A new framework to approach Vizing's conjecture

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Abstract

We introduce a new setting for dealing with the problem of the domination number of the Cartesian product of graphs related to Vizing's conjecture. The new framework unifies two different approaches to the conjecture. The most common approach restricts one of the factors of the product to some class of graphs and proves the inequality of the conjecture then holds when the other factor is any graph. The other approach utilizes the so-called Clark-Suen partition for proving a weaker inequality that holds for all pairs of graphs. We demonstrate the strength of our framework by improving the bound of Clark and Suen as follows: $\gamma(X \Box Y) \geq \max\{\frac{1}{2}\gamma(X)\gamma_t(Y), \frac{1}{2}\gamma_t(X)\gamma(Y)\}$, where γ stands for the domination number, γ_t is the total domination number, and $X \,\Box\, Y$ is the Cartesian product of graphs X and Y.

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1 Introduction

Let $X = (V(X), E(X))$ be a finite, simple graph. Let $A \subseteq V(G)$. The *open neighborhood* of A is the set, $N(A)$, of vertices in G that have a neighbor in A, and the *closed neighborhood* of A is the set $N[A]$ defined by $N[A] = N(A) \cup A$. For subsets of vertices, A and S, we say that A *dominates* S if $S \subseteq N[A]$; that is, if each vertex of S is in A or is adjacent to some vertex of A. (We also say that *vertices* of A *dominate* (vertices of) S.) The *domination number* of X is the smallest cardinality, denoted $\gamma(X)$, of a set that dominates $V(X)$. If A dominates $V(X)$, we will also say that A dominates the graph X and that A is a *dominating set* of X.

If S and T are subsets of vertices in X, then T *totally dominates* S in X if $S \subseteq N(T)$, that is, each vertex in S is adjacent to a vertex in T. Similarly, we say that a vertex t *totally dominates* a vertex s if $st \in E(X)$. A set T is a *total dominating set* of X if T totally dominates $V(X)$. The size of a minimum total dominating set of a graph X is called the *total domination number* of X, and is denoted by $\gamma_t(X)$.

The *Cartesian product* $X \,\square\, Y$ of graphs X and Y is the graph whose vertex set is $V(X) \times V(Y)$. Two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $X \,\square\, Y$ if either $x_1 = x_2$ and y_1y_2 is an edge in Y, or $y_1 = y_2$ and x_1x_2 is an edge in X. For a vertex x of X, the subgraph of $X \,\square\, Y$ induced by the set $\{\, (x, y) \mid y \in V(Y) \,\}$ is called a *Y-fiber* and is denoted by ${}^xY$. Similarly, for $y \in V(Y)$, the *X-fiber*, X^y, is the subgraph induced by $\{\, (x, y) \mid x \in V(X) \,\}$. We will also use the fiber notation X^y and ${}^xY$ to refer to the set of vertices in these subgraphs; the meaning should be clear from the context. It is clear that all X-fibers are isomorphic to X and all Y-fibers are isomorphic to Y. As usual, the *projection to* X is the map $p_X : V(X \,\square\, Y) \to V(X)$ defined by $p_X(x, y) = x$. Similarly, the *projection to* Y is the map $p_Y : V(X \,\square\, Y) \to V(Y)$ defined by $p_Y(x, y) = y$.

A subset S of vertices in a graph X is a *packing* if the closed neighborhoods of vertices in S are pairwise disjoint. The *packing number* $\rho(X)$ is the maximum cardinality of a packing. The *clique cover number* of a graph X, denoted $\theta(X)$, is the minimum number of complete subgraphs of X whose union is $V(X)$. The graph X is called a *BG-graph* if X is a spanning subgraph of some graph X' such that $\gamma(X) = \theta(X') = \gamma(X')$.

The following conjecture concerning the domination number of a Cartesian product was made by V. G. Vizing in 1968.

Conjecture 1. ([14]) *For every pair of graphs X and Y,*

$$\gamma(X \,\square\, Y) \geq \gamma(X)\,\gamma(Y). \tag{1}$$

We say that a graph X *satisfies Vizing's conjecture* if (1) holds for every graph Y. The only classes of graphs that are known to satisfy Vizing's conjecture are BG-graphs [2], chordal graphs [1], those with domination number 3 [13, 4], Type $\mathcal{X}$ graphs [9] and those whose fair domination number and domination number are equal [6]. See the survey [5] for other results on the conjecture.

In this paper, we present a new framework for studying the domination number of the Cartesian product of graphs. Our framework partitions the vertex set of the Cartesian product into cells, each of which is assigned one of four colors determined by properties of the vertices in the cell with respect to a given dominating set in the Cartesian product. Using the resulting framework we present transparent proofs of several previously known results related to Vizing's conjecture, including the classical result due to Barcalkin and German [2] and the important result due to Clark and Suen [7], as well as new improved results. These results by Barcalkin-German and Clark-Suen were considered as completely different approaches to the conjecture, since the first one proves that the conjectured bound holds for a (large) class of graphs, while the second one provides a proof of a weaker inequality related to the conjecture. The new framework thus unifies most of the previous partial results on the conjecture and combines them into a more transparent form.

We proceed as follows. In Section 2 we discuss our new framework. Thereafter, we present some key preliminary lemmas in Section 3. In Section 4 we give our main results and show that our new framework encompasses several of the most important results to date on Vizing's conjecture, including results due to Clark and Suen [7], Barcalkin and German [2], Suen and Tarr [12], Brešar [3], and Zerbib [15]. Further we show that if X and Y are arbitrary isolate-free graphs, then $\gamma(X \,\square\, Y) \geq \max\{\frac{1}{2}\gamma(X)\gamma_t(Y), \frac{1}{2}\gamma_t(X)\gamma(Y)\}$.

2 A Framework for Domination in Cartesian Products

For our initial framework we use that of the proof of the bound due to Clark and Suen in [7]. Let X be a graph with $\gamma(X) = k$ and let $\{u_1, \ldots, u_k\}$ be a minimum dominating set of X. Consider a partition $\pi = \{\pi_1, \ldots, \pi_k\}$ of $V(X)$ chosen so that $u_i \in \pi_i$ and $\pi_i \subseteq N[u_i]$ for each i. Let $Y_i = \pi_i \times V(Y)$. For a vertex y of Y the set of vertices $\pi_i \times \{y\}$ is called a *cell*, and is denoted shortly by π_i^y. (We may say that the cell π_i^y belongs to Y_i, and from the other perspective it also belongs to the fiber X^y.)

Let D be a minimum dominating set of $X \,\square\, Y$. As usual, let $[k] = \{1, \ldots, k\}$. For $i \in [k]$, let $D_i = D \cap Y_i$. Similarly, we denote $D^y = D \cap X^y$ for $y \in V(Y)$. The cell π_i^y is *blue* if $\pi_i^y \cap D \neq \emptyset$ and π_i^y is dominated by D^y (the latter condition will also be expressed as π_i^y is *horizontally dominated* by D.) The cell π_i^y is *green* if $\pi_i^y \cap D \neq \emptyset$ and π_i^y is not horizontally dominated by D. Finally, we call the cell *red* if it is horizontally dominated by D, and no vertex of π_i^y is dominated by D_i. All the remaining cells in X^y are colored *white*. Note that exactly the cells with color blue and green contain vertices of D. We color the vertices in $D \cap \pi_i^y$ blue if the cell π_i^y is blue. If a cell π_i^y is green, then we color the vertices in $D \cap \pi_i^y$ green. This coloring of the vertices of D is a partition of D into subsets of blue vertices and green vertices. We note that within the Cartesian product $X \,\square\, Y$ the only vertices that have a color are the blue vertices (the vertices from D that belong to a blue cell), and the green vertices (the vertices of D that belong to a green cell).

To illustrate this method of coloring cells consider the following example. The graph X has order 8 with vertex set $\{x_1, x_2, \ldots, x_8\}$ and $Y = C_4$ with $V(Y) = \{y_1, y_2, y_3, y_4\}$. See Figure 1. Note that edges in the Cartesian product $X \,\square\, Y$ are not drawn in the figure to simplify viewing.

The domination number of X is 3, and $\{x_2, x_4, x_7\}$ is a minimum dominating set of X. We use the partition $\{\pi_1, \pi_2, \pi_3\}$, where $\pi_1 = \{x_1, x_2, x_3\}, \pi_2 = \{x_4, x_5\}$ and $\pi_3 = \{x_6, x_7, x_8\}$. In this example $\gamma(X \,\square\, Y) = 8$ and a minimum dominating set in $X \,\square\, Y$ is denoted by the solid vertices. Using the definitions above we find that there are two blue cells, $\pi_1^{y_1}$ and $\pi_2^{y_3}$, and there is one red cell, $\pi_2^{y_1}$. Cells $\pi_1^{y_2}, \pi_3^{y_1}, \pi_3^{y_2}, \pi_3^{y_3}$, and $\pi_3^{y_4}$ are green; the remaining cells $\pi_1^{y_3}, \pi_1^{y_4}, \pi_2^{y_2}$, and $\pi_2^{y_4}$ are white.

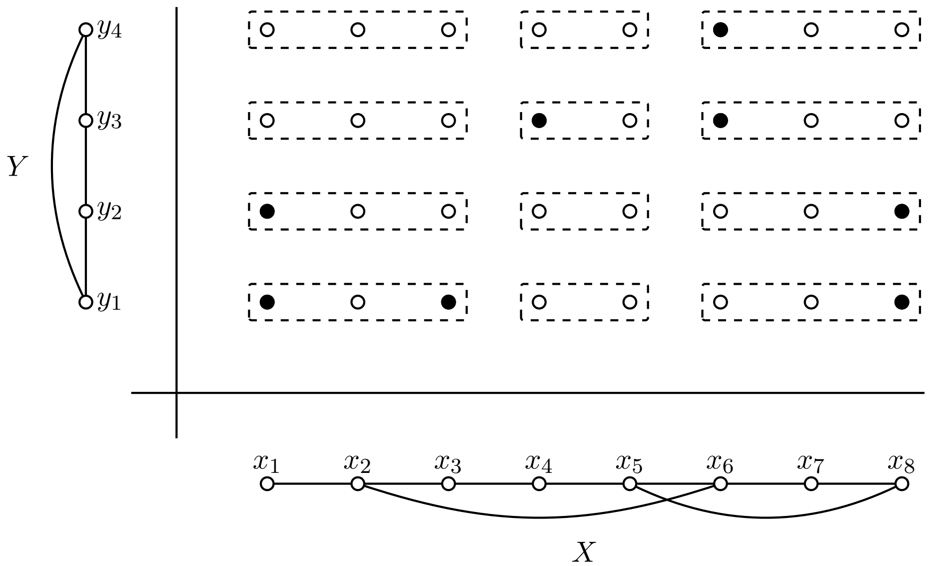

Figure 1: Cell Coloring Example

In the projection $p_Y(Y_i)$ of the cells of Y_i to Y, the vertex $y = p_Y(\pi_i^y)$ receives the color of the cell π_i^y for each $i \in [k]$. Let B_i, G_i and R_i be the resulting set of vertices in Y colored blue, green and red, respectively, for $i \in [k]$.

We also introduce the following notation concerning the size of some of the sets having certain colors. Let b'_y be the number of blue cells in the fiber X^y, let b'_i the number of blue cells in Y_i, and let b' the total number of all blue cells in $X \,\square\, Y$. We define analogously g'_y, g'_i, g', associated with the green cells and r'_y, r'_i, and r', associated with red cells. Next, let b_y and b_i denote the number of blue vertices in X^y and Y_i, respectively, and let b be the total number of blue vertices in $X \,\square\, Y$. In an analogous way we define g_y, g_i and g, associated with the green vertices. Clearly, $|D| = b + g$. Since each blue cell contains at least one blue vertex, we get $b_y \geq b'_y$, $b_i \geq b'_i$ and $b \geq b'$, and analogously, $g_y \geq g'_y$, $g_i \geq g'_i$ and $g \geq g'$.

3 Key Preliminary Lemmas

In this section, we present some key preliminary lemmas.

Lemma 2. *$b' + g' + r' \geq \gamma(X)\gamma(Y)$.*

Proof. For any $i \in [\gamma(X)]$, we consider the cells of Y_i. Recall that $p_Y(Y_i)$ denotes the projection of Y_i onto Y. We color Y so that the vertex $y = p_Y(\pi_i^y)$ receives the color of the cell π_i^y. We claim that the set S of vertices in Y that received (by the projection) one of the colors blue, green or red, forms a dominating set of Y. To see this it suffices to show that every vertex in Y that received color white (by the projection) is dominated by a vertex that received color blue or green.

Indeed, a white vertex $y \in V(Y)$ is projected from a white cell π_i^y. By definition the white cell π_i^y, is either not dominated by $D \cap X^y$, or it is dominated by $D \cap X^y$ but contains at least one vertex that is dominated by D_i. In either case this white cell π_i^y contains a vertex (u, y) that is dominated by a vertex $(u, y') \in D$, which is clearly blue or green. Hence $y' = p_Y(\pi_i^{y'})$ received color blue or green and it dominates y in Y, as claimed. Therefore, S is a dominating set of Y and it is clear that $|S| = b'_i + g'_i + r'_i$, which implies $b'_i + g'_i + r'_i \geq \gamma(Y)$. Summing up over all i between 1 and $\gamma(X)$, we get

$$b' + g' + r' = \sum_{i=1}^{\gamma(X)} (b'_i + g'_i + r'_i) \geq \gamma(X)\gamma(Y). \qquad \square$$

We proceed further with some additional notation. For $i \in [\gamma(X)]$, we consider the red cells in Y_i. In the projection $p_Y(Y_i)$ to Y, let R_i be the set of vertices in Y colored red. Let s'_i be the minimum number of vertices in Y needed to dominate the set R_i. Let

$$s' = \sum_{i=1}^{\gamma(X)} s'_i.$$

Using this definition, we can improve Lemma 2 as follows.

Corollary 3. $b' + g' + s' \geq \gamma(X)\gamma(Y)$.

Proof. For any $i \in [\gamma(X)]$, we consider the cells of Y_i, and, as before, in the projection $p_Y(Y_i)$ to Y, the vertex $y = p_Y(\pi_i^y)$ receives the color of the cell π_i^y. Let B_i, G_i and R_i be the set of vertices in Y colored blue, green and red, respectively. We note that $|B_i| = b'_i$, $|G_i| = g'_i$, and $|R_i| = r'_i$. Let S_i be a minimum set of vertices in Y needed to dominate the set R_i, and so $|S_i| = s'_i$. We claim that the set $T = B_i \cup G_i \cup S_i$ forms a dominating set of Y. Analogously as in the proof of Lemma 2, every white vertex is dominated by a blue or green vertex. By definition of the set S_i, every red vertex is dominated by a vertex in S_i. Therefore, T is a dominating set of Y, which implies $b'_i + g'_i + s'_i \geq \gamma(Y)$. Summing up over all i between 1 and $\gamma(X)$, we get

$$b' + g' + s' = \sum_{i=1}^{\gamma(X)} (b'_i + g'_i + s'_i) \geq \gamma(X)\gamma(Y). \qquad \square$$

Lemma 4. $r' \leq b - b' + g$.

Proof. For $y \in V(Y)$, we consider the blue and green vertices in the fiber X^y. These $b_y + g_y$ vertices dominate all vertices in the red and blue cells in X^y. In the projection $p_X(X^y)$ of X^y onto X we note that the vertices projected from the blue and green vertices, together with the vertices u_i from every projected white and green cell π_i^y, form a dominating set of X. Clearly, the number of white and green cells in X^y is $\gamma(X) - b'_y - r'_y$. Hence, $\gamma(X) \leq b_y + g_y + (\gamma(X) - b'_y - r'_y)$, or, equivalently, $r'_y \leq b_y - b'_y + g_y$. Summing over all vertices $y \in V(Y)$, we infer that $r' \leq b - b' + g$. $\square$

4 Main Results

As an immediate consequence of Lemma 2 and 4, we have the following fundamental result due to Clark and Suen [7].

Theorem 5. ([7]) $\gamma(X \,\Box\, Y) \geq \frac{1}{2}\gamma(X)\gamma(Y)$.

Proof. Let D be a minimum dominating set of $X \,\Box\, Y$. By Lemma 2 and 4, the following holds.

$$\begin{array}{rcl} \gamma(X)\gamma(Y) & \overset{(Lemma\ 2)}{\leq} & b' + g' + r' \\ & \overset{(Lemma\ 4)}{\leq} & b' + g' + (b - b' + g) \\ & = & b + g + g' \\ & \leq & 2(b+g) \\ & = & 2|D|, \end{array}$$

implying that $\gamma(X \,\Box\, Y) = |D| \geq \frac{1}{2}\gamma(X)\gamma(Y)$. $\square$

Recall that a graph X is *decomposable*, as defined by Barcalkin and German in [2], if its vertex set can be partitioned into $\gamma(X)$ subsets each of which induces a clique. In particular, note that for any minimum dominating set of $X \,\Box\, Y$, where Y is an arbitrary graph, and in the setting in which the sets in π induce cliques, there will be no green vertices. Hence the next result may be viewed as an extension of the classical result of Barcalkin and German.

Theorem 6. *If there is no green cell, then $\gamma(X \,\Box\, Y) \geq \gamma(X)\gamma(Y)$. In particular, the conjecture is true for decomposable graphs.*

Proof. Suppose there is no green cell, and so $g = 0$. Let D be a minimum dominating set of $X \,\Box\, Y$. By Lemma 2 and 4,

$$\gamma(X \,\Box\, Y) = |D| = b \geq b' + r' \geq \gamma(X)\gamma(Y). \qquad \square$$

By our new coloring technique, we show next that the inequality given by Clark and Suen in Theorem 5 is a strict inequality and can be improved in two different ways. Our framework can be used to prove the result of Suen and Tarr [12] as well as to show that for each pair of graphs the fraction 1/2 can be replaced by a larger rational number. For this purpose, we need the following additional lemma.

Lemma 7. *Renaming the graphs X and Y if necessary, we may assume that $b \geq \gamma(X)$.*

Proof. Suppose that the projection $p_X(D)$ of D onto X is not equal to $V(X)$. Thus there is some vertex $x \in X$ such that the Y-fiber, ${}^xY$, contains no vertex of D. Let y be an arbitrary vertex of Y. If the X-fiber, X^y, contains no vertex of D, then the vertex (x, y) is not dominated by D in $X \,\square\, Y$, which is a contradiction. Hence, the X-fiber X^y contains at least one vertex of D, implying that the projection $p_Y(D)$ of D onto Y is the entire set $V(Y)$. Renaming the graphs X and Y if necessary, we may therefore assume that the projection $p_X(D)$ is the set $V(X)$. For each vertex u_i belonging to the minimum dominating set $\{u_1, \ldots, u_k\}$ of X, let (u_i, y) be a vertex of D that belongs to the Y-fiber, ${}^{u_i}Y$. We note that the cell π_i^y is a blue cell, implying that there are at least $k = \gamma(X)$ blue cells. Hence, $b \geq \gamma(X)$. $\square$

We illustrate next that the coloring framework can be used to prove the important result on Vizing's conjecture by Suen and Tarr [12] that was historically the first improvement of Theorem 5.

Theorem 8. ([12]) *For any graphs X and Y,*

$$\gamma(X \,\square\, Y) \geq \frac{1}{2}\gamma(X)\gamma(Y) + \frac{1}{2}\min\{\gamma(X), \gamma(Y)\}\,.$$

Proof. By Lemma 7 we may assume that $b \geq \gamma(X)$. Following the proof of Theorem 5 we get

$$\gamma(X) + \gamma(X)\gamma(Y) \leq b + \gamma(X)\gamma(Y) \leq 2b + g + g' \leq 2|D|\,.$$

The conclusion of the theorem follows immediately. $\square$

4.1 Results in terms of the Total Domination Number

In this section, we use the coloring framework to obtain bounds in terms of the total domination number. Having identified that inequalities involving b', g' and r' are useful in deducing bounds, we now show how to bound a linear combination of them in terms of the domination number of X and the total domination number of Y. For this purpose, we need the following additional lemma.

Lemma 9. *$2b' + g' + r' \geq \gamma(X)\gamma_t(Y)$.*

Proof. For each $i \in [\gamma(X)]$, we consider the cells of Y_i. Recall that in the projection $p_Y(Y_i)$ to Y, the vertex $y = p_Y(\pi_i^y)$ receives the color of the cell π_i^y. Further, recall that B_i, G_i and R_i is the set of vertices in Y colored blue, green and red, respectively. We note that $|B_i| = b_i'$, $|G_i| = g_i'$, and $|R_i| = r_i'$. For each vertex $y \in B_i \cup R_i$, let t_y be a neighbor of y in Y, and let

$$T_i = \bigcup_{y \in B_i \cup R_i} \{t_y\}.$$

We note that $|T_i| \le |B_i| + |R_i| = b'_i + r'_i$. We claim that the set $B_i \cup G_i \cup T_i$ is a total dominating set of Y. Every white vertex in Y has a neighbor in $B_i \cup G_i$, and every blue and red vertex has a neighbor in T_i. Further, let y be an arbitrary green vertex in Y, and let π_i^y be the corresponding green cell in Y_i. Since the cell π_i^y is not horizontally dominated, it contains a vertex (u, y) that is dominated in $X \,\square\, Y$ by a vertex $(u, y') \in D_i$ for some neighbor y' of y. Since the vertex (u, y') is green or blue, the cell $\pi_i^{y'}$ is green or blue. Therefore, the vertex y' received the color green or blue, implying that y is totally dominated by a vertex of $B_i \cup G_i$. Hence also every green vertex in Y has a neighbor in $B_i \cup G_i$. Therefore, the set $B_i \cup G_i \cup T_i$ is a total dominating set of Y, as claimed. Recall that $|B_i| = b'_i$, $|G_i| = g'_i$, and $b'_i + r'_i \ge |T_i|$. Thus, $2b'_i + g'_i + r'_i \ge |B_i \cup G_i \cup T_i| \ge \gamma_t(Y)$. Summing up over all i between 1 and $\gamma(X)$, we get

$$2b' + g' + r' = \sum_{i=1}^{\gamma(X)} (2b'_i + g'_i + r'_i) \ge \gamma(X)\gamma_t(Y). \qquad \square$$

Theorem 10. *For any graphs X and Y,*

$$\gamma(X \,\square\, Y) \ge \max\left\{ \frac{1}{2}\gamma(X)\gamma_t(Y), \frac{1}{2}\gamma_t(X)\gamma(Y) \right\}.$$

Proof. Let D be a minimum dominating set of $X \,\square\, Y$. By Lemma 4 and Lemma 9, the following holds.

$$\begin{array}{rcl}
\gamma(X)\gamma_t(Y) & \stackrel{(Lemma\ 9)}{\le} & 2b' + g' + r' \\
& \stackrel{(Lemma\ 4)}{\le} & 2b' + g' + (b - b' + g) \\
& = & (g + g') + (b + b') \\
& \le & 2(b + g) \\
& = & 2|D|,
\end{array}$$

implying that $\gamma(X \,\square\, Y) = |D| \ge \frac{1}{2}\gamma(X)\gamma_t(Y)$. Interchanging the roles of X and Y, the desired result follows. $\square$

Since $\gamma_t(Y) \ge \gamma(Y)$, Theorem 10 provides an improvement of Theorem 5 due to Clark and Suen [7]. While the result could also be proved by using the standard Clark-Suen technique, our proof serves to illustrate the use of the coloring framework for attacking Vizing's conjecture.

In order to see that Theorem 10 provides an improvement over Theorem 5 we now present an infinite family of graphs $\{X_k\}$, none of which is known to satisfy Vizing's conjecture. Furthermore for a positive integer k, $\gamma(X_k) = k + 3$ whereas $\gamma_t(X_k) = 2k + 3$. Applying Theorem 10 we see that

$$\begin{array}{rcl}
\gamma(X_k \,\square\, Y) & \ge & \frac{1}{2}\gamma_t(X_k)\gamma(Y) \\
& = & \frac{1}{2}(2k + 3)\gamma(Y) \\
& = & \frac{1}{2}(\gamma(X_k) + k)\gamma(Y) \\
& = & \frac{1}{2}\gamma(X_k)\gamma(Y) + \frac{k}{2}\gamma(Y).
\end{array}$$

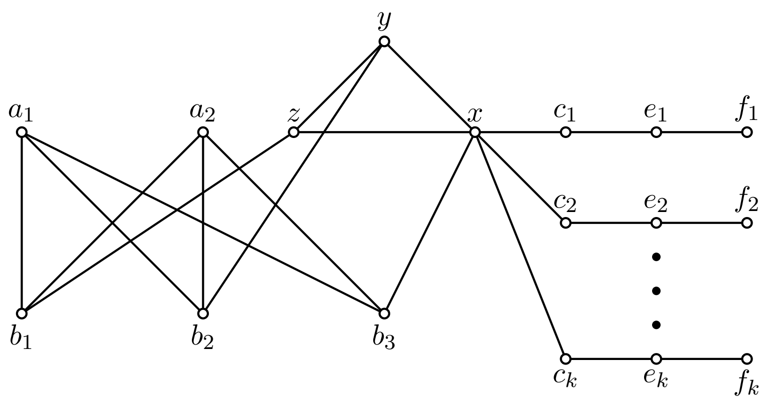

Figure 2: The graph X_k

For a positive integer k we let X_k be the graph of order $3k+8$ shown in Figure 2, where there are k copies of a path of order 3 attached at vertex x. It is easy to verify that the subgraph Q of X_k induced by the set of vertices $\{a_1, a_2, b_1, b_2, b_3, x, y, z\}$ has domination number 3 and total domination number 3. In addition, if any missing edge is added to Q, the resulting graph has domination number 2. From this it follows that the clique cover number of Q is 4, and thus Q is not a BG-graph.

We now proceed to show that X_k is not a BG-graph. That is, it is not possible to add some missing edges to X_k and produce a graph with domination number $k+3$ and clique cover number $k+3$. Suppose for the sake of contradiction that there is a subset $M \subseteq E(\overline{X_k})$ such that if $X_k^+ = X_k + M$, then $\gamma(X_k^+) = k+3 = \theta(X_k^+)$. Let $C = \{c_1, \ldots, c_k\}$, let $E = \{e_1, \ldots, e_k\}$ and let $F = \{f_1, \ldots, f_k\}$. If $a_1 e_i \in M$ for some $i \in [k]$, then $\{a_2, x\} \cup E$ is a dominating set of X_k^+ which is a contradiction. Similarly, $a_2 e_i \notin M$ for any $i \in [k]$. If $b_1 e_i \in M$ for some $i \in [k]$, then $\{b_2, x\} \cup E$ is a dominating set of X_k^+ of cardinality $k+2$, which is again a contradiction. A similar argument shows that neither of b_2 nor b_3 is adjacent in X_k^+ to any vertex of E. If $a_1 f_i \in M$ for some $i \in [k]$, then $\{a_2, f_i, x\} \cup (E - \{e_i\})$ is a set of cardinality $k+2$ that dominates X_k^+, which is a contradiction. A parallel argument holds when $a_2 f_i \in M$. Similarly, a contradiction is produced if either b_1 or b_2 is adjacent in X_k^+ to a vertex of F.

Consider the partition of $V(X_k^+)$ into $k+3$ cliques of X_k^+. Note that at most one of a_1 and a_2 is in a clique with b_3 since a_1 and a_2 are not adjacent in X_k^+. Without loss of generality we assume that a_1 and b_3 belong to different cliques. Since no vertex of $E \cup F$ is adjacent to a_1 in X_k^+, if a vertex w belongs to the clique containing a_1, then $w \in \{b_1, b_2\} \cup C$. Since b_1 and b_2 belong to different cliques, we can assume without loss of generality that any such vertex w is in $C \cup \{b_1\}$.

Suppose first that x and b_3 belong to the same clique. Note that x, z and b_2 belong to three distinct cliques in the partition. In this case by adding a single vertex from each of the remaining $k-1$ cliques not containing a_1 to $\{x, z, b_2\}$ we get a dominating set of cardinality $k+2$. Finally suppose that x are b_3 are in different cliques of the partition. By choosing x,

a_2 and whichever of b_2 and b_3 is not in the same clique as a_2 we can form a dominating set of size $k+2$ by adding a single vertex from each of the other $k-1$ cliques not containing a_1. This final contradiction proves that no such partition of $V(X_k^+)$ into $k+3$ cliques exists in X_k^+. That is, X_k is not a BG-graph.

Note that X_k is not a chordal graph. Furthermore, X_k is not of Type $\mathcal{X}$ and the fair domination number of X_k appears to be less than $k+3$. Therefore, it was not known whether X_k satisfies Vizing's conjecture.

4.2 Results in terms of the Packing Number

In this section, our aim is to show that the coloring framework can be used to prove previously published results on Vizing's conjecture involving bounds in terms of the packing number. For this purpose we first present two preliminary lemmas.

Lemma 11. *If A is an arbitrary subset of vertices of a graph X, then there is a set $S \subseteq V(X)$ such that S dominates A and $|S| \le \frac{1}{2}(|A| + \rho(X))$.*

Proof. Consider the collection of all partitions of A, say $(A_1, \dots, A_k, A_{k+1})$, such that A_{k+1} is a 2-packing in X and for each $i \in [k]$, A_i consists of two vertices of distance at most 2 in X. (We allow the possibilities that $A_{k+1} = \emptyset$ or $A_{k+1} = A$.) Choose such a partition with k as large as possible. By choice, $|A_{k+1}| \le \rho(X)$. For $i \in [k]$, let z_i be a vertex that dominates A_i, and define S by $S = \{z_1, \dots, z_k\} \cup A_{k+1}$. It is clear that S dominates A and

$$|S| \le k + |A_{k+1}| = \frac{|A| - |A_{k+1}|}{2} + |A_{k+1}| = \frac{|A| + |A_{k+1}|}{2} \le \frac{|A| + |\rho(X)|}{2}. \qquad \square$$

Our next lemma bounds the parameter s' in terms of the domination number of X and the packing number of Y.

Lemma 12. *$s' \le \frac{1}{2}(r' + \gamma(X)\rho(Y))$.*

Proof. We adopt the notation employed in the proof of Lemma 3. Let $i \in [\gamma(X)]$. By Lemma 11 applied to the graph Y the set R_i can be dominated by a subset S_i of Y such that $s'_i = |S_i| \le \frac{1}{2}(|R_i| + \rho(Y))$. Summing up over all i between 1 and $\gamma(X)$, we get

$$s' = \sum_{i=1}^{\gamma(X)} s'_i \le \sum_{i=1}^{\gamma(X)} \frac{1}{2}(r'_i + \rho(Y)) = \frac{1}{2}(r' + \gamma(X)\rho(Y)). \qquad \square$$

We are now in a position to show how the coloring framework can be used to prove the following result, which was first proven in [3].

Theorem 13. *If X and Y are arbitrary graphs, then*

$$\gamma(X \,\square\, Y) \geq \max\left\{\left(\frac{2\gamma(X)-\rho(X)}{3}\right)\gamma(Y), \left(\frac{2\gamma(Y)-\rho(Y)}{3}\right)\gamma(X)\right\}.$$

Proof. Interchanging the roles of X and Y, it suffices for us to prove the inequality $\gamma(X \,\square\, Y) \geq (2\gamma(Y)-\rho(Y))\gamma(X)/3$. Let D be a minimum dominating set of $X \,\square\, Y$. By Lemmas 4, 3 and 12, the following holds.

$$\begin{array}{rcl}
|D| & = & b+g \\
 & \geq & \dfrac{b+g}{2}+\dfrac{b'+g'}{2} \\
 & \geq & \dfrac{b-b'+g}{2}+b'+\dfrac{g'}{2} \\
 & \overset{(Lemma\ 4)}{\geq} & \dfrac{r'}{2}+b'+\dfrac{g'}{2} \\
 & \overset{(Lemma\ 12)}{\geq} & s'-\dfrac{1}{2}\gamma(X)\rho(Y)+b'+\dfrac{g'}{2} \\
 & = & b'+g'+s'-\dfrac{1}{2}\gamma(X)\rho(Y)-\dfrac{g'}{2} \\
 & \overset{(Lemma\ 3)}{\geq} & \gamma(X)\gamma(Y)-\dfrac{1}{2}\gamma(X)\rho(Y)-\dfrac{g'}{2}.
\end{array}$$

Therefore, since $|D| = g+b \geq g'$, we infer the following.

$$\frac{3}{2}|D| = |D| + \frac{1}{2}|D| \geq |D| + \frac{1}{2}g' \geq \gamma(X)\gamma(Y) - \frac{1}{2}\gamma(X)\rho(Y), \tag{2}$$

or, equivalently,

$$|D| \geq \frac{\gamma(X)}{3}(2\gamma(Y)-\rho(Y)). \qquad \square$$

4.3 Results involving Domination Numbers of X and Y

We close with the following stronger result than Theorem 8. We remark that this result was first proven by Zerbib [15] using the standard Clark-Suen technique. As before we use our coloring framework except that the dominating set of X is chosen differently here. However, the coloring, projecting and counting proceeds exactly as in the underlying framework, noting that the color of the resulting cells determines the color of the set of vertices in the dominating set D of the Cartesian product $X \,\square\, Y$.

Theorem 14. *For any graphs X and Y,*

$$\gamma(X \,\square\, Y) \geq \frac{1}{2}\gamma(X)\gamma(Y) + \frac{1}{2}\max\{\gamma(X), \gamma(Y)\}.$$

Proof. Renaming the graphs X and Y if necessary, we may assume that $\gamma(X) \geq \gamma(Y)$. In what follows, we adopt precisely the notation defined in Section 2 for our initial framework, except that here $k \geq \gamma(X)$ and we define the dominating set $\{u_1, \ldots, u_k\}$ of X differently. Let D_X be the projection $p_X(D)$ of the set D to X. Since D is a (minimum) dominating set of $X \,\square\, Y$, the set D_X is a dominating set of X. Among all subsets of vertices of D_X that form a dominating set of X, let $U = \{u_1, \ldots, u_k\}$ be one of minimum size. Possibly, $U = D_X$. Since U is a dominating set of X, we note that $\gamma(X) \leq |U| = k$. We proceed further with the following two claims.

Claim 1. $b' + g' + r' \geq k\gamma(Y)$.

Proof. Proceeding exactly as in the proof of Lemma 2, we have that $b'_i + g'_i + r'_i \geq \gamma(Y)$ holds for every $i \in [k]$. Summing up over all i between 1 and k, we get $b' + g' + r' = \sum_{i=1}^{k} (b'_i + g'_i + r'_i) \geq k\gamma(Y)$. (□)

Claim 2. $r' \leq b - b' + g$.

Proof. Proceeding as in the proof of Lemma 4, for each vertex $y \in V(Y)$ we consider the set, U_y say, consisting of the projection $p_X(D \cap X^y)$ of $D \cap X^y$ onto X together with the set of all vertices u_i from every projected white and green cell π_i^y; that is,

$$U_y = p_X(D \cap X^y) \cup \{u_i \in U \mid \pi_i^y \text{ is a white or green cell}\}.$$

Since the set U_y is a dominating set of X and since $U_y \subseteq D_X$, by the minimality of U we have $|U| \leq |U_y|$. Thus, $k = |U| \leq |U_y| = b_y + g_y + (k - b'_y - r'_y)$, or, equivalently, $r'_y \leq b_y - b'_y + g_y$. Summing over all vertices $y \in V(Y)$, we infer that $r' \leq b - b' + g$. (□)

We now return to the proof of Theorem 14. For each vertex $u_i \in U$, let (u_i, y) be a vertex of D that belongs to the Y-fiber, ${}^{u_i}Y$. Since the cell π_i^y is a blue cell, there are at least k blue cells, implying that $b \geq k$. By Claim 1 and 2, the following holds.

$$\begin{aligned}
\gamma(X) + \gamma(X)\gamma(Y) &\leq k + k\gamma(Y) \\
&\leq b + k\gamma(Y) \\
&\stackrel{(Claim\ 1)}{\leq} b + (b' + g' + r') \\
&\stackrel{(Claim\ 2)}{\leq} b + (b' + g' + (b - b' + g)) \\
&= 2b + g + g' \\
&\leq 2(b + g) \\
&= 2|D|,
\end{aligned}$$

implying that $\gamma(X \,\square\, Y) = |D| \geq \frac{1}{2}\gamma(X)\gamma(Y) + \frac{1}{2}\max\{\gamma(X), \gamma(Y)\}$. □

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