Mathematics 460 TEST I Due: 9-25-17

You are allowed to work on this test during any single, continuous 72 hour time interval. In other words, after you first look at the problems on the sheets that follow you must stop working on them 72 hours later. Putting your solutions into LaTeX is included in these 72 hours. The test must be turned in at the beginning of class on Monday, September 25, 2017. You are **not** allowed to use any mathematical software, but **you may** use a calculator to do ordinary multiplication and addition. You are to work alone without any online sources. That is, this work is to be your own! You **are** allowed to consult your class notes and your textbook - but no other books.

Since you are allowed to use your textbook while working on the test problems, please supply adequate references to theorems, corollaries, etc. that you use to justify your reasoning on a problem.

Use LaTeX to write up your solutions and remember to print on one side of the paper only. (The .tex file for the set of problems is available online on the "Written Assignments" tab.) Remember to show your work as completely as possible and to write using correct mathematical notation.

Record on the lines below the day and time you started working on the test and the day and time you stopped working on the test.

Start: ________________________

Finish: ________________________

Print your name here

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Sign below to pledge that you followed the directions above.

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Each problem is worth 15 points.

1(i). Prove that the group $G = \langle \mathbb{Z}_{17}^*, \cdot_{17} \rangle$ is a cyclic group by finding a generator a.

(ii). Use the generator you found in part (i) to determine all of the generators of G.

(iii) Make a list of all the **distinct** subgroups of G.

2. Let F be a subgroup of a group G, let b be a fixed element of the group G and let $K = \{\, b^{-1}ab \mid a \in F \,\}$. Prove that K is a subgroup of G.

3. Let $A = \{m + n\sqrt{3} \mid m, n \in \mathbb{Z}\}$ and let $B = \{2^r 5^s \mid r, s \in \mathbb{Z}\}$. $\langle A, + \rangle$ and $\langle B, \cdot \rangle$ are binary algebraic structures, where $+$ is ordinary addition of real numbers and $\cdot$ is ordinary multiplication of real numbers. Show that $\langle A, + \rangle$ is isomorphic to $\langle B, \cdot \rangle$.

4. Two different exercises in the textbook ask you to prove that if G is a finite group then for every $a \in G$, there exists a positive integer n such that $a^n = e$. You may assume those two exercises. Prove the following stronger statement. If G is a finite group, then there exists a positive integer M such that for every $b \in G$, $b^M = e$.

5. Consider the binary algebraic structure $\langle \mathbb{Z}, * \rangle$ where the binary operation $*$ is defined by $a * b = a + b - 3$. (The $+$ and $-$ used in this definition are ordinary addition and subtraction of integers.)

(i) Prove that $\langle \mathbb{Z}, * \rangle$ is a group.

(ii) Find the element x in this group such that $(2 * 5^{-1}) * (x * 4) = 1 * 0^{-1}$. (Of course, in such an equation a^{-1} means the inverse of a in the group $\langle \mathbb{Z}, * \rangle$.)

6. Assume that G is a finite group and $b \in G$ such that b has order n. Prove that for every $a \in G$, the element aba^{-1} also has order n.

7. Let G be a group (written multiplicatively) and let H and K be subgroups of G. Suppose that H is not a subset of K and K is not a subset of H. Prove that $H \cup K$ is not a subgroup of G.

8. The two sets $\mathbb{C}^*$ and $\mathbb{R}^*$ have the same cardinality (that is, there is a one-to-one and onto function $f : \mathbb{C} \to \mathbb{R}$). Prove that the two groups $\mathbb{C}^*$ (with binary operation multiplication of non-zero complex numbers) and $\mathbb{R}^*$ (with binary operation multiplication of non-zero real numbers) are **not** isomorphic.

9. Let G be a cyclic group of order 90 with generator a. Solve the following problems regarding this group G. Be sure to supply the reference for any result from the text that you are using to justify your answers.

(i) Find $|\langle a^{35} \rangle|$.

(ii) List five generators, other than a, for G.

(iii) G has a subgroup H of order 15. Give three distinct generators for H.

10. Let G be a group such that $(ab)^3 = a^3b^3$ and $a^2b^2 = b^2a^2$ for every $a, b \in G$. Prove that G is abelian.