Prove that the symmetric difference is an associative operation; that is, for any sets A, B and C, we have $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

We are assuming that the three sets A, B and C are all subsets of a fixed universal set U. In the proof we use the definition of symmetric difference (see the top of page 69), the distributive law, DeMorgan's law, and the following fact repeatedly: for any two subsets X and Y of U, $X - Y = X \cap \overline{Y}$. You should verify this yourself. Combining the first and last of these we see that $X \triangle Y = (X \cap \overline{Y}) \cup (Y \cap \overline{X})$.

$$
\begin{aligned}
A \triangle (B \triangle C) &= \left(A \cap (\overline{B \triangle C})\right) \cup \left((B \triangle C) \cap \overline{A}\right) \\
&= \left(A \cap \left[\overline{(B \cap \overline{C}) \cup (C \cap \overline{B})}\right]\right) \cup \left(\left[(B \cap \overline{C}) \cup (C \cap \overline{B})\right] \cap \overline{A}\right) \\
&= \left(A \cap (\overline{B \cap \overline{C}}) \cap (\overline{C \cap \overline{B}})\right) \cup \left((B \cap \overline{C}) \cap \overline{A}\right) \cup \left((C \cap \overline{B}) \cap \overline{A}\right) \\
&= A \cap \left((\overline{B} \cup C) \cap (\overline{C} \cup B)\right) \cup \left((B \cap \overline{C}) \cap \overline{A}\right) \cup \left((C \cap \overline{B}) \cap \overline{A}\right) \\
&= A \cap \left[\left(\overline{B} \cap (\overline{C} \cup B)\right) \cup \left(C \cap (\overline{C} \cup B)\right)\right] \cup \left((B \cap \overline{C}) \cap \overline{A}\right) \cup \left((C \cap \overline{B}) \cap \overline{A}\right) \\
&= A \cap \left[(\overline{B} \cap \overline{C}) \cup (\overline{B} \cap B) \cup (C \cap \overline{C}) \cup (C \cap B)\right] \cup \left((B \cap \overline{C}) \cap \overline{A}\right) \cup \left((C \cap \overline{B}) \cap \overline{A}\right) \\
&= A \cap \left[(\overline{B} \cap \overline{C}) \cup \emptyset \cup \emptyset \cup (C \cap B)\right] \cup \left((B \cap \overline{C}) \cap \overline{A}\right) \cup \left((C \cap \overline{B}) \cap \overline{A}\right) \\
&= A \cap \left[(\overline{B} \cap \overline{C}) \cup (C \cap B)\right] \cup \left((B \cap \overline{C}) \cap \overline{A}\right) \cup \left((C \cap \overline{B}) \cap \overline{A}\right) \\
&= (A \cap \overline{B} \cap \overline{C}) \cup (A \cap C \cap B) \cup (B \cap \overline{C} \cap \overline{A}) \cup (C \cap \overline{B} \cap \overline{A})
\end{aligned}
$$

Similarly, since

$$X \triangle Y = (X \cap \overline{Y}) \cup (Y \cap \overline{X}) = (Y \cap \overline{X}) \cup (X \cap \overline{Y}) = Y \triangle X\,,$$

we see that

$$
\begin{aligned}
(A \triangle B) \triangle C &= C \triangle (A \triangle B) \\
&= \left(C \cap (\overline{A \triangle B})\right) \cup \left((A \triangle B) \cap \overline{C}\right) \\
&= \left(C \cap \left[\overline{(A \cap \overline{B}) \cup (B \cap \overline{A})}\right]\right) \cup \left(\left[(A \cap \overline{B}) \cup (B \cap \overline{A})\right] \cap \overline{C}\right) \\
&= \left(C \cap (\overline{A \cap \overline{B}}) \cap (\overline{B \cap \overline{A}})\right) \cup \left((A \cap \overline{B}) \cap \overline{C}\right) \cup \left((B \cap \overline{A}) \cap \overline{C}\right) \\
&= C \cap \left((\overline{A} \cup B) \cap (\overline{B} \cup A)\right) \cup \left((A \cap \overline{B}) \cap \overline{C}\right) \cup \left((B \cap \overline{A}) \cap \overline{C}\right) \\
&= C \cap \left[\left(\overline{A} \cap (\overline{B} \cup A)\right) \cup \left(B \cap (\overline{B} \cup A)\right)\right] \cup \left((A \cap \overline{B}) \cap \overline{C}\right) \cup \left((B \cap \overline{A}) \cap \overline{C}\right) \\
&= C \cap \left[(\overline{A} \cap \overline{B}) \cup (\overline{A} \cap A) \cup (B \cap \overline{B}) \cup (B \cap A)\right] \cup \left((A \cap \overline{B}) \cap \overline{C}\right) \cup \left((B \cap \overline{A}) \cap \overline{C}\right) \\
&= C \cap \left[(\overline{A} \cap \overline{B}) \cup \emptyset \cup \emptyset \cup (B \cap A)\right] \cup \left((A \cap \overline{B}) \cap \overline{C}\right) \cup \left((B \cap \overline{A}) \cap \overline{C}\right) \\
&= C \cap \left[(\overline{A} \cap \overline{B}) \cup (B \cap A)\right] \cup \left((A \cap \overline{B}) \cap \overline{C}\right) \cup \left((B \cap \overline{A}) \cap \overline{C}\right) \\
&= (C \cap \overline{A} \cap \overline{B}) \cup (C \cap B \cap A) \cup (A \cap \overline{B} \cap \overline{C}) \cup (B \cap \overline{A} \cap \overline{C})
\end{aligned}
$$

Now, since union and intersection are commutative operations on sets, we get that

$$\begin{aligned}
A \triangle (B \triangle C) &= (A \cap \overline{B} \cap \overline{C}) \cup (A \cap C \cap B) \cup (B \cap \overline{C} \cap \overline{A}) \cup (C \cap \overline{B} \cap \overline{A}) \\
&= (C \cap \overline{A} \cap \overline{B}) \cup (C \cap B \cap A) \cup (A \cap \overline{B} \cap \overline{C}) \cup (B \cap \overline{A} \cap \overline{C}) \\
&= (A \triangle B) \triangle C
\end{aligned}$$

If you think about this for a moment you can see that the symmetric difference of 3 sets (in any order) is the set consisting of those elements that belong to exactly one of the three sets or to all three sets. Can you give a verbal description of the set of elements that comprise

$$A_1 \triangle A_2 \triangle \cdots \triangle A_n$$

for an arbitrary positive integer n?