On Cartesian products having a minimum dominating set that is a box or a stairway

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Abstract

In this note we characterize the pairs of graphs G and H, for which $\gamma(G \,\Box\, H)$ equals $\min\{|V(G)|, |V(H)|\}$. Notably, assuming that $|V(G)| \le |V(H)|$, G can be an arbitrary graph, and H is a join $L \oplus F$, where L is any spanning supergraph of the graph $\mathcal{L}(G : A_1, \ldots, A_\ell)$, which is determined by a partition $(A_1, \ldots, A_\ell)$ of $V(G)$ and F is any graph such that $|V(F)| \ge |V(G)| - \ell$. Furthermore we give some sufficient and some necessary conditions for pairs of graphs G and H to satisfy $\gamma(G \,\Box\, H) = \min\{\gamma(G)|V(H)|, |V(G)|\gamma(H)\}$.

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1 Introduction

V.G. Vizing [9] conjectured that the domination number of the Cartesian product of two graphs is at least the product of their domination numbers; that is, $\gamma(G \,\Box\, H) \ge \gamma(G)\gamma(H)$. While the conjecture is still open after almost 50 years, numerous attempts

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to prove it initiated the study of related problems in Cartesian products of graphs, which also yielded several partial results that support the truth of the conjecture; see the recent survey [1].

It is easy to prove the following upper and lower bounds for the domination number of such a Cartesian product; see [3] and [8]:

$$\min\{|V(G)|, |V(H)|\} \leq \gamma(G \,\square\, H) \leq \min\{\gamma(G)|V(H)|, |V(G)|\gamma(H)\}. \tag{1}$$

Any pair of complete graphs or any graph G and its complement H show the lower bound is sharp. If at least one of G or H is the complement of a complete graph, then the upper bound is attained. The motivation for the current work is an attempt to characterize pairs G and H that demonstrate the sharpness of the bounds in (1).

In the next section we present the main definitions and notation used in the paper. In particular, we introduce the so-called box dominating sets and stairway dominating sets, which appear as minimum dominating sets when the upper and the lower bounds in (1) are realized, respectively. Section 3 contains the main result, which is a characterization of the pairs of graphs G and H for which $\gamma(G \,\square\, H) = \min\{|V(G)|, |V(H)|\}$. In Section 4 we present a sufficient and a necessary condition for graphs G and H to achieve the upper bound in (1). The former condition involves the concept of k-rainbow domination from [2].

2 Definitions and Preliminaries

All graphs considered here will be finite, simple graphs. We follow the notation of [5]. In particular, the *open neighborhood* of a vertex x in a graph G is the set $N(x) = \{u \in V(G) \,|\, xu \in E(G)\}$ and its *closed neighborhood* is the set $N[x] = N(x) \cup \{x\}$. The open (respectively, closed) neighborhood of a subset S of $V(G)$ is the set $N(S) = \cup_{u \in S} N(u)$ (respectively, $N[S] = N(S) \cup S$). Recall that the degree of a vertex $x \in V(G)$ is defined as $\deg(x) = |N(x)|$, and $\Delta(G) = \max_{v \in V(G)} \deg(v)$.

A *dominating set* of a graph G is a subset $S \subseteq V(G)$ such that each vertex in G is either in S or is adjacent to a vertex in S. This condition is equivalent to requiring $x \in N[S]$ for every vertex x of G. We also say that S *dominates* G. More generally, given two sets $A, B \subset V(G)$ we will say that A *dominates* B if for any vertex $y \in B$ there exists a vertex $x \in A$ such that $y \in N[x]$. The *domination number* of a graph G is the minimum cardinality of a dominating set of G. This number is denoted $\gamma(G)$, and any dominating set with this cardinality is called a *minimum dominating set* of G.

Given a dominating set $D \subseteq V(G)$ of a graph G, we can speak about *private neighbors* of a vertex $x \in D$ with respect to D; they are the vertices in G that are dominated only by x among vertices from D. If x is not adjacent to any vertex of D then we say that x is its *own private neighbor*. We also speak about *external private neighbors* of x with respect to D, and the corresponding *external private neighborhood* is defined as $Epn(x, D) = \{y \,|\, N(y) \cap D = \{x\} \wedge y \notin D\}$.

The *Cartesian product* of two graphs G and H is the graph $G \Box H$, whose vertex set is the (set) Cartesian product $V(G) \times V(H)$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \Box H$ if either $g_1 = g_2$ and $h_1 h_2 \in E(H)$, or $g_1 g_2 \in E(G)$ and $h_1 = h_2$. The graphs G and H are called the *factors* of $G \Box H$. For a fixed $h \in V(H)$ the subgraph of $G \Box H$ induced by $\{(g, h) \mid g \in V(G)\}$ is isomorphic to G. This subgraph is called the *G-fiber through h* and is denoted G^h. In an entirely similar way the *H-fiber through g*, for a fixed $g \in V(G)$, is the subgraph ${}^gH$ induced by $\{(g, h) \mid h \in V(H)\}$. The *projection onto* G from $G \Box H$ is the map $p_G : V(G \Box H) \to V(G)$ defined by $p_G(g, h) = g$, and the *projection* onto H is defined by $p_H(g, h) = h$. The *join* of G and H is the graph $G \oplus H$ constructed from the disjoint union of G and H by adding the set of edges $\{gh \mid g \in V(G) \text{ and } h \in V(H)\}$. It will be convenient to allow one of the graphs, say H, to be empty, in which case $G \oplus H = G$.

A subset B of $V(G \Box H)$ is called a *box* in the Cartesian product $G \Box H$ if $B = X \times Y$ where $X \subseteq V(G)$ and $Y \subseteq V(H)$. It follows directly from the definition that $B \subseteq V(G \Box H)$ is a box in $G \Box H$ if and only if $B = p_G(B) \times p_H(B)$. If a minimum dominating set D of $G \Box H$ is a box in $G \Box H$, we call D a *box dominating set* or a BDS for short. Fig. 1 presents the Cartesian product of graphs G and H (note that edges in the product are omitted for clarity). Note that black vertices mark a BDS of this product graph, i.e. there exists no dominating set of $G \Box H$ with less than 10 vertices.

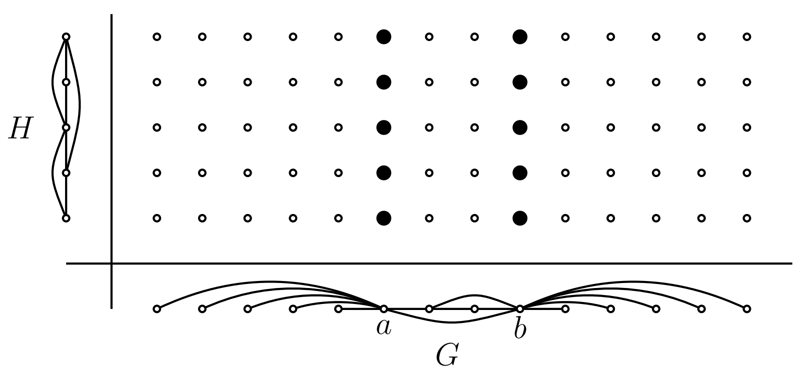

Figure 1: Box dominating set

A set $S \subset V(G \Box H)$ is called a *stairway* in the Cartesian product $G \Box H$ if $|S \cap V(G^h)| = 1$ for every $h \in V(H)$, or $|S \cap V({}^gH)| = 1$ for every $g \in V(G)$. (Observe that by choosing an appropriate order in which vertices of G are drawn, a picture of such a set S indeed resembles a stairway.) If a minimum dominating set D of $G \Box H$ is a stairway in $G \Box H$, then we call D a *stairway dominating set* or an SDS for short. Fig. 2 presents two examples of Cartesian products with a stairway dominating set. Note that the second factor in the figure on the right-hand side is the join of two graphs.

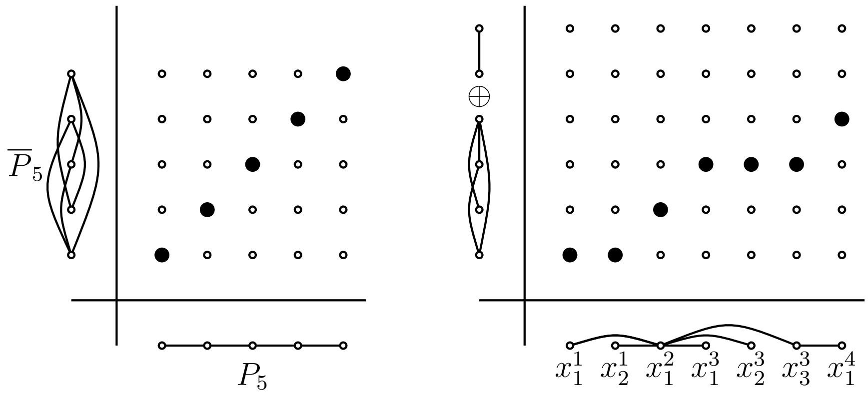

Figure 2: Stairway dominating set

In general, if D_1 and D_2 are dominating sets of G and H, respectively, then $D_1 \times D_2$ need not be a dominating set of $G \,\square\, H$. In fact, the following propositions give necessary conditions for any dominating set of a Cartesian product.

Proposition 1 *If D is any dominating set of $G \,\square\, H$, then $p_G(D)$ dominates G and $p_H(D)$ dominates H.*

Proof. Let D be a dominating set of $G \,\square\, H$ and let g be any vertex of G. Every vertex of the H-fiber ${}^gH$ is in $N[D]$, and therefore $g \in p_G(D)$ or $g \in N(p_G(D))$. Hence, $p_G(D)$ dominates G. Similarly, every G-fiber is dominated by D, and thus $p_H(D)$ dominates H. ■

Indeed, a stronger condition holds. The image of a dominating set in a Cartesian product under at least one of the projection maps is "large."

Proposition 2 *If D is any dominating set of $G \,\square\, H$, then $p_G(D) = V(G)$ or $p_H(D) = V(H)$.*

Proof. Let D be any subset of $V(G \,\square\, H)$. If there exists $g \in V(G) \setminus p_G(D)$ and $h \in V(H) \setminus p_H(D)$, then $(g, h) \notin N[D]$. Consequently, D is not a dominating set of $G \,\square\, H$. ■

An immediate consequence of Proposition 2 is that

$$\gamma(G \,\square\, H) \geq \min\{|V(G)|, |V(H)|\}\,.$$

3 The Lower Bound

In this section we give a complete characterization of those pairs of graphs G and H that achieve the lower bound in (1), which is the same as to say that $G \Box H$ contains a stairway dominating set. We need yet one more definition.

For any partition $(A_1, \ldots, A_\ell)$ of $V(G)$, we denote by $\mathcal{L}(G : A_1, \ldots, A_\ell)$ the graph that has vertex set $\{A_1, \ldots, A_\ell\}$ and edge set $\{A_iA_j \mid A_i \not\subseteq N(A_j) \text{ or } A_j \not\subseteq N(A_i)\}$.

Theorem 3 *Let G and H be arbitrary graphs such that $|V(G)| \leq |V(H)|$. The product $G \Box H$ has a stairway dominating set of cardinality $|V(G)|$ if and only if H is a join $L \oplus F$, where L is any spanning supergraph of $\mathcal{L}(G : A_1, \ldots, A_\ell)$ determined by a partition $(A_1, \ldots, A_\ell)$ of $V(G)$ and F is any graph such that $|V(F)| \geq |V(G)| - \ell$.*

Proof. To simplify the notation we will assume that the order of G is n and the order of H is $n+k$ for some nonnegative integer k. Assume first that $\gamma(G \Box H) = n$. Let D be a minimum dominating set of $G \Box H$. By Proposition 2, we may assume that $p_G(D) = V(G)$. Indeed, if $k > 0$, then this must be the case. On the other hand, if $k = 0$, then the two possibilities are symmetric.

Now, denote $p_H(D) = L = \{a_1 \ldots, a_\ell\}$, and choose the notation of vertices of G as $x_1^1, \ldots, x_{s_1}^1, \ldots, x_1^\ell, \ldots, x_{s_\ell}^\ell$ such that

$$D = \bigcup_{i=1}^{\ell} (\{x_1^i, \ldots, x_{s_i}^i\} \times \{a_i\}).$$

(Notation is as in Fig. 2.) Clearly, $\sum_{i=1}^{\ell} s_i = n$, and in each H-fiber ${}^{x_k^i}H$ there is exactly one vertex in D, namely (x_k^i, a_i). (This yields that D is an SDS of $G \Box H$.) By letting F be the graph induced by $V(H) \setminus V(L)$, we immediately derive that $|V(F)| = |V(H)| - |V(L)| \geq |V(G)| - |V(L)|$, and no other condition is imposed on F. Since for arbitrary $a_i \in V(L)$ and $u \in F$, vertices (x_1^i, a_i) and (x_1^i, u) must be adjacent, it follows that a_i and u are adjacent in H, hence H is the join of L and F.

Consider the partition of $V(G)$ by the sets $A_i = \{x_1^i, \ldots, x_{s_i}^i\}$ for all $i \in \{1, \ldots, \ell\}$. Suppose that A_i does not dominate A_j. (The case when A_j does not dominate A_i is symmetric.) Hence there exists $x_k^j \in A_j$, which is not adjacent to any $x_t^i \in A_i$. This implies that in $G \Box H$, the vertex (x_k^j, a_i) is not adjacent to any of the vertices $(x_t^i, a_i) \in D$, hence it is not dominated within the G-fiber G^{a_i}. Since D is a dominating set of the Cartesian product $G \Box H$, it must be the case that (x_k^j, a_i) is dominated within the H-fiber, in which there is only one vertex from D, namely (x_k^j, a_j). Hence a_i and a_j are adjacent in L. This observation completes the proof of this direction, because we have shown that L is isomorphic to a spanning supergraph of $\mathcal{L}(G : A_1, \ldots, A_\ell)$ determined by the partition $(A_1, \ldots, A_\ell)$.

For the converse it suffices to observe that

$$\bigcup_{i=1}^{\ell} (A_i \times \{a_i\})$$

is a dominating set of $G \square H$ of cardinality n. ■

Remark 1 *In the case when $\ell = |V(G)|$, each A_i contains exactly one vertex of G, say x_i, and the condition that one of A_i, A_j does not dominate the other coincides with the condition that x_i and x_j are not adjacent. In this case, it follows that L is isomorphic to a supergraph of the complement, $\overline{G}$, of G.*

Although stairway dominating sets are of the smallest possible cardinality among minimum dominating sets of Cartesian products of pairs of graphs with given orders, Cartesian products enjoying such a dominating set satisfy the inequality in Vizing's conjecture. Indeed, if $F \neq \emptyset$ then H is the join $L \oplus F$ of two graphs, and so $\gamma(H) \leq 2$. It is well-known that such graphs satisfy the conjecture [1]. On the other hand, if $F = \emptyset$, H is a supergraph of $\overline{G}$. In this case, by the result from [6, 7], $\gamma(G)\gamma(\overline{G}) \leq |V(G)|$, hence

$$\gamma(G)\gamma(H) \leq \gamma(G)\gamma(\overline{G}) \leq |V(G)| = \gamma(G \square H).$$

4 The Upper Bound

In this section we present two results in relation with the upper bound in (1). The first one is a sufficient condition for a Cartesian product of graphs not to have a BDS, while the second one is a sufficient condition for a product to have a BDS.

Lemma 4 *If $G \square H$ has a box dominating set D, then $p_H(D)$ is a minimum dominating set of H or $p_G(D)$ is a minimum dominating set of G.*

Proof. Assume that D is a BDS of $G \square H$. By Proposition 2 we may assume without loss of generality that $p_G(D) = V(G)$. By Proposition 1 it follows that $p_H(D)$ dominates H. Let S be any dominating set of H. Clearly, $V(G) \times S$ dominates $G \square H$. Since D is a BDS of $G \square H$,

$$|V(G)||S| \geq \gamma(G \square H) = |D| = |p_G(D) \times p_H(D)| = |V(G)||p_H(D)|\,.$$

Consequently, $|p_H(D)| \leq |S|$, and it follows that $p_H(D)$ is a minimum dominating set of H. ■

Proposition 5 *Let G and H be graphs such that $\gamma(G)|V(H)| \geq \gamma(H)|V(G)|$. If H has a minimum dominating set D_H with a vertex $v \in D_H$ such that $|Epn(v, D_H)| < \Delta(G)$, then $G \square H$ has no box dominating set.*

Proof. Assume that G and H are as described in the statement of the theorem, but for the sake of contradiction assume that $G \square H$ has a BDS D. Since $\gamma(G)|V(H)| \geq \gamma(H)|V(G)|$ we may also assume that $p_G(D) = V(G)$. Since D is a BDS it follows that $D = p_G(D) \times p_H(D) = V(G) \times p_H(D)$. Lemma 4 implies that $p_H(D)$ is a minimum dominating set of H, and hence $|p_H(D)| = |D_H|$. It follows that $V(G) \times D_H$ is also a minimum dominating set of $G \,\square\, H$.

We may assume that $Epn(v, D_H) \neq \emptyset$, since $V(G) \times D_H$ is a minimum dominating set of $G \square H$. Indeed, if $Epn(v, D_H) = \emptyset$, then $(V(G) \times D_H) \setminus \{(u, v)\}$ dominates $G \square H$ for any $u \in V(G)$ that is not isolated in G (such u exists because of the condition $0 \leq |Epn(v, D_H)| < \Delta(G)$).

Note that for any $a \in V(G)$ the external private neighborhood of the vertex (a, v) with respect to the dominating set $V(G) \times D_H$ in $G \,\square\, H$ is $\{a\} \times Epn(v, D_H)$. In other words, all vertices in the H-fiber ${}^aH$, except for the vertices from $\{a\} \times Epn(v, D_H)$ and possibly also (a, v), are already dominated by $\{a\} \times (D_H \setminus \{v\})$. Let $x \in V(G)$ be a vertex with $\deg(x) = \Delta(G)$. Note that the set $N(x) \times Epn(v, D_H)$ can also be dominated in $G \,\square\, H$ by the set $\{x\} \times Epn(v, D_H)$, and

$$|\{x\} \times Epn(v, D_H)| = |Epn(v, D_H)| < \Delta(G) = |N(x) \times \{v\}|.$$

Hence the set $D' = ((V(G) \times D_H) \setminus (N(x) \times \{v\})) \cup (\{x\} \times Epn(v, D_H))$ is a dominating set of $G \,\square\, H$ with $|D'| < |V(G) \times D_H| = |D|$ (note that in D' the vertices from $N(x) \times \{v\}$ are dominated by the vertex $(x, v) \in D'$). This is a contradiction with D being a minimum dominating set, which completes the proof. ■

The conditions on G and H in Proposition 5 are not necessary for the Cartesian product $G \,\square\, H$ not to have a BDS. This can be seen by letting $G = K_2$ and $H = C_5$. For this pair of graphs $\gamma(G)|V(H)| \geq \gamma(H)|V(G)|$. Yet, for every minimum dominating set D of H and any $x \in D$, $|Epn(v, D)| = 1 = \Delta(G)$. However, $\gamma(C_5 \,\square\, K_2) = 3$, and consequently $G \,\square\, H$ does not have a BDS.

For the sufficient condition in the other direction we need the notion of rainbow domination, as introduced in [2]. Let G be a graph and let f be a function that assigns to each vertex a set of colors chosen from the set $\{1, \ldots, k\}$; that is, $f \colon V(G) \to \mathcal{P}(\{1, \ldots, k\})$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$, we have

$$\bigcup_{u \in N(v)} f(u) = \{1, \ldots, k\},$$

then f is called a *k-rainbow dominating function* (kRDF) of G. The *weight*, $w(f)$, of a function f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. Given a graph G, the minimum weight of a kRDF is called the *k-rainbow domination number of* G, which we denote by $\gamma_{\mathrm{r}k}(G)$. (The 1-rainbow domination number of G is just the domination number of G.)

Rainbow domination in a graph G has a natural connection with the study of $\gamma(G \,\square\, K_k)$. It is easy to verify that for $k \geq 1$ and for any graph G,

$$\gamma_{\mathrm{r}k}(G) = \gamma(G \,\square\, K_k).$$

In particular, for any graph G for which $\gamma_{\mathrm{r}k}(G) = k\gamma(G)$ it follows that $\gamma(G \,\Box\, K_k) = k\gamma(G)$, and so $G \,\Box\, K_k$ has a box dominating set. In the next proposition we will show even more.

Proposition 6 *Let G be a graph such that $\gamma_{\mathrm{r}k}(G) = k\gamma(G)$, for some $k \geq 2$. If H is any graph of order k, then $\gamma(G \,\Box\, H) = k\gamma(G)$ and $G \,\Box\, H$ has a box dominating set.*

Proof. Since H is a spanning subgraph of K_k, $G \,\Box\, H$ is a spanning subgraph of $G \,\Box\, K_k$. Clearly $\gamma(G \,\Box\, H) \geq \gamma(G \,\Box\, K_k)$, and as above we note that $\gamma(G \,\Box\, K_k) = k\gamma(G)$. Altogether, using (1), we have $\gamma(G \,\Box\, H) = \gamma(G)|V(H)|$, which implies that $G \,\Box\, H$ has a box dominating set. ■

To see that the converse of Proposition 6 does not hold, consider the graph G in Figure 3. Let $f\colon V(G) \to \mathcal{P}(\{1,\ldots,4\})$ be defined by $f(a_1) = \{1\}$, $f(a_2) = \{2\}$, $f(a_3) = \{3\}$, $f(b_1) = \{1\}$, $f(b_2) = \{1\}$, $f(b_3) = \{1\}$, $f(b) = \{4\}$, and $f(a) = \emptyset$. Clearly, f is a 4-rainbow dominating function, and thus $\gamma_{r4}(G) \leq 7 < 4\gamma(G)$. Now, let $H = P_4$ with vertex set $\{u, v, w, x\}$ where v is of degree 2 with neighbors u of degree 1 and w of degree 2. It is straightforward to verify that $\gamma(G \,\Box\, H) = 8$ and $\{a, b\} \times V(H)$ is a BDS.

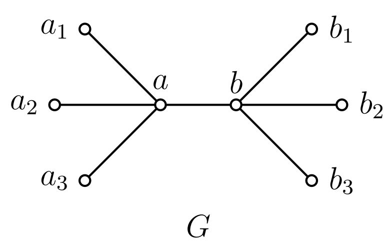

Figure 3: A graph G with $\gamma_{r4}(G) < 4\gamma(G)$

References

[1] B. Brešar, P. Dorbec, W. Goddard, B. L. Hartnell, M. A. Henning, S. Klavžar, and D. F. Rall, Vizing's conjecture: a survey and recent results, J. Graph Theory 69 (2012) 46–76.

[2] B. Brešar, M. A. Henning, and D. F. Rall, Rainbow domination in graphs, Taiwanese J. Math. 12 (2008) 213–225.

[3] M. El-Zahar, and C. M. Pareek, Domination number of products of graphs, Ars Combin. 31 (1991) 223–227.

[4] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[5] W. Imrich, S. Klavžar, and D. F. Rall, *Topics in Graph Theory: Graphs and Their Cartesian Product*, A K Peters Ltd., Wellesley, MA, 2008.

[6] F. Jaeger, and C. Payan, Relations du type Nordhaus-Gaddum pour le nombre d'absorption d'un graphe simple, C.R. Acad. Sc. (A) 274 (1972) 728–730.

[7] C. Payan, and N. H. Xuong, Domination-balanced graphs, J. Graph Theory 6 (1982) 23–32.

[8] V. G. Vizing, The cartesian product of graphs, Vyčisl. Sistemy 9 (1963) 30–43.

[9] V. G. Vizing, Some unsolved problems in graph theory, Uspehi Mat. Nauk 23 (6(144)) (1968) 117–134.