Prove that the symmetric difference is an associative operation; that is, for any sets \( A, B \) and \( C \), we have \( A \triangle (B \triangle C) = (A \triangle B) \triangle C \).

We are assuming that the three sets \( A, B \) and \( C \) are all subsets of a fixed universal set \( U \). In the proof we use the definition of symmetric difference (see the top of page 69), the distributive law, DeMorgan’s law, and the following fact repeatedly: for any two subsets \( X \) and \( Y \) of \( U \), \( X – Y = X \cap Y^c \). You should verify this yourself. Combining the first and last of these we see that \( X \triangle Y = (X \cap Y^c) \cup (Y \cap X^c) \).

\[
A \triangle (B \triangle C) = (A \cap (B \triangle C)) \cup ((B \triangle C) \cap A)
= (A \cap [(B \cap C^c) \cup (C \cap B^c)]) \cup ((B \cap C^c) \cap A) \cup ((C \cap B^c) \cap A)
= (A \cap (B \cap C^c) \cap (C \cap B^c)) \cup ((B \cap C^c) \cap A) \cup ((C \cap B^c) \cap A)
= A \cap ((B \cap C^c) \cap (C \cap B^c)) \cup ((B \cap C^c) \cap A) \cup ((C \cap B^c) \cap A)
= A \cap [((B \cap C^c) \cup (C \cap B^c)) \cup ((B \cap C^c) \cap A) \cup ((C \cap B^c) \cap A)]
= A \cap [(B \cap C^c) \cup B \cup C \cap B \cup C \cap B^c \cup B \cap C^c \cap A \cup (C \cap B \cap A)]
= A \cap [(B \cap C^c) \cup (C \cap B) \cup (C \cap B \cap A) \cup (C \cap B) \cap A \cup (C \cap B) \cap A]
= (A \cap B \cap C^c) \cup (A \cap C \cap B) \cup (B \cap C \cap A) \cup (C \cap B \cap A)

Similarly, since

\[
X \triangle Y = (X \cap Y^c) \cup (Y \cap X^c) = (Y \cap X^c) \cup (X \cap Y) = Y \triangle X,
\]
we see that

\[
(A \triangle B) \triangle C = C \triangle (A \triangle B)
= (C \cap (A \triangle B)) \cup ((A \triangle B) \cap C)
= (C \cap [(A \cap B^c) \cup (B \cap A^c)]) \cup ((A \cap B^c) \cap C) \cup ((B \cap A^c) \cap C)
= (C \cap (A \cap B^c) \cap (B \cap A^c)) \cup ((A \cap B^c) \cap C) \cup ((B \cap A^c) \cap C)
= C \cap ((A \cup B) \cap (B \cup A)) \cup ((A \cap B) \cap C) \cup ((B \cap A) \cap C)
= C \cap [(A \cap B) \cup (B \cap A)] \cup ((A \cap B) \cap C) \cup ((B \cap A) \cap C)
= C \cap [(A \cap B) \cup (B \cap A)] \cup ((A \cap B) \cap C) \cup ((B \cap A) \cap C)
= (C \cap A \cap B) \cup (C \cap B \cap A) \cup (A \cap B \cap C) \cup (B \cap A \cap C)
Now, since union and intersection are commutative operations on sets, we get that

\[ A \triangle (B \triangle C) = (A \cap \overline{B} \cap \overline{C}) \cup (A \cap C \cap B) \cup (B \cap \overline{C} \cap \overline{A}) \cup (C \cap \overline{B} \cap \overline{A}) \]
\[ = (C \cap A \cap B) \cup (C \cap B \cap A) \cup (A \cap B \cap C) \cup (B \cap A \cap C) \]
\[ = (A \triangle B) \triangle C \]

If you think about this for a moment you can see that the symmetric difference of 3 sets (in any order) is the set consisting of those elements that belong to exactly one of the three sets or to all three sets. Can you give a verbal description of the set of elements that comprise

\[ A_1 \triangle A_2 \triangle \cdots \triangle A_n \]

for an arbitrary positive integer \( n \)?