Basic Information About Sets and Interval Notation

A **set** is a collection of distinct objects. The objects in the collection are usually referred to as the **elements** of the set. If there is a small, finite, number of elements in a given set, then we can completely describe the set by listing its elements—separated by commas—inside set brackets $\{\quad\}$. For example, the set $X = \{a, b, c, d, e, f, g\}$ is the collection consisting of the first 7 lower-case letters of the alphabet. We often give a name (usually an upper-case letter; in this example X) to a set so that we can easily refer to it without having to write out the entire list of its elements each time.

The important feature of a particular set A is which objects are in A, and which objects are not in A. This means that two sets are the same (or **equal**) when they have exactly the same objects. The notation $r \in A$, which is read "r is an element of A", means that r is one of the objects that belongs to the set A. $s \notin A$ is read "s is not an element of A", and means exactly what it says. The order in which the elements are listed does not matter. Thus, all of $\{a, b, c, d, e, f, g\}$, $\{e, c, a, g, f, b, d\}$ and $\{a, e, b, c, d, f, g\}$ are the same set, X, and so we write

$$X = \{a, b, c, d, e, f, g\} = \{e, c, a, g, f, b, d\} = \{a, e, b, c, d, f, g\}.$$

It will often be necessary to talk about a set that has so many elements that it would be impossible—or at least a waste of time—to list all of its elements. When this is the case we use what is called **set-builder** notation. This involves using a **dummy variable** and a precise description or a property that a generic object—represented by the dummy variable—must satisfy, in order to be an element of the set. For example, suppose we wanted to refer to the sets G, S and B of individuals who won gold, silver or bronze medals, respectively, for the United States at the Beijing Olympics. We may not even know the names of all the appropriate individuals, but we can still specify these sets as illustrated here.

- G is the set of all persons who won a gold medal for the United States at the Beijing Olympics. The set-builder (mathematical) notation used to write this is $G = \{x : x \text{ is a person who won a gold medal for the United States at the Beijing Olympics.}\}$ This is read as "G is the set of all x, **such that** x is a person ... Beijing Olympics." Notice how in this set-builder notation the colon (":") is read "such that." Some people use a vertical bar ("|") instead of a colon.

Exercise 1. Write set-builder notation for the sets S and B defined above.

Exercise 2. Write set-builder notation for $\{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$.

If X is any set, then by a **subset** of X we mean a set Y such that each element of Y is also an element of X. In other words Y is a subset of X (this is written as $Y \subseteq X$) precisely when it is the case that if $a \in Y$ is true, then $a \in X$ is also true. For example, if C denotes the set of students registered for our class, then there are many different subsets of C that we could define. We might let

$$F = \{f : f \in C \text{ and } f \text{ is a female}\};$$

or

$$M = \{m : m \in C \text{ and } m \text{ is a male}\};$$

or

$$E = \{x : x \in C \text{ and } x \text{ is eighteen years old}\}.$$

Just by the way we defined these sets F, M and E it follows that each is a subset of C.

Suppose that for each integer n between 15 and 25, inclusive, we define

$$S_n = \{c \,:\, c \in C \text{ and } c \text{ is } n \text{ years old}\}\,.$$

In this new naming notation $E = S_{18}$. From experience I suspect that some of these sets, though clearly defined, do not have any members. For example, I suspect that E_{15}, E_{25} as well as E_{24} and several others, do not actually contain any students.

This is okay, and it is not a problem with the set-builder definition of these sets. It will be convenient to have a way to indicate the fact that a set has no elements. If a set does not contain any elements, then it is called the **empty set** and is denoted by $\emptyset$. It turns out that for the purposes of analyzing combinatorial games, the empty set will be one of the most important sets.

Notice that $\emptyset$ is a subset of every set X; that is $\emptyset \subseteq X$ for every set X. To see why this is so let's consider what we have to show. The definition of subset given above indicates that we must show that if $a \in \emptyset$ is true, then $a \in X$. But, since the statement $a \in \emptyset$ is never true, we never have to check to see whether $a \in X$. Thus, $\emptyset \subseteq X$.

If we have two given sets there are a number of ways to use them to specify a new set. For example, in the case above where we were considering the set C of students in our class, we might be interested in the subset of C consisting of those students who are 19 years old and are female. What we have done is to put two requirements—both of which must be satisfied—on a student to belong to the subset we are interested in. In terms of our previously defined sets we are requiring a student to belong to both E_{19} and to F to be in this new subset. This is an example of the intersection of two sets.

In particular, for sets A and B, the set

$$A \cap B = \{x \,:\, x \in A \text{ and } x \in B\}$$

is called the **intersection** of A and B. The intersection of A and B consists of all elements belonging to both A and B.

The **union** of A and B is the set

$$A \cup B = \{x \,:\, x \in A \text{ or } x \in B\}$$

and is the set consisting of all elements belonging to A, or to B, or to both. (By the way, unless we explicitly indicate otherwise, the word "or" will be used in this inclusive way—one or the other, or both—in our class.)

Most of the sets we will use in calculus are sets of real numbers. The set of all real numbers is denoted $\mathbb{R}$. Certain kinds of subsets of $\mathbb{R}$ will be especially important. Those are the intervals. If $a < b$, then the **closed interval from a to b**, written as $[a, b]$, is the set of all real numbers between a and b, inclusive (i.e., including a and b). In set-builder notation this is

$$[a, b] = \{x \,:\, a \le x \le b\}\,.$$

If we want to speak of this set without including one (or both) of its endpoints we write

$$[a, b) = \{x \,:\, a \le x < b\} \quad (a, b] = \{x \,:\, a < x \le b\} \quad \text{or} \quad (a, b) = \{x \,:\, a < x < b\}\,.$$

(a, b) is called an **open interval**. We will often need to speak of all the numbers larger than some fixed number a. There is an interval notation for this as well. It uses some mathematical symbols, ∞ or $-\infty$, that are **not** themselves numbers but are used in several different contexts to convey certain information. Here are some examples of intervals whose notation involves these symbols. You should be able to see how to generalize from these examples.

- $(-2, \infty) = \{t : -2 < t\}$
- $[\frac{2}{3}, \infty) = \{z : z \geq \frac{2}{3}\}$
- $(-\infty, -9] = \{x : x \leq -9\}$
- $(-\infty, \infty) = \{r : r \text{ is a real number}\} = \mathbb{R}$

Now, to test your understanding of these ideas and notations convince yourself of the correctness of each of the following statements:

1. $(-\infty, 6] \cup (-3, 18] = (-\infty, 18]$
2. $(-3, 8] \cup [6, 19] = (-3, 19]$
3. $(-\infty, 5) \cap [0, 7) = [0, 5)$
4. $(-1, \infty) \cap [2, 4) = [2, 4)$
5. $(-10, -3] \cap [0, 1] = \emptyset$
6. $[-10, 10] \cup (-3, 5] = [-10, 10]$
7. $[5, 5] = \{5\}$
8. $[5, 5) = \emptyset$