see Fig. 3 again) where as before, the attachment of $S$ to $P_6$ is made in the vertex $a_i$ (of $S$). It can be checked that $\gamma_g(T_6^{(4)}[S]) = 11$, hence $T_6^{(4)}[S]$ is not a 3/5-tree. One could also try to attach $S$ to $P_6$ at the vertex $a_i'$, but also in this case the game domination number of the resulting graph is 11. This example demonstrates that one would need to be very careful in an attempt to extend the set of basic trees.

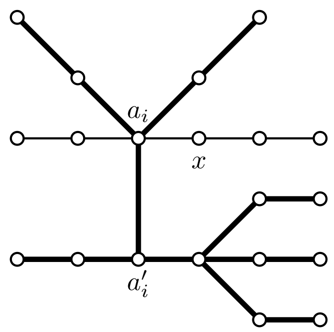


Figure 3: Trees $S$ (in bold) and $T_6^{(4)}[S]$ (the whole graph)

# 3 Generalized constructions

We begin this section with a construction that yields graphs whose game domination number is at least 3/5 of their order in which trees with certain special properties are glued to vertices of an arbitrary graph. In Theorem 3.3 we then present an infinite family of such trees obtained by gluing two trees to $K_2$. Strengthening the conditions that these two trees must satisfy leads us to introduce attachable trees. In Proposition 3.5 we then prove that the trees constructed in the previous section (in Theorem 2.1) are attachable. To be able to attach trees to general graphs in Theorem 3.7 we impose an additional condition on attachable trees and call them special.

Let $G$ be an arbitrary graph on $n$ vertices $v_1, v_2, \ldots, v_n$, let $H_i$, $1 \le i \le n$, be a connected graph of order $m_i \ge 2$, and let $x_i \in V(H_i)$. We denote by

$$G\big[H_1[x_1], H_2[x_2], \ldots, H_n[x_n]\big]$$

the graph of order $\sum_{i=1}^n m_i$ formed by identifying $x_i$ and $v_i$ for $1 \le i \le n$. Whenever all vertices $x_i$ from $H_i$ are clear from the context we simplify this notation to $G\big[H_1, H_2, \ldots, H_n\big]$.

**Proposition 3.1** *Let $G$ be an arbitrary graph of order $n$ and for each $1 \le i \le n$ let $T_i$ be a tree containing a vertex $x_i$ such that $\gamma_g(T_i) = \gamma_g(T_i|x_i)$. Then*

$$\gamma_g(G\big[T_1[x_1], T_2[x_2], \ldots, T_n[x_n]\big]) \ge \sum_{i=1}^n \gamma_g(T_i)\,.$$

**Proof.** To prove the proposition it suffices to give a strategy for Staller that will ensure for each $i$ at least $\gamma_g(T_i)$ moves are made in $T_i$. Her strategy is to follow Dominator in whichever $T_j$ he plays if possible. In this way she guarantees that whenever Dominator makes a move in some $T_j$ the last move previously made in $T_j$ was made by Staller. On the other hand, it can happen that Staller can make consecutive moves in the game restricted to some $T_j$. This is possible when a move by Dominator ended the game restricted to a different $T_i$. It follows that when the game has ended, the game restricted to $T_j$ is the version of either Game 1 or Game 2 in which Dominator may have passed some moves. By Theorem 1.4, in the case of Game 1 the number of moves made in $T_j$ is at least $\gamma_g(T_j|x_j) = \gamma_g(T_j)$, while in Game 2 at least $\gamma_g'(T_j|x_j) \ge \gamma_g(T_j|x_j) = \gamma_g(T_j)$ moves were made in $T_j$. □

In the construction of Proposition 3.1 trees $T_i$ cannot be replaced by arbitrary graphs. To see this, note first that for any vertex $x$ of $C_6$ we have $\gamma_g(C_6) = \gamma_g(C_6|x)$. Observe that $\gamma_g\left(G\big[C_6[x], C_6[x] \ldots, C_6[x]\big]\right) = 2n+1 < 3n = \sum_{i=1}^n \gamma_g(C_6)$ where $n$ is the order of $G$.

**Corollary 3.2** *Let $G$ be a graph of order $n$ and for each $1 \le i \le n$ let $T_i$ be a tree containing a vertex $x_i$ with $\gamma_g(T_i) = \gamma_g(T_i|x_i) = 3|V(T_i)|/5$. For $\widetilde{G} = G\big[T_1[x_1], T_2[x_2], \ldots, T_n[x_n]\big]$ we have*

$$\gamma_g(\widetilde{G}) \ge \frac{3}{5}|V(\widetilde{G})|\,.$$

As noted in Section 2, the path of order 5 and the fork are two examples of trees that fulfil the assumption of Corollary 3.2. Hence given an arbitrary graph $G$ and attaching to each of its vertices either a fork or a $P_5$ in appropriate vertices (corresponding to $a_i$ as in Fig. 1) we get a large family of graphs that attain the conjectured 3/5-bound, unless this family contains a counter-example. More generally, attaching at each vertex of a graph an arbitrary 3/5-tree that fulfils the assumption of Proposition 3.1 either yields a larger 3/5-graph or a counterexample to Conjecture 1.2.

We next present an infinite family of trees that can be attached in the same way as $P_5$ and fork that attain the bound in Corollary 3.2. A vertex $x$ in a graph $G$ is called an *optimal start vertex* if Dominator has an optimal strategy for Game 1 such that $d_1 = x$.

**Theorem 3.3** *Let $T_1$ and $T_2$ be trees, and for $1 \le i \le 2$ let $x_i \in V(T_i)$ such that $\gamma_g(T_i) = \gamma_g(T_i|x_i)$. If $x_1$ is an optimal start vertex of $T_1$ and $\gamma_g(T_2) = \gamma_g'(T_2)$, then*

$$\gamma_g(K_2\big[T_1[x_1], T_2[x_2]\big]) = \gamma_g(T_1) + \gamma_g(T_2).$$

*In addition, $x_1$ is an optimal start vertex of $K_2\big[T_1[x_1], T_2[x_2]\big]$.*

**Proof.** Set $H = K_2\big[T_1[x_1], T_2[x_2]\big]$.

For the upper bound we will prove that Dominator has a strategy on $H$ that requires at most $\gamma_g(T_1) + \gamma_g(T_2)$ moves. Let $d_1 = x_1$. After the first move he always follows (playing optimally) Staller in the same subgraph $T_i$ in which the previous move was made, as long as this is possible. Note that Staller might jump from one subtree to the other in the course of the game. In particular, Staller may be the first to play in $T_2$. In this case the game restricted to $T_2$ corresponds to Game 2 on $T_2|x_2$. We are assuming that $\gamma_g'(T_2) = \gamma_g(T_2) = \gamma_g(T_2|x_2)$.

If Staller is the last one to play in one of the subtrees (that is, Staller makes a move in one of the subtrees that finishes the game restricted to that subtree), then the corresponding game in the other subtree $T_i$, is a Staller-pass game. Note that even if all the moves were made in $T_1$ until the game in that subtree ended on a move of Staller, then Dominator will actually be the first to play in $T_2|x_2$. This situation can also be thought of as a Staller-pass game in which Staller passed on her first move. Since $T_i$ is a tree, by Lemma 1.5, the Continuation Principle, and our hypothesis, it follows that

$$\hat{\gamma}_g'(T_i|x_i) = \gamma_g'(T_i|x_i) \le \gamma_g'(T_i) = \gamma_g(T_i)\,.$$

We conclude that $\gamma_g(H) \le \gamma_g(T_1) + \gamma_g(T_2)$.

The lower bound $\gamma_g(H) \ge \gamma_g(T_1) + \gamma_g(T_2)$ follows immediately from Proposition 3.1. Finally, from the above strategy of Dominator on $H$, in which he started the game by playing $x_1$, we infer that $x_1$ is an optimal start vertex of $H$. □

Let $T$ be a tree and $x \in V(T)$. Then we say that $(T, x)$ is an *attachable tree* (with the attaching vertex $x$) provided that

(i) $x$ is an optimal start vertex in $T$,

(ii) $\gamma_g(T|x) = \gamma_g(T)$, and

(iii) $\gamma_g'(T) = \gamma_g(T)$.

With this definition in hand we can state:

**Corollary 3.4** *If $(T_1, x_1)$ and $(T_2, x_2)$ are attachable trees, then*

$$\gamma_g(K_2\big[T_1[x_1], T_2[x_2]\big]) = \gamma_g(T_1) + \gamma_g(T_2).$$

We already know that $(P_5, a_i)$ and $(F, a_i)$ are attachable trees. They are in fact the first two trees that are obtained by Theorem 2.1. We now show that eventually any tree constructed in that result is such:

**Proposition 3.5** *If $T$ is a 3/5-tree as constructed in Theorem 2.1 with $x$ as specified there, then $(T,x)$ is attachable.*

**Proof.** Since in all cases the strategy of Dominator in the proof of Theorem 2.1 is to play $x$ as the first move, the condition (i) to be attachable is fulfilled.

To prove (ii), note first that $\gamma_g(T|x) \le \gamma_g(T)$ follows from the Continuation Principle. To prove the reverse inequality, let Game 1 be played on $T$. We are going to show that Staller has a strategy on $T|x$ that lasts at least as many moves as the game played on $T$ which will imply that $\gamma_g(T|x) \ge \gamma_g(T)$. Suppose first that $d_1 = x$. Then Staller copies this move to $T|x$. At this point the set of vertices dominated is the same in both games. Staller replies optimally in the imaginary game and copies her move to the real game. Continuing in this way, the game on $T|x$ will last the same number of moves as the game on $T$. Assume next that $d_1 \neq x$. Then Staller copies this move to the imaginary game on $T|x$. Note that this move is legal in $T|x$. Then Staller plays $x$. Observe that this move is then a legal move of Staller also in the real game on $T$ because $x$ has two (non-adjacent) neighbors. Moreover, by the strategy of Staller as described in the proof of Theorem 2.1, this move of Staller is an optimal move in the real game. At this point once again the set of vertices dominated is the same in both games and arguing as in the first case the number of moves will be the same in both games.

To prove (iii), we first observe that $\gamma_g'(T) \ge \gamma_g(T)$ follows by Theorem 1.4. To prove that $\gamma_g'(T) \le \gamma_g(T)$ also holds, we need to give a strategy of Dominator in Game 2 that guarantees that the game lasts no more than $\frac{3}{5}|V(T)|$ moves. The basic strategy of Dominator is to follow Staller in trees $T_i$ as long as possible. Moreover, if the first move of Staller in some $T_i$ is not $u_i$, then Dominator responds by playing $u_i$ (which is, as we know from the proof of Theorem 2.1, an optimal move). Note that in this way on each $T_i$ one of Game 1, Game 2, or Staller-pass will be played. In each case (in the first two cases because $T_i$ is $P_5$ or $F$, and in the last case due to Lemma 1.5), at most $\frac{3}{5}|V(T_i)|$ moves will be played on $T_i$. Suppose that $s$ is played at some stage of the game by Staller. Then Dominator responds with a move on the neighbor of $s'$ if possible. If this is not possible, $s'$ cannot be played in the rest of the game; in this case, Dominator makes an optimal move in some $T_j$ in which the game is not yet finished. The strategy of Dominator is parallel after Staller plays $t$. Similarly, if Staller plays $x$, Dominator simply replies with an optimal move in some $T_j$. By this strategy of Dominator, at most three vertices from $\{s, s', x, t, t'\}$ are played. □

**Corollary 3.6** *Let $(T_1, x_1)$ and $(T_2, x_2)$ be any (attachable) trees constructed in Theorem 2.1. Then $K_2\big[T_1[x_1], T_2[x_2]\big]$ is a 3/5-tree.*

To extend Theorem 3.3 from $K_2$ to an arbitrary graph $G$, we need the following additional assumption on attachable trees. We say that an attachable tree $(T,x)$ is *special* if for any optimal first move of Staller in Game 2 that is different from $x$, Dominator can optimally reply with $d_1 = x$.

**Theorem 3.7** *Let $G$ be a connected graph of order $n$ and for each $1 \le i \le n$, let $(T_i, x_i)$ be a special attachable tree. Then*

$$\gamma_g(G\big[T_1[x_1], T_2[x_2], \ldots, T_n[x_n]\big]) = \sum_{i=1}^{n} \gamma_g(T_i).$$

**Proof.** We only need to prove $\gamma_g(G\big[T_1[x_1],T_2[x_2],\ldots,T_n[x_n]\big]) \le \sum_{i=1}^n \gamma_g(T_i)$ because the reverse inequality follows from Proposition 3.1. That is, we need to provide a strategy for Dominator that limits the number of moves made in each $T_i$ to $\gamma_g(T_i)$. His strategy is to play $d_1 = x_1$, and then to follow Staller in whichever $T_j$ she plays, if possible. If this is not possible and the game is not yet over, Dominator plays in some $T_i$ in which the game is not yet finished. If no vertex on that $T_i$ has been played, Dominator plays $x_i$. This is an optimal move because $(T_i, x_i)$ is attachable. Otherwise he plays optimally in $T_i$. Note that it is possible that in such a $T_i$ the last two moves played in $T_i$ were both made by Dominator, in other words, Staller-pass game is played on $T_i$. Suppose next that the first move in some $T_j$ was made by Staller. We distinguish two cases.

Assume first that this first move in $T_j$ (played by Staller) was optimal with respect to the game restricted to $T_j$. If this move was not $x_j$, then Dominator replies by playing $x_j$. This is optimal since $(T_j, x_j)$ is a special attachable tree. If on the other hand her first move was $x_j$ then Dominator replies with an optimal move in $T_j$.

Suppose next that Staller played the first move in $T_j$ which is not optimal with respect to $T_j$ (but is of course optimal with respect to the entire graph). Then Dominator replies with an optimal move with respect to $T_j$ which will guarantee that after at most $\gamma_g(T_j) - 1$ moves played on $T_j$ all its vertices will be dominated. Hence even if eventually $x_j$ is played by Staller, at most $\gamma_g(T_j)$ vertices of $T_j$ are played.

Using this strategy, Dominator can guarantee that on each $T_j$ one of Game 1, Game 2, or Staller-pass game will be played. In each of these cases his goal will be reached using Lemma 1.5 and the definition of the special attachable tree. $\square$

The reader is invited to verify that $P_5$ and $F$ are special attachable trees as well as are $T_7^{(3)}[P_5, P_5]$ and $T_7^{(4)}[P_5, P_5]$.

For an example of an attachable tree that is not special consider the lower right tree from Fig. 4, call it $T$. Let $x$ be a vertex of degree 3 in $T$. Then one can check that $(T, x)$ is attachable. On the other hand, let $y$ be a vertex of $T$ at distance 5 from $x$. (So $y$ is one of the leaves eccentric to $x$.) Then the first move $s_1' = y$ of Staller followed by $d_1' = x$ enables Staller to enforce 13 vertices to be played. Therefore, $(T, x)$ is not special.

# 4 All 3/5-trees up to 20 vertices

Using computer all 3/5-trees of order $n = 5k$ were obtained for $k \le 4$. We present the complete list of these trees and show how each of them can be obtained by the constructions from previous sections. Some of these 3/5-trees can be obtained in more than one way.

The unique 3/5-tree on 5 vertices is $P_5$. Using the notation of Section 2, we can write $P_5 = T_5^{(3)}[\ ]$, that is, we attach no tree to $P_5$.

There are two 3/5-trees on 10 vertices. The first one is the fork $F$. Note that $F = T_6^{(3)}[P_5]$. The second one is obtained from two copies of $P_5$ by connecting their central vertices $x$ and $x'$ by an edge. In other words, this is the graph $K_2\big[P_5[x], P_5[x']\big]$, in the simplified notation $K_2\big[P_5, P_5\big]$. In the following we will use this simplified notation, always using the convention that $P_5$ and $F$ are attached in vertices $a_i$ as in Fig. 1.

On 15 vertices we have four 3/5-trees. They are

$$S = K_2\big[F, P_5\big], \quad P_3\big[P_5, P_5, P_5\big], \quad T_7^{(4)}[P_5, P_5], \quad \text{and} \quad T_7^{(3)}[P_5, P_5] = T_6^{(3)}[F],$$

see Fig. 3 again for $S$. Fig. 4 displays all 3/5-trees on 20 vertices, there are exactly 10 of them.

The first row shows three trees that can be expressed as $T_k^{(\ell)}$ trees, they are $T_7^{(3)}[F, P_5]$, $T_7^{(4)}[F, P_5]$, and $T_7^{(3)}[P_5, F]$. Note that the second and the third can also be represented as $T_8^{(5)}[P_5, P_5, P_5]$ and $T_8^{(3)}[P_5, P_5, P_5]$, respectively, cf. the figure. The second row of the figure contains three 3/5-trees that are obtained using Theorem 3.3, where the $K_2$ is indicated in bold. The last line contains four trees that are obtained using Theorem 3.7 where the graphs $G$ (twice $P_3$, $K_{1,3}$, and $P_4$) are shown in bold.

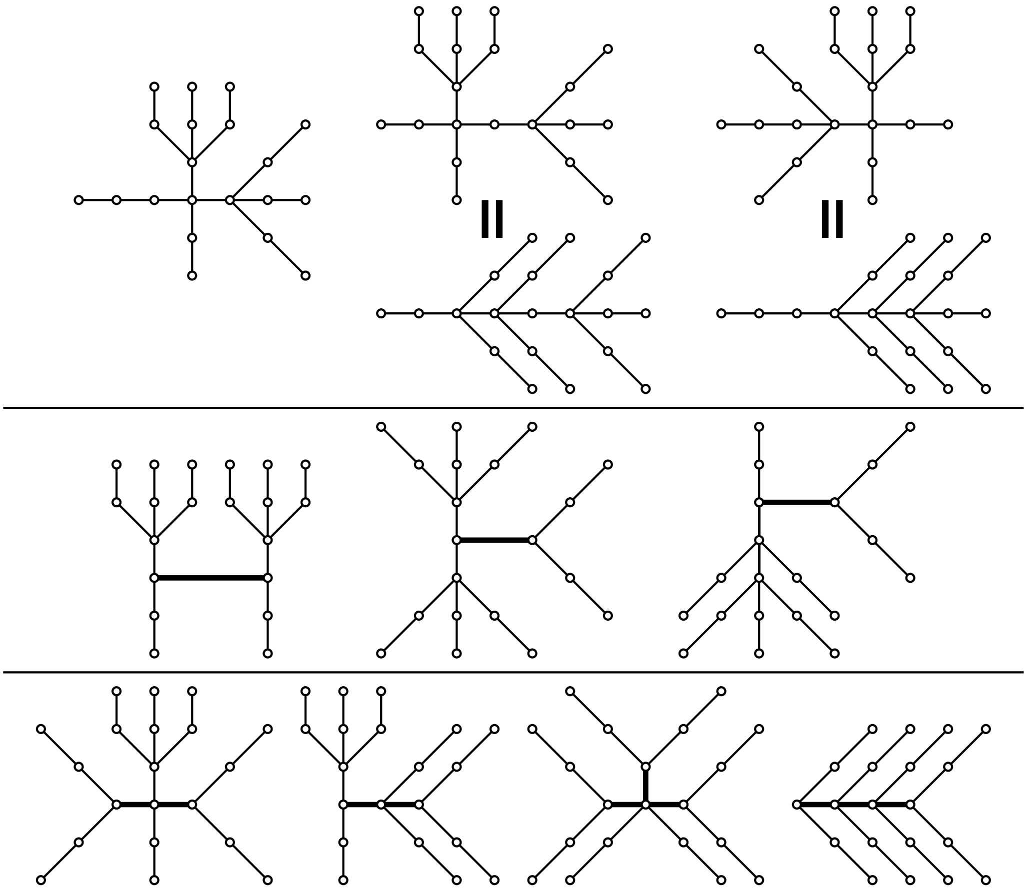

Figure 4: All 3/5-trees on 20 vertices.

Note that the left tree in the middle of Fig. 4, that is, the graph $K_2[F, F]$, can be obtained from the tree $T_6^{(4)}[S]$ shown in Fig. 3 by removing the edge $a_i a_i'$ and adding the edge $a_i' x$. So these two trees are very similar but $\gamma_g(T_6^{(4)}[S]) = 11$ and $\gamma_g(K_2[F, F]) = 12$.

In the rest of the section the computational approach is explained. We used the list of trees due to Brendan McKay [10]. To be able to do the computations for $n = 20$, the following algorithm was designed. During the course of the game a vertex is called *saturated* if every vertex in its closed neighborhood is dominated by the set of vertices already played. For any graph $G$ the algorithm finds $\gamma_g(G|S)$ or $\gamma_g'(G|S)$ for any $S \subseteq V(G)$ by the following recursive formulas:

$$\gamma_g(G|S) = 1 + \min\{\gamma_g'(G|S \cup N[v]) \mid v \text{ is not saturated}\}\,, \tag{1}$$

$$\gamma_g'(G|S) = 1 + \max\{\gamma_g(G|S \cup N[v]) \mid v \text{ is not saturated}\}\,. \tag{2}$$

For every set $S$, the algorithm memorizes $\gamma_g(G|S)$ if it was Dominator who played last, and memorizes $\gamma_g'(G|S)$ if Staller played last. On each step the algorithm checks if the result for some particular set $S$ is already known. Otherwise the algorithm picks some vertex that is not saturated and uses Equations (1) and (2). The algorithm stops when all possibilities are exhausted. The final output is then $\gamma_g(G) = \gamma_g(G|\{\ \})$ and $\gamma_g'(G) = \gamma_g(G|\{\ \})$. The actual program is written in C++, here is its pseudocode:

```
Procedure GD(game, G, S):
G graph,
S set of dominated vertices
  if S == V(G):  return 0
  if game == 1 and γg(G|S) is known:  return γg(G|S)
  else if game == 2 and γ'g(G|S) is known:  return γ'g(G|S)
  else
     results = empty list
     foreach v ∈ V(G):
        if v is not saturated:
           add 1+GD(3-game, G, S ∪ N[v]) to results
     if game == 1:  remember γg(G|S) = min(results)
     else:  remember γ'g(G|S) = max(results)
```

Using an independently designed algorithm, in particular by generating the required lists of trees directly, Bill Kinnersley found the same set of 3/5-trees of order at most 20.

# 5 Concluding remarks

We have verified by computer that all the trees constructed by Theorem 2.1 on up to and including 30 vertices have the property that $s_1' = x$ is the unique optimal move for Staller in Game 2. Hence by Proposition 3.5 all these trees are special attachable. Hence we pose:

**Question 5.1** *Are all attachable trees constructed by Theorem 2.1 special?*

Let $T$ be the following tree on 35 vertices. Start with $P_7$ and identify its third vertex with the vertex $x$ of $T_7^{(3)}[P_5, P_5]$ as constructed in Theorem 2.1 and repeat this for its fifth vertex. We have checked that $\gamma_g(T) = 21$. This naturally gives rise to the following

**Question 5.2** *Can the set of basic trees be enlarged?*

We close with another question.

**Question 5.3** *Are the trees constructed by Corollary 3.6 attachable?*

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