Mathematics 460 Final Exam 12-11-17 Name:

Your final exam must be turned in by noon on Monday, December 11, 2017. You are **not** allowed to use any mathematical software, but **you may** use a calculator to do ordinary multiplication and addition. You are to work alone without any online sources. That is, this work is to be your own! You **are** allowed to consult your class notes and your textbook - but no other books. Since you are allowed to use your textbook while working on the test problems, please supply adequate references to theorems, corollaries, etc. that you use to justify your reasoning on a problem.

Problems will be added to this exam as we progress toward the end of the term. No problems will be added after Tuesday, December 5. Until then please check back on the "Written Assignments" tab to find the updated version of the exam.

Use LaTeX to write up your solutions and remember to print on one side of the paper only. (The .tex file for the set of problems is available online on the "Written Assignments" tab.) Remember to show your work as completely as possible and to write using correct mathematical notation.

Print your name here

____

Sign below to pledge that you followed the directions above.

____

1. [10 points] Let $\alpha = (2,5,6,8,4)(1,8,3,7)$, let $\beta = (1,5,2,8,6,3,4)$ in S_8, let $\sigma = \alpha\beta$ and let $H = \langle\sigma\rangle$. Find the **smallest** positive integer n such that $\mathbb{Z}_n$ has a **proper** subgroup K, which is isomorphic to H. Find one specific isomorphism $f : H \to K$ and give $f(\sigma)$ and $f(\sigma^3)$.

2. [10 points] Let G be a group of order p^n, where p is a prime and n is a positive integer such that $n \geq 2$. Prove that G has a subgroup of order p. Be sure you do **not** assume that G is abelian.

3. [10 points] Let $\{H_k\}_{k\in\mathbb{Z}^+}$ be a sequence of subgroups of a group G. Suppose that for each $k \geq 1$, $H_k \subset H_{k+1}$ but $H_k \neq H_{k+1}$. Let $H = \bigcup_{k\in\mathbb{Z}^+} H_k$. Prove that H is a subgroup of G but that H is not cyclic.

4. [10 points] Show that the group $\langle\mathbb{Q}^+, \cdot\rangle$, the positive rational numbers under multiplication, is not cyclic.

5. [10 points] Let R be a commutative ring with unity and let a and b be elements of R. Prove that if ab is a unit, then both a and b are units.

6. [10 points] An element a of a group is a **square** if there exists an element b in the group so that $a = b^2$. Let G be an abelian group and let H be a subgroup of G. If every element of H is a square and every element of the factor group G/H is a square, prove that every element of G is a square.

7. [10 points] Recall that when R is a ring with unity, we denote the group of units in R by $U(R)$. (The binary operation on $U(R)$ is the multiplication in the ring R. See Exercise #37 on page 176.) Let S and T be rings with unity, and let $R = S \times T$, the direct product of the two **rings** S and T. Prove that $U(R) = U(S) \times U(T)$, the direct product of the two **groups** $U(S)$ and $U(T)$.

8. [10 points] Let $R = \mathbb{Z}_7 \times \mathbb{Z}_{20}$, the direct product of the two rings $\mathbb{Z}_7$ and $\mathbb{Z}_{20}$. Classify the group $U(R)$ in the form given by FTFGAG, Theorem 11.12. (Please do problem #7 above before you do this one. Even if you do not do #7, you can use it in doing this one.)

9. [10 points] Let R be a commutative ring with no divisors of 0, and let G be the abelian group $\langle R, +\rangle$. Assume that every element of G has finite order. Prove that all of the nonzero elements of G have the **same** order. (Do not assume that R is a finite ring.)

10. [10 points] Find all integer solutions to the congruence $56x \equiv 70 \pmod{455}$ using the techniques from Section 20 in the text.