Mathematics 460　　**TEST II**　　**Due: 10-30-17**

The test must be turned in by the beginning of class on Monday, October 30, 2017. You are **not** allowed to use any mathematical software, but **you may** use a calculator to do ordinary multiplication and addition. You are to work alone without any online sources. That is, this work is to be your own! You **are** allowed to consult your class notes and your textbook - but no other books. Since you are allowed to use your textbook while working on the test problems, please supply adequate references to theorems, corollaries, etc. that you use to justify your reasoning on a problem.

Use LaTeX to write up your solutions and remember to print on one side of the paper only. (The .tex file for the set of problems is available online on the "Written Assignments" tab.) Remember to show your work as completely as possible and to write using correct mathematical notation.

Print your name here

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Sign below to pledge that you followed the directions above.

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1. In both parts of this problem $G = S_{14}$, the symmetric group on 14 letters. Consider Theorem 9.8.

(a) [5 points] For each $k \in \{6, 10, 11, 12, 15, 18, 20, 21\}$ give a specific $\alpha_k \in G$ such that α_k has order k.

(b) [5 points] Find the largest integer m such that G has an element of order m. Justify your answer. Give a specific permutation $\beta \in G$ such that β has order m.

2. [10 points] Find all the abelian groups, up to isomorphism, of order 784. Write them in the form given in the statement of The Fundamental Theorem of Finitely Generated Abelian Groups in our textbook. For each group H in your list, determine whether H has an element of order 56. If it does, then explicitly give one such element. If H does not have an element of order 56 but has a subgroup of order 56, then explicitly give one such subgroup.

3. [10 points] Count the number of $\alpha \in S_7$ such that $\alpha^2 = id$, where id is the identity element of the group S_7. Use counting techniques and briefly explain how you get your numbers. (Do **not** make a list of such permutations and count them.)

4. [10 points] Let G be a group of order 1020, let $H \leq G$ and $K \leq G$ such that $(G : H) = 20$ and $(G : K) = 30$. Prove that $H \cap K$ is a cyclic subgroup of G. (You do not need to prove that $H \cap K$ is a subgroup of G; that was done in the textbook and we proved it in class. Just give an appropriate reference for that part. You only need to prove that $H \cap K$ is cyclic.)

5. [10 points] Let G be an arbitrary group of order 121 with identity element e. Prove that either G is cyclic or $x^{11} = e$ for every x in G. You are **not** allowed to assume that G is abelian.

6. [10 points] Let $G = \langle a \rangle$ be a cyclic group of order 28 (written multiplicatively) and let H be the subgroup of G generated by a^{12}. Make an explicit list of the distinct left cosets of H in G. (Write each left coset in the form bH for some b and also as a set of elements from the group G.)

7. [5 points each]

a. Let $x = (6, (1, 3, 7, 4)(2, 6, 5)(8, 9))$. Find the order of x in the direct product $\mathbb{Z}_{14} \times S_{10}$.

b. Let $w = ((1, 3, 2)(2, 5, 1, 4)(7, 1, 6)(9, 7, 10, 2)\ ,\ 28)$. Find the order of w in the direct product $S_{12} \times \mathbb{Z}_{60}$.

c. Find an element of largest possible order in the direct product $U_{12} \times \mathbb{Z}_{100}$. Justify your answer.

8. Let G be a group where the binary operation is written multiplicatively. For $n \in \mathbb{Z}^+$, let $P_n(G) = \{a^n \mid a \in G\}$.

a. [5 points] Prove that if G is abelian and n is any positive integer, then $P_n(G)$ is a subgroup of G.

b. [5 points] Show that the assumption that the group is abelian is needed in part **a.** by computing $P_3(S_3)$ and showing it is not a subgroup of S_3.

9. Recall that for a nonempty set X, the group of all permutations of X is denoted S_X. Let's use I to denote the identity element of this group. Now let $X = \mathbb{Z}^+$, the set of positive integers, and let $T = \{\sigma \in S_X : \{n \in X : \sigma(n) \neq n\} \text{ is a finite set}\}$. That is, a permutation σ of X belongs to the set T if and only if σ fixes all but finitely many positive integers.

(a) [5 points] Prove that T is a subgroup of S_X.

(b) [5 points] Prove that for any $\alpha \in T$, the subgroup $\langle \alpha \rangle$ is a finite subgroup of S_X.

(c) [5 points] For each positive integer n give a specific $p \in T$ such that p has order n.

(d) [5 points] Let f and g be the permutations in S_X defined by

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is odd,} \\ x-1 & \text{if } x \text{ is even.} \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } x = 1, \\ x+1 & \text{if } x \text{ is even,} \\ x-1 & \text{if } x \text{ is odd and } x \geq 3. \end{cases}$$

Determine whether $fT = gT$. Justify your answer.